The scale, tidy subgroups and flat groups

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Abstract These notes discuss the scale, tidy subgroups, subgroups associated with endomorphisms and flat groups on totally disconnected locally compact (t.d.l.c) groups. The first section discusses the structure theory of subgroups which are *minimizing* for an endomorphism and introduces the *scale* of an endomorphism. The second section discusses the applications and properties of the scale function. Section 3 discusses other subgroups which may be associated with endomorphisms in a unique way. Section 4 discusses flat groups of automorphisms, the flat rank and various results about flat groups. The final section discusses the geometry of t.d.l.c groups.

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1 Subgroups tidy for an endomorphism

1.1 Introduction

A group G is *locally compact* if it has a locally compact topology such that the group operations are continuous. Locally compact groups have a structure theory; there exists a short exact sequence

$$G_0 \hookrightarrow G \twoheadrightarrow G/G_0$$

where G_0 is the connected component of the identity and G/G_0 is totally disconnected [21].

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If G is a connected locally compact group and $\mathscr{N}$ is a neighbourhood of the identity, then there exists a compact normal subgroup U contained in $\mathscr{N}$ such that G/U is a Lie group [20]. In other words, connected groups are approximated by Lie groups. This was the solution to Hilbert's fifth problem.

Matrix groups over a locally compact field are important examples of locally compact groups. Every topological field is either connected or totally disconnected, and the group $SL(n,\mathbb{F})$ is connected or totally disconnected depending on whether $\mathbb{F}$ is. The connected locally compact fields are $\mathbb{R}$ and $\mathbb{C}$. Every totally disconnected topological field is either discrete or non-discrete. The non-discrete totally disconnected locally compact fields are the p-adics and their finite extensions, which have characteristic zero, and the formal Laurent series over finite fields, which have positive characteristic [8].

The focus of these notes is non-discrete totally disconnected locally compact groups.

1.2 Totally disconnected groups

From now on, suppose that G is a totally disconnected locally compact group and that $\mathscr{N}$ is a neighbourhood of the identity. Then there exists a compact open subgroup O contained in $\mathscr{N}$ [12]. This subgroup is not necessarily normal. We denote by $\mathrm{COS}(G)$ the set of compact open subgroups of G. The compact open subgroups are also called *0-dimensional groups* because they have inductive dimension equal to zero.

All compact metrizable totally disconnected spaces are homeomorphic to a Cantor set [14]. There are therefore no topological invariants–such as dimension–which are useful for distinguishing them.

Definition 1. Let G be a totally disconnected locally compact (*t.d.l.c.*) group. An *endomorphism* is a continuous homomorphism on G. The set of endomorphisms on G form a semi-group under composition, denoted by $\mathsf{End}(G)$. An *automorphism* on G is an endomorphism which is a bijection with a continuous inverse. The group of automorphisms on G is denoted by $\mathsf{Aut}(G)$.

1.3 Endomorphisms and minimizing subgroups

Suppose that α is an endomorphism of G and that U is a compact open subgroup. Then the set $\alpha(U)\cap U$ is open in the subspace topology on $\alpha(U)$ and $\alpha(U)$ is compact. Hence $[\alpha(U):\alpha(U)\cap U]$ is finite. The following definition was made for automorphisms in [24, 25] and for endomorphisms in [28].

Definition 2. Suppose that α is an endomorphism on a t.d.l.c. group G. The scale of α is

$$s(\alpha) = \min\{[\alpha(U) : U \cap \alpha(U)] : U \in \mathrm{COS}(G)\}.$$

A compact open subgroup U is said to be *minimizing for* α if

$$s(\alpha) = [\alpha(U) : U \cap \alpha(U)].$$

Suppose that α is an endomorphism of G. For each compact open subgroup U, let

$$U_+ = \{x \in U : \exists\{x_n\}_{n\in\mathbb{N}} \subset U \text{ with } x_0 = x \text{ and } \alpha(x_{n+1}) = x_n \ \forall n \in \mathbb{N}\}$$

and let

$$U_- = \{x \in U : \alpha(x) \in U \ \forall n \in \mathbb{N}\}.$$

Lemma 1. *Suppose that* α *is an automorphism of* G *and that* U *is a compact open subgroup of* G*. Then*

$$U_+ = \bigcap_{k\geq 0} \alpha^k(U)$$

and

$$U_- = \bigcap_{k\geq 0} \alpha^{-k}(U)$$

Remark 1. These expressions for U_- and U_+ are usually given as the definition in the case of automorphisms. The fact that endomorphisms may lack an inverse is why the definitions must be changed to accommodate endomorphisms.

In the above context, note that α expands U_+ and contracts U_-. See Figure 1 for an illustration of an automorphism α with $s(\alpha) = 3$ and $s(\alpha^{-1}) = 2$. The figure is not accurate for endomorphisms which may have range much thinner than U.

The following characterisation of minimising subgroups in terms of their structure is given in [28].

