A survey of elementary totally disconnected locally compact groups

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Abstract The class of elementary totally disconnected locally compact (t.d.l.c.) groups is the smallest class of t.d.l.c. second countable (s.c.) groups which contains the second countable profinite groups and the countable discrete groups and is closed under taking closed subgroups, Hausdorff quotients, group extensions, and countable directed unions of open subgroups. This class appears to be fundamental to the study of t.d.l.c. groups. In these notes, we give a complete account of the basic properties of the class of elementary groups. The approach taken here is more streamlined than previous works, and new examples are sketched.

1 Introduction

In the general study of totally disconnected locally compact (t.d.l.c.) groups, one often wishes to avoid discrete groups and compact t.d.l.c., equivalently profinite, groups. For example, considering finitely generated groups as lattices in themselves is unenlightening, and the scale function on a profinite group is trivial. However, non-discreteness and non-compactness are often not enough by themselves. For example, every finitely generated group is a lattice in a non-discrete t.d.l.c. group simply by taking a direct product with an infinite profinite group. We thus wish to study t.d.l.c. groups that are 'sufficiently non-discrete.'

What we mean by 'sufficiently non-discrete' is that there is a suitably rich interaction between the topological structure and the large-scale structure of the group in question. With this in mind, let us consider examples. Certainly discrete groups

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have weak interaction between topological and large-scale structure, since they have trivial topological structure. The profinite groups have the opposite problem: they have local structure but trivial large-scale structure. On the other hand, compactly generated t.d.l.c. groups which are non-discrete and topologically simple have a rich interaction between topological and large-scale structure; examples of these include the Neretin groups, $\mathrm{Aut}(T_n)^+$ for T_n the n-regular tree, the simple algebraic groups, and many others. Thinking further, it is clear that abelian groups and compact-by-discrete groups have much weaker interaction between topological and large-scale structure than that of the aforementioned simple groups, so these groups should be collected with the profinite groups and discrete groups. At this point, it seems natu-

Fig. 1 Interaction between large-scale structure and topological structure

Weak interaction	**Strong interaction**
Profinite groups, discrete groups	$\mathrm{Aut}(T_n)$
abelian groups	The Neretin groups
profinite-by-discrete groups, e.g. $A_5^{\mathbb{Z}} \rtimes \mathbb{Z}$	$SL_n(\mathbb{Q}_p)$

ral to conclude that any 'elementary' combination of groups with weak interaction should again have weak interaction. We thus arrive to the central definition of these notes:

Definition 1. The class of **elementary groups** is the smallest class $\mathscr{E}$ of t.d.l.c.s.c. groups such that

(i) $\mathscr{E}$ contains all second countable profinite groups and countable discrete groups.
(ii) $\mathscr{E}$ is closed under taking closed subgroups.
(iii) $\mathscr{E}$ is closed under taking Hausdorff quotients.
(iv) $\mathscr{E}$ is closed under taking group extensions.
(v) If G is a t.d.l.c.s.c. group and $G = \bigcup_{i\in\mathbb{N}} O_i$ where $(O_i)_{i\in\mathbb{N}}$ is an $\subseteq$-increasing sequence of open subgroups of G with $O_i \in \mathscr{E}$ for each i, then $G \in \mathscr{E}$. We say that $\mathscr{E}$ is **closed under countable increasing unions**.

The operations $(\mathrm{ii}) - (\mathrm{v})$ are often called the **elementary operations**.

Remark 1. We restrict to the second countable t.d.l.c. groups. This is a mild and natural assumption which makes our discussion much easier. Any notion of being 'elementary' must be 'regional' in the sense that it reduces to compactly generated subgroups, and compactly generated groups are second countable modulo a compact normal subgroup. Generalizing our notion of elementary groups to the non-second countable setting thus adds little to the theory.

In these notes, we explore the class of elementary groups. In particular, the class is shown to enjoy strong permanence properties and to admit a well-behaved, ordinal valued rank function. This rank function, aside from being an important tool to study elementary groups, gives a quantitative measure of the level of interaction between topological and large scale structure in a given elementary group.

Remark 2. The primary reference for these notes is [15]; the reader may also wish to consult the nice survey of M. Cesa and F. Le Maître [5]. The general approach developed in these notes is different from that of [15]. Our approach follows that of [16]; in loc. cit., the class of elementary amenable *discrete* groups is studied, but the parallels are obvious.

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2 Preliminaries

For G a t.d.l.c. group, we shall use $\mathscr{U}(G)$ to denote the set of compact open subgroups of G. For K a subgroup of a group G, we use $\langle\langle K\rangle\rangle_G$ to denote the normal subgroup generated by K in G. When clear from context, we drop the subscript. If H is an open subgroup of G, we write $H \leq_o G$. If H is a closed subgroup of G such that G/H is compact in the quotient topology, we say that H is a **cocompact** subgroup and write $H \leq_{cc} G$. It is a classical result that cocompact closed subgroups of compactly generated t.d.l.c. groups are themselves compactly generated.

2.1 Ordinals

Ordinal numbers are used frequently in these notes; [8] contains a nice introduction to ordinal numbers and ordinal arithmetic. Recalling that a **well-order** is a total order with no infinite descending chains, the easiest definition of an ordinal number is due to J. von Neumann: Each ordinal is the well-ordered set of all smaller ordinals with $0 := \emptyset$. For example, $2 = \{0,1\}$ and $3 = \{0,1,2\}$. Ordinal numbers are in particular well-orders themselves. For example, 2 is the two element well-order, and the first transfinite ordinal is $\omega := \mathbb{N}$. The second transfinite ordinal, $\omega + 1$, is the well-order given by a copy of $\mathbb{N}$ followed by one point. The first uncountable ordinal is denoted by ω_1. An important feature of ω_1, which is often used implicitly, is that there is no countable cofinal subset. That is to say, there is no countable sequence of countable ordinals $(\alpha_i)_{i\in\mathbb{N}}$ such that $\sup_{i\in\mathbb{N}} \alpha_i = \omega_1$. We stress that ω_1 is much larger than any countable ordinal. Ordinals such as ω^ω or ω^{ω^ω} are still strictly smaller than ω_1. Indeed, one can never reach ω_1 via arithmetic combinations of countable ordinals.

Given ordinals α and β, the ordinal $\alpha + \beta$ is the well-order given by a copy of α followed by a copy of β. Observe that the well-orders $1 + \omega$ and $\omega + 1$ are thus not equal, since the former is order isomorphic to ω while the latter is not, hence addition is *non-commutative*. Multiplication and exponentiation can be defined similarly.

We shall not use ordinal arithmetic in a complicated way. The reader is free to think of ordinal arithmetic as usual arithmetic keeping in mind that *it is non-commutative*.

Ordinals of the form $\alpha+1$ for some ordinal α are called **successor ordinals**. A **limit ordinal** is an ordinal which is not of the form $\alpha+1$ for some ordinal α. The ordinals ω, $\omega+\omega$, and ω_1 are examples of limit ordinals. We stress that our definition implies 0 is a limit ordinal.