Theorem 1 (The Structure of Minimising Subgroups). *A compact open subgroup* U *of a locally compact totally disconnected group is minimizing for an endomorphism* α *if and only if it satisfies:*

(TA) $U = U_+U_-$.
(TB1) $U_{++} = \bigcup_{n\geq 0} \alpha^n(U_+)$ *is closed.*
(TB2) The sequence of integers $[\alpha^{n+1}(U_+) : \alpha^n(U_+)]$ *is constant.*

When these conditions are satisfied $s(\alpha) = [\alpha(U_+) : U_+]$.

Remark 2. The property (TB2) is not needed if α is an automorphism.

Remark 3. It is immediate from the definition that $\alpha(U_+) \geq U_+$. It follows that U_{++} is a subgroup for this reason.

Remark 4. One problem with working with endomorphisms instead of automorphisms is the fact that if α is an endomorphism, then $\alpha(A \cap B)$ is in general not necessarily equal to $\alpha(A) \cap \alpha(B)$.

It is immediate that U_+, U_- and U_{++} are subgroups, that $\alpha(U_+) \geq U_+$, and $\alpha(U_-) \leq U_-$, U_+ and U_- are all closed.

Definition 3. Let G be a totally disconnected locally compact group and let α be an endomorphism on G. A compact open subgroup U is called *tidy above for* α if it satisfies (TA) and *tidy below for* α if it satisfies both (TB1) and (TB2). If U is both tidy above and tidy below for α, then it is simply called *tidy*.

The motivation for the names ‘tidy above’ and ‘tidy below’ comes from a *tidying procedure*. Given any compact open subgroup U, the tidying procedure produces a tidy compact open subgroup. There are two steps to the procedure. The first step produces a compact open subgroup, V, which is tidy above. The second step takes this tidy above subgroup and produces a new compact open subgroup, W, which is both tidy above and tidy below. Each step of the tidying process reduces the index, so that

$$[\alpha(U) : \alpha(U) \cap U] \geq [\alpha(V) : \alpha(V) \cap V] \geq [\alpha(W) : \alpha(W) \cap W].$$

If U and V are tidy subgroups for an endomorphism α, then they have the same index,

$$[\alpha(U) : \alpha(U) \cap U] = [\alpha(V) : \alpha(V) \cap V].$$

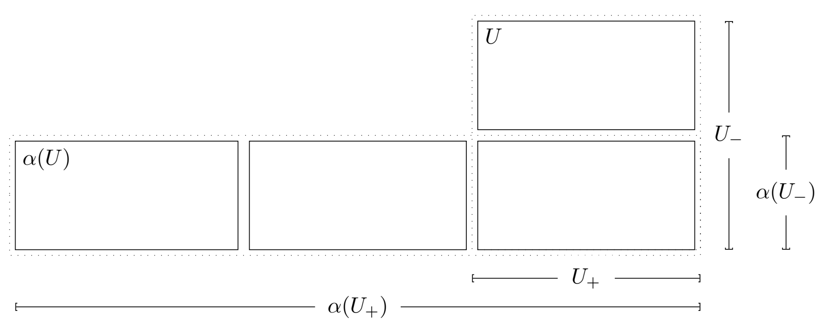

Fig. 1: Illustration of an automorphism α with $s(\alpha) = 3$ and $s(\alpha^{-1}) = 2$.

Lemma 2 (Tidying procedure part one). *Let α be an endomorphism on G and let U be a compact open subgroup in G. Then there exists a natural number n such that*

$$V = U_{-n} = \bigcap_{k=0}^{n} \alpha^{-n}(U) = \{u \in U : \alpha^k(u) \in U \text{ for } 0 \leq k \leq n\}$$

is tidy above.

Proof. We suppose that α is an automorphism, and only give the proof in that case. We first note that each U_{-n} is open, because it is a finite intersection of open subgroups. Recall that

$$U_+ = \bigcap_{k \geq 0} \alpha^k(U)$$

and

$$U_- = \bigcap_{k \geq 0} \alpha^{-k}(U).$$

Let

$$U_k = \bigcap_{0 \leq j \leq k} \alpha^j(U)$$

and see Figure 2 for an ilustration of these sets. Then

$$\alpha(U_+) = \bigcap_{0 \leq j \leq k} \alpha(U_j).$$

(That this modest claim fails for endomorphisms is one reason that definitions and arguments must be modified.) Furthermore, $\{\alpha(U_j)\}_{j=0}^{\infty}$ is a decreasing sequence of compact open subgroups. Since $\alpha(U_+)U$ is an open neighbourhood of $\alpha(U_+)$, there must be a natural number n such that $\alpha(U_n) \subseteq U_+U$.

Since $\alpha(U_n) \subseteq \alpha(U_+)U$, if y is in $\alpha(U_n)$, then $y = zu$, for some z in $\alpha(U_+)$ and u in U. Then $u = z^{-1}y$ is in $\alpha(U_n) = \bigcap_{j \geq 1}^{n+1} \alpha^j(U)$, which implies that u is in U_{n+1}. In fact, $\alpha(U_n) \subseteq \alpha(U_+)U_{n+1}$.