An important feature of ordinals is that they allow us to extend induction arguments transfinitely. Transfinite induction proceeds just as the familiar induction with one additional step: One must check the inductive claim holds for limit ordinals λ given that the claim holds for all ordinals $\alpha<\lambda$. In the induction arguments in these notes, the limit case of the argument will often be trivial.

2.2 Descriptive-set-theoretic trees

We will require the notion of a descriptive-set-theoretic tree. This notion of a tree differs from the usual graph-theoretic definition; it is similar to the notion of a rooted tree used in the study of branch groups. The definitions given here are restricted to the collection of finite sequences of natural numbers; see [7, 2.A] for a general account.

Denote the collection of finite sequences of natural numbers by $\mathbb{N}^{<\mathbb{N}}$. For sequences $s:=(s_0,\dots,s_n)\in\mathbb{N}^{<\mathbb{N}}$ and $r:=(r_0,\dots,r_m)\in\mathbb{N}^{<\mathbb{N}}$, we write $s\sqsubseteq r$ if s is an **initial segment** of r. That is to say, $n\leq m$ and $s_i=r_i$ for $0\leq i\leq n$. The empty sequence, denoted by $\emptyset$, is considered to be an element of $\mathbb{N}^{<\mathbb{N}}$ and is an initial segment of any $t\in\mathbb{N}^{<\mathbb{N}}$. We define the **concatenation** of s with r to be

$$s^\frown r:=(s_0,\dots,s_n,r_0,\dots,r_m).$$

For $t=(t_0,\dots,t_k)\in\mathbb{N}^{<\mathbb{N}}$, the **length** of t, denoted by $|t|$, is the number of coordinates; i.e. $|t|:=k+1$. If $|t|=1$, we write t as a natural number, as opposed to a sequence of length one. For $0\leq i\leq|t|-1$, we define $t(i):=t_i$. For an infinite sequence $\alpha\in\mathbb{N}^{\mathbb{N}}$, we set $\alpha\restriction_n:=(\alpha(0),\dots,\alpha(n-1))$, so $\alpha\restriction_n\in\mathbb{N}^{<\mathbb{N}}$ for any $n\geq 0$.

Definition 2. A set $T\subseteq\mathbb{N}^{<\mathbb{N}}$ is a **tree** if it is closed under taking initial segments. We call the elements of T the **nodes** of T. If $s\in T$ and there is no $n\in\mathbb{N}$ such that $s^\frown n\in T$, we say s is a **leaf** or **terminal node** of T. An **infinite branch** of T is a sequence $\alpha\in\mathbb{N}^{\mathbb{N}}$ such that $\alpha\restriction_n\in T$ for all n. If T has no infinite branches, we say that T is **well-founded**.

For T a well-founded tree, there is an ordinal valued rank, denoted by ρ_T, on the nodes of T defined inductively as follows: If $s\in T$ is terminal, $\rho_T(s):=0$. For a non-terminal node s,

$$\rho_T(s):=\sup\left\{\rho_T(r)+1\mid s\sqsubset r\in T\right\}.$$

The reader is encouraged to verify that this function is defined on all nodes of a well-founded tree. The **rank** of a well-founded tree T is defined to be

$$\rho(T) := \sup\{\rho_T(s)+1 \mid s \in T\}.$$

When T is the empty tree, $\rho(T)=0$, and for all other well-founded trees, it is easy to verify that $\rho(T)=\rho_T(\emptyset)+1$. We thus see that $\rho(T)$ is always either a successor ordinal or zero. We extend ρ to ill-founded trees by declaring $\rho(T)=\omega_1$ for T an ill-founded tree.

There is an important, well-known relationship between the rank ρ_T on the nodes of T and the rank ρ on well-founded trees; we give a proof for completeness. For T a tree and $s \in T$, we put $T_s := \{r \in \mathbb{N}^{<\mathbb{N}} \mid s^\frown r \in T\}$. The set T_s is the tree obtained by taking the elements in T that extend s and deleting the initial segment s from each.

Lemma 1. *Suppose that $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a well-founded tree and $s \in T$. Then*

(1) $\rho_T(s)+1=\rho(T_s)$ and
(2) $\rho(T)=\sup\{\rho(T_i) \mid i \in T\}+1$.

Proof. Fixing $s \in T$, we first argue by induction on $\rho_{T_s}(r)$ that $\rho_{T_s}(r)=\rho_T(s^\frown r)$. For the base case, $\rho_{T_s}(r)=0$, the node r is terminal in T_s. The node $s^\frown r$ is thus terminal in T, hence $\rho_T(s^\frown r)=0$.

Suppose that the inductive claim holds for all $r \in T_s$ with $\rho_{T_s}(r)<\beta$ and say that $\rho_{T_s}(r)=\beta$. We now deduce that

$$\begin{aligned}\rho_{T_s}(r) &= \sup\{\rho_{T_s}(t)+1 \mid r \sqsubset t \in T_s\}\\ &= \sup\{\rho_T(s^\frown t)+1 \mid s^\frown r \sqsubset s^\frown t \in T\}\\ &= \sup\{\rho_T(t)+1 \mid s^\frown r \sqsubset t \in T\}\\ &= \rho_T(s^\frown r)\end{aligned}$$

where the second equality follows from the inductive hypothesis. Our induction is complete.

Taking $r=\emptyset$, we deduce that $\rho_{T_s}(\emptyset)=\rho_T(s)$. Therefore, $\rho(T_s)=\rho_{T_s}(\emptyset)+1=\rho_T(s)+1$, which verifies (1). Claim (2) follows from (1).

3 Elementary groups and well-founded trees

Classes defined by axioms, such as $\mathscr{E}$, are often studied via induction on the class formation axioms. In the case of $\mathscr{E}$, this approach has the unfortunate side-effect of cumbersome and technical proofs. We thus begin by characterizing $\mathscr{E}$ in terms of well-founded descriptive-set-theoretic trees. This gives an elegant and natural approach to the class of elementary groups.

To motivate our characterization, consider a game in which a friend builds a t.d.l.c.s.c. group and asks you to determine if it is or is not elementary. Since your friend built the group, there must be some way in which the group can be disassembled. You could thus, in principle, devise a general strategy to disassemble the group which halts exactly when the group is elementary

Our characterization will be exactly such a strategy. Our decomposition strategy will alternate between eliminating discrete quotients and passing to compactly generated open subgroups. These operations will "undo" the closure properties (iv) and (v). A priori, there are other elementary operations that must also be "undone." It will turn out that it is indeed enough to only consider (iv) and (v). (This is unsurprising in view of [10].)

3.1 Decomposition trees

Eliminating discrete quotients is accomplished by taking the discrete residual.

Definition 3. For a t.d.l.c. group H, the **discrete residual** of H is

$$\mathrm{Res}(H) := \bigcap \{O \mid O \trianglelefteq_o H\}.$$

The discrete residual is a closed characteristic subgroup of H. The quotient $H/\mathrm{Res}(H)$ also has a special structure. A t.d.l.c. **SIN group** is a t.d.l.c. group which admits a basis at 1 of compact open *normal* subgroups; note that t.d.l.c. SIN groups are elementary.

Proposition 1 ([4, Corollary 4.1]). *For G a compactly generated t.d.l.c. group, the quotient $G/\mathrm{Res}(G)$ is a SIN group.*

To reduce to compactly generated open subgroups, we define a second operation. Let G be a t.d.l.c.s.c. group and $U \in \mathscr{U}(G)$. Fix γ a choice of a countable dense subset of *every* closed subgroup of G; we call γ a **choice function** for G. Formally, γ is a map that sends a closed subgroup $H \leq G$ to a countable dense subset $\{h_i\}_{i\in\mathbb{N}}$ of H; the axiom of choice ensures such a γ exists. If L is a closed subgroup of G, then the restriction of γ to closed subgroups of L obviously induces a choice function for L. We will abuse notation and say that γ is a choice function for L.

For $H \leq G$ closed and $n \in \mathbb{N}$, we now define

$$R_n^{(U,\gamma)}(H) := \langle U \cap H, h_0, \dots, h_n \rangle$$

where the $h_0, \dots, h_n$ are the first $n+1$ elements of the countable dense set of H picked out by γ. For each $n \in \mathbb{N}$, the subgroup $R_n^{(U,\gamma)}(H)$ is a compactly generated open subgroup of H. Furthermore, $R_n^{(U,\gamma)}(H) \leq R_{n+1}^{(U,\gamma)}(H)$ for all n, and

$$H = \bigcup_{i\in\mathbb{N}} R_i^{(U,\gamma)}(H).$$

The subgroups $R_n^{(U,\gamma)}(H)$ thus give a canonical increasing exhaustion of H by compactly generated open subgroups.

We now define a tree $T_{(U,\gamma)}(G)$ and associated closed subgroups G_s of G for each $s \in T_{(U,\gamma)}(G)$. Put

- $\emptyset \in T_{(U,\gamma)}(G)$ and $G_\emptyset := G$.
- Suppose we have defined $s \in T_{(U,\gamma)}(G)$ and $G_s \leq G$. If $G_s \neq \{1\}$ and $n \in \mathbb{N}$, then put $s^\frown n \in T_{(U,\gamma)}(G)$ and set

$$G_{s^\frown n} := \mathrm{Res}\left(R_n^{(U,\gamma)}(G_s)\right).$$

Definition 4. For G a t.d.l.c.s.c. group, $U \in \mathscr{U}(G)$, and γ a choice function for G, we call $T_{(U,\gamma)}(G)$ the **decomposition tree** of G with respect to U and γ.

The decomposition tree is always non-empty, and the subgroup associated to any terminal node is the trivial group. We make one further observation; the proof is straightforward and therefore left to the reader. Recall that T_s is the tree below the node s in the tree T; precisely, $T_s = \{r \in \mathbb{N}^{<\mathbb{N}} \mid s^\frown r \in T\}$. For a decomposition tree $T_{(U,\gamma)}(G)$, we shall write $T_{(U,\gamma)}(G)_s$, instead of the more precise $(T_{(U,\gamma)}(G))_s$, for the tree below s.

Observation 1 *For any $s \in T_{(U,\gamma)}(G)$, $T_{(U,\gamma)}(G)_s = T_{(G_s \cap U,\gamma)}(G_s)$. Further, for $r \in T_{(G_s \cap U,\gamma)}(G_s)$, the associated subgroup $(G_s)_r$ is the same as the subgroup $G_{s^\frown r}$ associated to $s^\frown r \in T_{(U,\gamma)}(G)$.*

Remark 3. By classical results in descriptive set theory, the choice function γ can indeed be constructed in a Borel manner using selector functions; see [7, (12.13)]. The advantage of using selector functions to produce γ is that the assignment $G \mapsto T_{(U,\gamma)}(G)$ is Borel, when considered as a function between suitable parameter spaces. This allows for further descriptive-set-theoretic analysis of the class of elementary groups. See [16] for an example of such an analysis in the space of marked groups.

The decomposition tree plainly depends on the choices of compact open subgroup U and choice function γ, so there is no hope the decomposition tree outright is an invariant of the group. However, a decomposition tree comes with an ordinal rank, and this rank is a group invariant. That is to say, the rank of a decomposition tree does not depend on the choices of compact open subgroup and choice function.

Proposition 2. *Suppose that G is a t.d.l.c.s.c. group, $U \in \mathscr{U}(G)$, and γ is a choice function for G. Suppose additionally that H is a t.d.l.c.s.c. group, $W \in \mathscr{U}(H)$, and δ is a choice function for H. If $\psi : H \to G$ is a continuous, injective homomorphism, then*

$$\rho(T_{(W,\delta)}(H)) \leq \rho(T_{(U,\gamma)}(G)).$$

Proof. We induct on $\rho(T_{(U,\gamma)}(G))$ simultaneously for all G, $U \in \mathscr{U}(G)$, and γ a choice function for G. The base case is obvious since $\rho(T_{(U,\gamma)}(G)) = 1$ implies

$G = \{1\}$. We may also ignore the case of $\rho(T_{(U,\gamma)}(G)) = \omega_1$, since the proposition obviously holds here.

Suppose $\rho(T_{(U,\gamma)}(G)) = \beta + 1$. For each i, the subgroup $R_i^{(W,\delta)}(H)$ is compactly generated, so there is $n(i)$ with $\psi\left(R_i^{(W,\delta)}(H)\right) \leq R_{n(i)}^{(U,\gamma)}(G)$. We thus have that

$$\psi(H_i) = \psi\left(\mathrm{Res}\left(R_i^{(W,\delta)}(H)\right)\right) \leq \mathrm{Res}\left(R_{n(i)}^{(U,\gamma)}(G)\right) = G_{n(i)}.$$

The map ψ thereby restricts to $\psi : H_i \to G_{n(i)}$. Lemma 1 and Observation 1 imply

$$\rho\left(T_{(G_{n(i)}\cap U,\gamma)}(G_{n(i)})\right) = \rho\left(T_{(U,\gamma)}(G)_{n(i)}\right) \leq \beta.$$

Applying the inductive hypothesis, we deduce that

$$\rho\left(T_{(H_i\cap W,\delta)}(H_i)\right) \leq \rho\left(T_{(G_{n(i)}\cap U,\gamma)}(G_{n(i)})\right).$$

Therefore,

$$\begin{aligned}\rho(T_{(W,\delta)}(H)) &= \sup_{i\in\mathbb{N}} \rho\left(T_{(H_i\cap W,\delta)}(H_i)\right) + 1\\ &\leq \sup_{i\in\mathbb{N}} \rho\left(T_{(G_{n(i)}\cap U,\gamma)}(G_{n(i)})\right) + 1\\ &\leq \rho(T_{(U,\gamma)}(G)),\end{aligned}$$

so $\rho(T_{(W,\delta)}(H)) \leq \beta + 1$. This finishes the induction, and we conclude the proposition.

Proposition 2 ensures that the rank of a decomposition tree is indeed a group-theoretic property.

Corollary 1. *For G a t.d.l.c.s.c. group, $U,W \in \mathscr{U}(G)$, and γ and δ choice functions for G, $\rho(T_{(U,\gamma)}(G)) = \rho(T_{(W,\delta)}(G))$. In particular, $T_{(U,\gamma)}(G)$ is well-founded for some U and γ if and only if $T_{(U,\gamma)}(G)$ is well-founded for all U and γ.*

In view of Corollary 1, we make a definition.

Definition 5. For a t.d.l.c.s.c. group G, the **decomposition rank** of G is

$$\xi(G) := \rho(T_{(U,\gamma)}(G))$$

for some (any) $U \in \mathscr{U}(G)$ and γ a choice function for G.

Decomposition trees are a strategy to disassemble t.d.l.c.s.c. groups. Requiring the resulting decomposition tree to be well-founded is the obvious halting condition for this decomposition strategy. With this in mind, we define the following class:

Definition 6. The class $\mathscr{W}\mathscr{F}$ is defined to be the class of t.d.l.c.s.c. groups G with $\xi(G) < \omega_1$. Equivalently, $\mathscr{W}\mathscr{F}$ is the collection of t.d.l.c.s.c. groups with some (equivalently every) decomposition tree well-founded.

Our goal is to show that indeed $\mathscr{WF} = \mathscr{E}$, verifying that well-founded decomposition trees exactly isolate the elementary groups; the notation "$\mathscr{WF}$" will be discarded after establishing $\mathscr{WF} = \mathscr{E}$. We shall argue for $\mathscr{E} \subseteq \mathscr{WF}$ by verifying that $\mathscr{WF}$ satisfies the same closure properties; the next section will make these verifications. The converse inclusion will be an easy induction argument.

3.2 The class $\mathscr{WF}$

Our analysis of the class $\mathscr{WF}$ is via induction on the decomposition rank, so we first establish a computation technique for the rank. This technique allows us to avoid discussing decomposition trees. To establish this technique, let us first recast Proposition 2; our restatement also gives a first closure property of $\mathscr{WF}$.

Proposition 3. *Suppose that G and H are t.d.l.c.s.c. groups. If $\psi : H \to G$ is a continuous, injective homomorphism, then $\xi(H) \leq \xi(G)$. In particular, if $H \leq G$, then $\xi(H) \leq \xi(G)$, so $\mathscr{WF}$ is closed under taking closed subgroups.*

Proposition 4. *Suppose $G \in \mathscr{WF}$ is non-trivial.*

(1) If $G = \bigcup_{i\in\mathbb{N}} O_i$ with $(O_i)_{i\in\mathbb{N}}$ an $\subseteq$-increasing sequence of compactly generated open subgroups of G, then $\xi(G) = \sup_{i\in\mathbb{N}} \xi(\mathrm{Res}(O_i)) + 1$.
(2) If G is compactly generated, then $\xi(G) = \xi(\mathrm{Res}(G)) + 1$.

Proof. For (1), fix $U \in \mathscr{U}(G)$ and a choice function γ for G. For each i, there is $n(i)$ such that $O_i \leq R^{(U,\gamma)}_{n(i)}(G)$, since O_i is compactly generated. Therefore,

$$\mathrm{Res}(O_i) \leq \mathrm{Res}\left(R^{(U,\gamma)}_{n(i)}(G)\right) = G_{n(i)},$$

and Proposition 2 implies $\xi(\mathrm{Res}(O_i)) \leq \xi(G_{n(i)})$. We conclude that

$$\sup_{i\in\mathbb{N}} \xi(\mathrm{Res}(O_i)) + 1 \leq \sup_{j\in\mathbb{N}} \xi(G_j) + 1 = \xi(G).$$

On the other hand, $(O_i)_{i\in\mathbb{N}}$ is an exhaustion of G by open subgroups, so for each j, there is $n(j)$ with $R^{(U,\gamma)}_j(G) \leq O_{n(j)}$. Therefore, $G_j \leq \mathrm{Res}(O_{n(j)})$, and applying Proposition 2 again,

$$\xi(G) = \sup_{j\in\mathbb{N}} \xi(G_j) + 1 \leq \sup_{i\in\mathbb{N}} \xi(\mathrm{Res}(O_i)) + 1.$$

Hence, $\xi(G) = \sup_{i\in\mathbb{N}} \xi(\mathrm{Res}(O_i)) + 1$, as required.

Claim (2) now follows immediately from (1) by taking the sequence $(O_i)_{i\in\mathbb{N}}$ with $O_i = G$ for all i.

We now begin in earnest to verify that $\mathscr{WF}$ satisfies the same closure properties as $\mathscr{E}$. A t.d.l.c. group G is **residually discrete** if $\mathrm{Res}(G) = \{1\}$. From the definition of a decomposition tree, we see that any decomposition tree for such a group has rank at most 2. We thus deduce the following proposition:

Proposition 5. *All residually discrete groups are elements of $\mathscr{WF}$. In particular, all second countable profinite groups and countable discrete groups are elements of $\mathscr{WF}$.*

We next consider countable unions; we prove a slightly more general result for later use.

Proposition 6. *Suppose G is a t.d.l.c.s.c. group and $(O_i)_{i\in\mathbb{N}}$ is an $\subseteq$-increasing exhaustion of G by compactly generated open subgroups. If $\xi(\mathrm{Res}(O_i)) < \omega_1$ for all i, then $G \in \mathscr{WF}$. In particular, $\mathscr{WF}$ is closed taking countable increasing unions.*

Proof. Fix $U \in \mathscr{U}(G)$ and γ a choice function for G. Via Observation 1, the tree $T_{(U,\gamma)}(G)$ is well-founded exactly when $T_{(G_j\cap U,\gamma)}(G_j)$ is well-founded for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, there is $i \in \mathbb{N}$ such that $R_j^{(U,\gamma)}(G) \leq O_i$, since $R_j^{(U,\gamma)}(G)$ is compactly generated. We deduce that $G_j \leq \mathrm{Res}(O_i)$. Proposition 3 now ensures that $\xi(G_j) < \omega_1$, and thus, $T_{(G_j\cap U,\gamma)}(G_j)$ is well-founded. We conclude that $G \in \mathscr{WF}$.

We now turn our attention to quotients and group extensions. Our arguments here require several preliminary results. The first observation is immediate from the relevant definitions.

Observation 2 *If G is a t.d.l.c. SIN group and $L \trianglelefteq G$, then G/L is a t.d.l.c. SIN group.*

Let us also note an easy fact about the discrete residual.

Lemma 2. *If G is a compactly generated t.d.l.c.s.c. group and $L \trianglelefteq G$, then $\mathrm{Res}(G/L) = \overline{\mathrm{Res}(G)L}/L$.*

Proof. Let $\pi : G \to G/L$ be the usual projection map. For every open normal $O \trianglelefteq G/L$, the subgroup $\pi^{-1}(O)$ is an open normal subgroup of O. Hence, $\mathrm{Res}(G) \leq \pi^{-1}(\mathrm{Res}(G/L))$, and we deduce that $\overline{\mathrm{Res}(G)L}/L \leq \mathrm{Res}(G/L)$.

Conversely, the group $(G/L)/(\overline{\mathrm{Res}(G)L}/L)$ is a quotient of the SIN group $G/\mathrm{Res}(G)$. Observation 2 ensures $(G/L)/(\overline{\mathrm{Res}(G)L}/L)$ is a SIN group and therefore residually discrete. We conclude that $\mathrm{Res}(G/L) \leq \overline{\mathrm{Res}(G)L}/L$, verifying the proposition.

A non-trivial permanence property of t.d.l.c. SIN groups will be needed. The argument requires the following easy application of the Baire category theorem, which we leave as an exercise: *Every element of a discrete normal subgroup of a t.d.l.c.s.c. group has an open centralizer.*

Lemma 3. *If G is a compactly generated t.d.l.c. group and $N \trianglelefteq_{cc} G$ is a SIN group, then G is a SIN group.*

Proof. Fix $U \in \mathscr{U}(G)$ and form the subgroup UN. Since N is a SIN group, we may find $W \in \mathscr{U}(N)$ with $W \leq U$ and $W \trianglelefteq N$. The normal closure $J := \overline{\langle\langle W \rangle\rangle}$ of W in UN is generated by U-conjugates of W, and thus $J \leq U$. Since N is cocompact in G, UN has finite index in G, so $N_G(J)$ has finite index in G. Letting $g_1, \dots, g_n$ list left coset representatives for $N_G(J)$ in G, we see that

$$\bigcap_{g \in G} gJg^{-1} = \bigcap_{i=1}^{n} g_i J g_i^{-1}.$$

Defining $K := \bigcap_{g \in G} gJg^{-1}$, it follows that $K \in \mathscr{U}(N)$ and that $K \trianglelefteq G$.

Passing to G/K, the image $\pi(N)$ is normal and discrete in G/K where $\pi : G \to G/K$ is the usual projection. The subgroup N is compactly generated, since cocompact in a compactly generated group, hence the subgroup $\pi(N)$ is finitely generated. Moreover, since each generator of $\pi(N)$ has an open centralizer, $\pi(N)$ has an open centralizer. Say that $Q \leq_o \pi(U)$ centralizes $\pi(N)$. Clearly, $Q \trianglelefteq Q\pi(N)$, and using that $\pi(N)$ is cocompact in G/K, we additionally see that $Q\pi(N)$ has finite index in G/K. Just as in the previous paragraph, there is $L \leq_o Q$ with $L \trianglelefteq G/K$. It now follows that $\pi^{-1}(L)$ is an open normal subgroup of G contained in U.

We conclude that inside every compact open subgroup U of G, we may find a compact open normal subgroup of G. That is to say, G is a SIN group.

Our final subsidiary result is important outside the immediate application, because it allows one to go from a closed normal subgroup to an *open* subgroup with the same rank.

Proposition 7 ([11, Lemma 2.9]). *If $G \in \mathscr{WF}$ with $N \trianglelefteq_{cc} G$ closed and non-trivial, then $\xi(G) = \xi(N)$.*

Proof. Fix $(O_i)_{i \in \mathbb{N}}$ a countable $\subseteq$-increasing exhaustion of G by compactly generated open subgroups of G and put $N_i := N \cap O_i$. Each N_i is open in N, and since $N_i \trianglelefteq_{cc} O_i$, it is also compactly generated. Proposition 3 ensures $N \in \mathscr{WF}$, and in view of Proposition 4, we infer that

$$\xi(N) = \sup_{i \in \mathbb{N}} \xi(\mathrm{Res}(N_i)) + 1.$$

We now consider the group $O_i/\mathrm{Res}(N_i)$. The subgroup $N_i/\mathrm{Res}(N_i)$ is a SIN group via Proposition 1, and it is cocompact in $O_i/\mathrm{Res}(N_i)$. Lemma 3 thus implies that $O_i/\mathrm{Res}(N_i)$ is also a SIN group, hence $O_i/\mathrm{Res}(N_i)$ is residually discrete. It now follows that $\mathrm{Res}(O_i) = \mathrm{Res}(N_i)$. Applying Proposition 4 again, we conclude that

$$\xi(G) = \sup_{i \in \mathbb{N}} \xi(\mathrm{Res}(O_i)) + 1 = \sup_{i \in \mathbb{N}} \xi(\mathrm{Res}(N_i)) + 1 = \xi(N),$$

verifying the lemma.

We are now prepared to show $\mathscr{WF}$ is closed under taking quotients; the proof is an instructive illustration of the utility of Proposition 7.

Proposition 8. *If $G \in \mathscr{W}\mathscr{F}$ and $L \trianglelefteq G$ is closed, then $G/L \in \mathscr{W}\mathscr{F}$ with $\xi(G/L) \leq \xi(G)$.*

Proof. Fix $U \in \mathscr{U}(G)$ and fix $(O_i)_{i\in\mathbb{N}}$ an $\subseteq$-increasing exhaustion of G by compactly generated open subgroups such that $U \leq O_0$.

We induct on $\xi(G)$ for the proposition. The case of $\xi(G) = 1$ is obvious, and it will be convenient to take $\xi(G) = 2$ as the base case. Proposition 4 ensures that $\mathrm{Res}(O_i) = \{1\}$ for all i, and in view of Proposition 1, we deduce that each O_i is a SIN group. Since the class of SIN groups is stable under taking Hausdorff quotients, $O_i/O_i \cap L$ is also a SIN group for all $i \in \mathbb{N}$. On the other hand, G/L is the union of the increasing sequence $(O_iL/L)_{i\in\mathbb{N}}$, and since $O_iL/L \simeq O_i/O_i \cap L$, each term of the sequence is a SIN group. Proposition 5 now ensures each O_iL/L is in $\mathscr{W}\mathscr{F}$, so we conclude that $G/L \in \mathscr{W}\mathscr{F}$ via Proposition 6. From Proposition 4, we deduce further that $\xi(G/L) \leq 2$.

Let us now suppose that $\xi(G) = \beta + 1$ with $\beta > 1$. In view of Proposition 4, each $R_i := \mathrm{Res}(O_i)$ has rank at most β. Furthermore, it cannot be the case that $R_i = \{1\}$ for all i, since then G has rank two. Throwing out finitely many O_i if needed, we may assume each R_i is non-trivial. Each R_i is then a non-trivial cocompact normal subgroup of UR_i, so Proposition 7 implies $\xi(UR_i) = \xi(R_i)$. Applying the inductive hypothesis, we infer that

$$UR_i/UR_i \cap L \simeq UR_iL/L$$

has rank at most β for each i. As $\overline{R_iL}/L$ is a closed subgroup of UR_iL/L, we deduce further that $\xi(\overline{R_iL}/L) \leq \beta$, via Proposition 3.

The quotient G/L is the increasing union of the compactly generated open subgroups $W_i := O_iL/L$. Lemma 2 shows that $\mathrm{Res}(W_i) = \overline{R_iL}/L$, so our work above implies $\xi(\mathrm{Res}(W_i)) \leq \beta$. Applying Proposition 6, we deduce that $G/L \in \mathscr{W}\mathscr{F}$, and via Proposition 4, $\xi(G/L) \leq \beta + 1$, completing the induction.

We next show $\mathscr{W}\mathscr{F}$ is closed under forming group extensions; our proof is inspired by a similar argument in [10].

Proposition 9 ([12, Lemma 7.4]). *Suppose*

$$\{1\} \to N \to G \to Q \to \{1\}$$

is a short exact sequence of t.d.l.c.s.c. groups. If N and Q are members of $\mathscr{W}\mathscr{F}$, then $G \in \mathscr{W}\mathscr{F}$ with

$$\xi(G) \leq \xi(N) + \xi(Q).$$

In particular, $\mathscr{W}\mathscr{F}$ is closed under group extensions.

Proof. We induct on $\xi(Q)$ for the proposition. The base case, $\xi(Q) = 1$, is obvious, so we suppose $\xi(Q) = \beta + 1$.

Let $\pi : G \to Q$ be the projection given in the short exact sequence, fix $(O_i)_{i\in\mathbb{N}}$ an $\subseteq$-increasing exhaustion of G by compactly generated open subgroups, and put $W_i := \pi(O_i)$. The sequence $(W_i)_{i\in\mathbb{N}}$ is an exhaustion of Q by compactly generated open subgroups. Fix $i \in \mathbb{N}$, form $R := \mathrm{Res}(O_i)$, and put $M := \overline{RN}$. The group M/N is a closed subgroup of $\mathrm{Res}(W_i)$, hence $\xi(M/N) \leq \beta$ via Proposition 3. The inductive hypothesis implies $M \in \mathscr{W}\mathscr{F}$ with $\xi(M) \leq \xi(N) + \beta$, and since $R \leq M$, a second application of Proposition 3 ensures that $\xi(R) \leq \xi(N) + \beta$. In view of Propositions 4 and 6, we conclude that $G \in \mathscr{W}\mathscr{F}$ with $\xi(G) \leq \xi(N) + \beta + 1$, verifying the inductive claim.

3.3 The return of elementary groups

We now argue that $\mathscr{W}\mathscr{F}$ is exactly the class of elementary groups. Our argument will have the added benefit of showing that some of the elementary operations used to define $\mathscr{E}$ are redundant.

Definition 7. The class $\mathscr{E}^*$ is the smallest class of t.d.l.c.s.c. groups such that the following hold:

(i) $\mathscr{E}^*$ contains all second countable profinite groups and countable discrete groups.
(ii) $\mathscr{E}^*$ is closed under taking group extensions of second countable profinite or countable discrete groups. That is, if G is a t.d.l.c.s.c. group and $H \trianglelefteq G$ is a closed normal subgroup with $H \in \mathscr{E}^*$ and G/H profinite or discrete, then $G \in \mathscr{E}^*$.
(iii) If G is a t.d.l.c.s.c. group and $G = \bigcup_{i\in\mathbb{N}} O_i$ where $(O_i)_{i\in\mathbb{N}}$ is an $\subseteq$-increasing sequence of open subgroups of G with $O_i \in \mathscr{E}^*$ for each i, then $G \in \mathscr{E}^*$.

Obviously $\mathscr{E}^*$ is contained in $\mathscr{E}$. It turns out this containment is indeed an equality.

Theorem 1. $\mathscr{E} = \mathscr{W}\mathscr{F} = \mathscr{E}^*$.

Proof. Since $\mathscr{E}^* \subseteq \mathscr{E}$, it suffices to show the inclusions $\mathscr{E} \subseteq \mathscr{W}\mathscr{F} \subseteq \mathscr{E}^*$. For the first inclusion, since $\mathscr{E}$ is defined to be the smallest class such that certain closure properties hold, it is enough to show that $\mathscr{W}\mathscr{F}$ satisfies the same properties. That $\mathscr{W}\mathscr{F}$ contains the profinite groups and discrete groups is given by Proposition 5. The class $\mathscr{W}\mathscr{F}$ is closed under taking closed subgroups, Hausdorff quotients, and countable increasing unions via Propositions 3, 8, and 6, respectively. Proposition 9 ensures $\mathscr{W}\mathscr{F}$ is closed under forming group extensions.

For the second inclusion, we argue by induction on $\xi(G)$. If $\xi(G) = 1$, then $G = \{1\}$ is plainly in $\mathscr{E}^*$. Suppose $H \in \mathscr{E}^*$ for all $H \in \mathscr{W}\mathscr{F}$ with $\xi(H) \leq \beta$ and consider $G \in \mathscr{W}\mathscr{F}$ with $\xi(G) = \beta + 1$. Fix $(O_i)_{i\in\mathbb{N}}$ an $\subseteq$-increasing exhaustion of G by compactly generated open subgroups. In view of Proposition 4, each O_i is such that $\xi(\mathrm{Res}(O_i)) \leq \beta$, so the inductive hypothesis implies $\mathrm{Res}(O_i) \in \mathscr{E}^*$. The quotient $O_i/\mathrm{Res}(O_i)$ is a SIN group via Proposition 1. We may then fix $W \trianglelefteq O_i/\mathrm{Res}(O_i)$ a compact open normal subgroup. Letting $\pi : O_i \to O_i/\mathrm{Res}(O_i)$ be the usual projection, $\mathrm{Res}(O_i)$ is a cocompact normal subgroup of $\pi^{-1}(W)$, and as $\mathscr{E}^*$ is closed under extensions of profinite groups, we deduce that $\pi^{-1}(W) \in \mathscr{E}^*$. On the other hand, the quotient $O_i/\pi^{-1}(W)$ is discrete. As $\mathscr{E}^*$ is closed under extensions of discrete groups, we can conclude that $O_i \in \mathscr{E}^*$. It now follows that $G \in \mathscr{E}^*$, completing the induction.

As an immediate consequence, we obtain a simpler characterization of elementary groups.

Corollary 2. *The class of elementary groups is the smallest class $\mathscr{E}$ of t.d.l.c.s.c. groups such that the following hold:*

(i) $\mathscr{E}$ contains all second countable profinite groups and countable discrete groups.
(ii) $\mathscr{E}$ closed under taking group extensions of second countable profinite or countable discrete groups; that is, if G is a t.d.l.c.s.c. group and $H \trianglelefteq G$ is a closed normal subgroup with $H \in \mathscr{E}$ and G/H profinite or discrete, then $G \in \mathscr{E}$.
(iii) If G is a t.d.l.c.s.c. group and $G = \bigcup_{i\in\mathbb{N}} O_i$ where $(O_i)_{i\in\mathbb{N}}$ is an $\subseteq$-increasing sequence of open subgroups of G with $O_i \in \mathscr{C}$ for each i, then $G \in \mathscr{E}$.

We note a second consequence, which is quite useful in the study of elementary groups.

Corollary 3. *If G is a non-trivial compactly generated elementary group, then G has a non-trivial discrete quotient.*

Proof. Via Theorem 1, G is a member of $\mathscr{W}\mathscr{F}$, and Proposition 4 implies $\xi(G) = \xi(\mathrm{Res}(G)) + 1$. We conclude that $\mathrm{Res}(G) \lneq G$, and thus, G has a non-trivial discrete quotient.

4 Examples and non-examples of elementary groups

We conclude with a discussion of examples and non-examples. In particular, we will exhibit a family of examples with unboundedly large finite rank and compactly generated examples with transfinite rank.

4.1 Non-examples

Our motivation to form the class of elementary groups is to make precise the class of groups with weak interaction between topological and large-scale structure. The groups which surely have strong interaction between topological and large-scale structure are the compactly generated t.d.l.c.s.c. groups which are non-discrete and simple. Our notion of an elementary group excludes these simple groups.

Proposition 10. *If G is a compactly generated t.d.l.c.s.c. group that is non-discrete and topologically simple, then G is not elementary.*

Proof. Since G is topologically simple and non-discrete, it has no non-trivial discrete quotients. In view of Corollary 3, that G is compactly generated ensures that it is non-elementary.

We note that there are many compactly generated t.d.l.c.s.c. groups that are topologically simple and non-discrete. For the n-regular tree T_n with $n \geq 3$, work of J. Tits [14] shows that there is an index two closed subgroup of $\mathrm{Aut}(T_n)$, denoted by $\mathrm{Aut}^+(T_n)$, that is topologically simple, compactly generated, and non-discrete. The projective special linear groups $PSL_n(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the p-adic numbers and $n \geq 2$ are further examples; cf. [1, 6]. There are in fact continuum many compactly generated t.d.l.c.s.c. groups that are topologically simple and non-discrete by work of S. Smith [13].

4.2 Finite rank examples

Our construction requires a couple of general notions. A group is called **perfect** if it is generated by commutators; a **commutator** is an element of the form $[g,h] := ghg^{-1}h^{-1}$ for group elements g and h. We also require the notion of a local direct product.

Definition 8. Suppose that $(G_i)_{i\in\mathbb{N}}$ is a sequence of t.d.l.c. groups and suppose that there is a distinguished compact open subgroup $U_i \leq G_i$ for each $i \in \mathbb{N}$. The **local direct product** of $(G_i)_{i\in\mathbb{N}}$ over $(U_i)_{i\in\mathbb{N}}$ is defined to be

$$\left\{ f : \mathbb{N} \to \bigsqcup_{i\in\mathbb{N}} G_i \mid f(i) \in G_i, \text{ and } f(i) \in U_i \text{ for all but finitely many } i \in \mathbb{N} \right\}$$

with the group topology such that $\prod_{i\in\mathbb{N}} U_i$ continuously embeds as an open subgroup. We denote the local direct product by $\bigoplus_{i\in\mathbb{N}}(G_i,U_i)$.

The following property of local direct products is an easy consequence of the definitions; we leave the proof to the reader.

Proposition 11. *If $(G_i)_{i\in\mathbb{N}}$ is a sequence of elementary groups with U_i a distinguished compact open subgroup for each i, then $\bigoplus_{\mathbb{N}}(G_i,U_i)$ is an elementary group.*

We are now ready to construct our groups. Let A_5 be the alternating group on five letters and let S be an infinite finitely generated perfect group. Form $H := S^{[5]} \rtimes A_5$ where $A_5 \curvearrowright S^{[5]}$ by shift and fix a transitive, free action of H on $\mathbb{N}$.

Lemma 4. *The normal subgroup of H generated by A_5 equals H.*

Proof. Identify S with the copy of S in $S^{[5]}$ supported on 0 and take $a \in A_5$ so that $a(0) \neq 0$. For $g,h \in S \leq S^{[5]}$, the element aga^{-1} has support disjoint from both g and h, hence aga^{-1} commutes with both g and h. An easy calculation now shows that

$[h,[g,a]] = [h,g]$. Since $[g,a] \in \langle\langle A_5 \rangle\rangle$, we deduce that $[h,[g,a]] \in \langle\langle A_5 \rangle\rangle$. The group $\langle\langle A_5 \rangle\rangle$ thus contains all commutators of S, and since S is perfect, $S \leq \langle\langle A_5 \rangle\rangle$. It now follows that $\langle\langle A_5 \rangle\rangle = H$.

Starting from the group H, we inductively define compactly generated elementary groups L_n with a distinguished $K_n \in \mathscr{U}(L_n)$ such that $\langle\langle K_n \rangle\rangle = L_n$. For the base case, $n = 1$, define $L_1 := H$ and $K_1 := A_5$. The group L_1 is compactly generated, K_1 is a compact open subgroup of L_1, and $\langle\langle K_1 \rangle\rangle = L_1$, via Lemma 4.

Suppose we have defined a compactly generated group L_n with a compact open subgroup K_n such that $\langle\langle K_n \rangle\rangle = L_n$. Let $(L_n^i)_{i\in\mathbb{N}}$ and $(K_n^i)_{i\in\mathbb{N}}$ list countably many copies of L_n and K_n and form the local direct product $\bigoplus_{i\in\mathbb{N}}(L_n^i, K_n^i)$. Taking the previously fixed action of H on $\mathbb{N}$, we have that $H \curvearrowright \bigoplus_{i\in\mathbb{N}}(L_n^i, K_n^i)$ by shift, so we may form

$$L_{n+1} := \bigoplus_{i\in\mathbb{N}}(L_n^i, K_n^i) \rtimes H.$$

The group L_{n+1} is a t.d.l.c. group under the product topology, and $K_{n+1} := K_n^{\mathbb{N}} \rtimes A_5$ is a compact open subgroup. Letting X be a compact generating set for L_n^0 and F be a finite generating set for H in L_{n+1}, one verifies that $X \times \prod_{i>0} K_n^i \cup F$ is a compact generating set for L_{n+1}. It is easy to further verify that $\langle\langle K_{n+1} \rangle\rangle_{L_{n+1}} = L_{n+1}$. This completes our inductive construction.

Proposition 12. *For each $n \geq 1$, $L_n \in \mathscr{E}$ with $\xi(L_n) \geq n+1$.*

Proof. In view of Proposition 11, an easy induction argument verifies that $L_n \in \mathscr{E}$ for all $n \geq 1$. For the lower bound on the rank, we argue by induction on n. For the base case, $L_1 = H$ is non-trivial and discrete. Since the trivial group has rank 1, Proposition 4 implies that $\xi(L_1) = 2$.

Suppose the inductive hypothesis holds up to n and consider L_{n+1}. We first compute $\mathrm{Res}(L_{n+1})$. Consider $O \trianglelefteq_o L_{n+1}$. Since $K_n^{\mathbb{N}}$ is a compact open subgroup of L_{n+1}, the subgroup O must contain

$$K_n^{[k,\infty]} := \{f : \mathbb{N} \to K_n \mid f(0) = \cdots = f(k) = 1\}$$

for some $k \in \mathbb{N}$. Since H acts transitively on $\mathbb{N}$ and O is normal, O indeed contains $K_n^{\mathbb{N}}$. Recalling that $\langle\langle K_n \rangle\rangle_{L_n} = L_n$, we conclude that

$$\bigoplus_{i\in\mathbb{N}}(L_n^i, K_n^i) = \langle\langle K_n^{\mathbb{N}} \rangle\rangle_{L_{n+1}} \leq O.$$

It now follows that $\mathrm{Res}(L_{n+1}) = \bigoplus_{i\in\mathbb{N}}(L_n^i, K_n^i)$.

In view of Proposition 4, $\xi(L_{n+1}) = \xi(\mathrm{Res}(L_{n+1})) + 1$, because L_{n+1} is compactly generated. The group L_n admits a continuous injection into $\mathrm{Res}(L_{n+1})$, so

$$\xi(\mathrm{Res}(L_{n+1})) \geq \xi(L_n) \geq n+1$$

via Proposition 3 and the inductive hypothesis. We conclude that $\xi(L_{n+1}) \geq n+2$, and the induction is complete.

It is indeed the case that $\xi(L_n) = n+1$ for all $n \geq 1$; one can devise a proof of this using the computation of the rank of a quasi-product given in [11]. The set $\{L_n \mid n \geq 1\}$ is thus a family of elementary groups with members of arbitrarily large finite decomposition rank. From this family, we obtain a first example of an elementary group with transfinite rank

Corollary 4. *The group $G := \bigoplus_{n\geq 1}(L_n, K_n)$ is elementary with $\xi(G) \geq \omega+1$.*

Proof. For each $n \geq 1$, there is a continuous injection $L_n \hookrightarrow G$. Via Proposition 3, $n+1 \leq \xi(G)$ for all $n \geq 1$, so $\omega \leq \xi(G)$. Since the decomposition rank is always a successor ordinal, we conclude that $\omega+1 \leq \xi(G)$.

Remark 4. The examples above demonstrate a strategy for finding examples of higher rank. Suppose that we have $H \in \mathscr{E}$ with rank α and suppose that we construct a compactly generated $G \in \mathscr{E}$ for which $H \hookrightarrow \mathrm{Res}(G)$. Applying Proposition 4, we then have that $\xi(G) \geq \alpha+1$. The problem, of course, is finding the group G. We stress that one should not expect general embedding theorems which produce such a G. Indeed, there are groups in $\mathscr{E}$ which do not embed into *any* compactly generated t.d.l.c.s.c. group; see [3].

4.3 Compactly generated elementary groups with transfinite rank

We here describe a technique which produces compactly generated elementary groups with transfinite rank. We omit proofs as they are somewhat technical; the full details of the construction will appear in a later article. The construction is inspired by ideas from [2, 9, 13], and the reader familiar with [9] and the theory of elementary groups can likely fill in the proofs.

Let T be the countable regular tree and fix δ an end of T. We orient the edges of T such that all edges point toward the end δ. The resulting directed graph is denoted by $\mathscr{T}$, and we call δ the **distinguished end** of $\mathscr{T}$. Given a countable set X, a **coloring** of $\mathscr{T}$ is a function $c : E\mathscr{T} \to X$ such that for each $v \in V\mathscr{T}$,

$$c_v := c\restriction_{\mathrm{inn}(v)} : \mathrm{inn}(v) \to X$$

is a bijection. The set $\mathrm{inn}(v)$ is the collection of directed edges with terminal vertex v. We call the coloring **ended** if there is a monochromatic directed ray which is a representative of the distinguished end δ; *we shall always assume our colorings are ended.* The coloring allows us to define the **local action** of $g \in \mathrm{Aut}(\mathscr{T})$ at $v \in V\mathscr{T}$:

$$\sigma(g,v) := c_{g(v)} \circ g \circ c_v^{-1} \in \mathrm{Sym}(X).$$

The local action allows us to isolate the groups we wish to consider. It shall be convenient to make a definition: A **t.d.l.c.s.c. permutation group** is a pair (G,X) where G is a t.d.l.c.s.c. group and X is a countable set on which G acts faithfully with compact open point stabilizers. We stress that X is assumed to be infinite.

Definition 9. Suppose that (G,X) is a t.d.l.c.s.c. permutation group with $U \in \mathscr{U}(G)$ and color the tree $\mathscr{T}$ by X. We define the group $E_X(G,U) \leq \mathrm{Aut}(\mathscr{T})$ as follows: $E_X(G,U)$ is the set of $g \in \mathrm{Aut}(\mathscr{T})$ such that $\sigma(g,v) \in G$ for all $v \in V\mathscr{T}$ and that $\sigma(g,v) \in U$ for all but finitely many $v \in V\mathscr{T}$.

It is easy to verify that $E_X(G,U)$ is an abstract group. With more care, one can also identify a natural t.d.l.c.s.c. group topology on $E_X(G,U)$. One first verifies that the vertex stabilizer $E_X(U,U)_{(v)}$ is compact in the topology on $\mathrm{Aut}(\mathscr{T})$. The group $\mathrm{Aut}(\mathscr{T})$ is given the topology of pointwise convergence; this topology is not locally compact, since the tree is locally infinite. One then argues for the following proposition:

Proposition 13. *For (G,X) a t.d.l.c.s.c. permutation group and $U \in \mathscr{U}(G)$, there is a t.d.l.c.s.c. group topology on $E_X(G,U)$ such that the inclusion $E_X(U,U)_{(v)} \hookrightarrow E_X(G,U)$ is continuous with a compact open image for any $v \in V\mathscr{T}$.*

The resulting t.d.l.c.s.c. group $E_X(G,U)$ yields the desired examples.

Theorem 2. *Suppose that (G,X) is a transitive t.d.l.c.s.c. permutation group. If G is compactly generated and elementary, then $E_X(G,U)$ is compactly generated and elementary with*

$$\xi(E_X(G,U)) \geq \xi(G) + \omega + 2$$

for any non-trivial $U \in \mathscr{U}(G)$.

Take G any infinite finitely generated group with a non-trivial finite subgroup U such that U has a trivial normal core in U. Letting $X := G/U$ and $G \curvearrowright X$ by left multiplication, the pair (G,X) is a t.d.l.c.s.c. permutation group. Theorem 2 now implies that $E_X(G,U)$ is elementary with rank at least $\omega+2$. (It is indeed the case that $E_X(G,U)$ has rank exactly $\omega+2$.)

Applying Theorem 2 repeatedly allows us to build elementary groups with even larger rank.

Corollary 5. *For each $0 \leq n < \omega$, there is a compactly generated elementary group L_n with $\xi(L_n) \geq \omega \cdot n + 2$.*

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