In order to complete the proof, we will prove the claim that

$$\alpha^l(U_n) = \alpha^l(U_+)U_{n+l}$$

for all non-negative integers l. We do so by induction.

$$\begin{aligned}
\alpha^{l+1}(U_n) &= \alpha(\alpha^l(U_+)U_{n+l}) \\
&= \alpha^{l+1}(U_+)\alpha(U_{n+l}) \\
&= \alpha^{l+1}(U_+)\alpha(U_+)U_{n+l+1} \\
&= \alpha^{l+1}(U_+)U_{n+l+1}.
\end{aligned}$$

Let y be an element of our compact open subgroup $V = U_{-n}$ and let

$$C_j = \{z \in V_+ : \alpha^j(y) \in \alpha^j(z)u\} \neq \varnothing.$$

Note that C_j is compact, and that C_{j+1} is contained within C_j for all natural numbers j. Now choose some z contained in the intersection $\bigcap_{j \geq 0} C_j$. Observe that z is in V_+, and that $z^{-1}y$ is in V_j, for all $j \geq 0$. This implies that $z^{-1}y$ is in V_-.

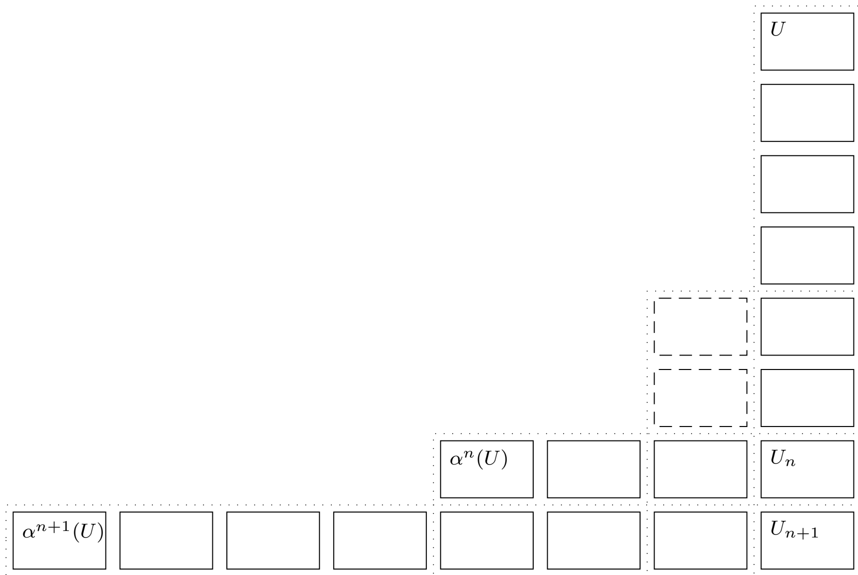

Fig. 2: Illustration of the subsets $U_n = \bigcap_{0\leq j\leq n} \alpha^j(U)$.

2 Scale of an endomorphism

Recall Definition 2, which states that if G is a t.d.l.c. group, then the scale of α is

$$s(\alpha) = \min\{[\alpha(U)\colon U\cap\alpha(U)] : U \in \mathrm{COS}\,(G)\}.$$

The scale defines a function from the endomorphisms of the group to the natural numbers. The scale of an endomorphism is 1 if and only if there exists a compact open subgroup U such that $\alpha(U)$ is a subgroup of U.

The scale of an endomorphism α satisfies

$$s(\alpha^n) = s(\alpha)^n \tag{1}$$

for all integers n. The proof of this fact is a consequence of the following lemma, which is [28, Proposition 16].

Lemma 3. *Suppose that U is a compact open subgroup which is tidy for an endomorphism α on a t.d.l.c. group G. Then U is tidy for α^n for every natural number n. Furthermore,*

$$s(\alpha^n) = [\alpha^n(U_+) : U_+].$$

To deduce (1) observe that

$$s(\alpha^n) = [\alpha^n(U_+) : U_+]$$

$$= \prod_{k=0}^{n-1} [\alpha^{k+1}(U_+) : \alpha^k(U_+)]$$
$$= [\alpha(U_+) : U_+]^n,$$

because $\{[\alpha^{k+1}(U_+) : \alpha^k(U_+)]\}$ is constant.

Another characterisation of the scale, known as *Møller's spectral radius formula*, may be derived from (1). This formula asserts that for any endomorphism α

$$s(\alpha) = \lim_{n\to\infty} [\alpha^n(U) : \alpha^n(U) \cap U]^{\frac{1}{n}} \tag{2}$$

where U is any compact open subgroup, not necessarily minimizing. (This formula is analogous to the spectral radius of a bounded linear operator T

$$\rho(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.)$$

Proof (Möller's Formula). Since (2) holds when U is tidy, by (1), it suffices to show that the $\lim_{n\to\infty}[\alpha^n(U) : \alpha^n(U) \cap U]^{\frac{1}{n}}$ is independent of U. For this, it suffices to show that the limit is the same for any compact open subgroup V containing U. To do this, consider

$$[\alpha^n(V) : \alpha^n(U) \cap U] = [\alpha^n(V) : \alpha^n(U)][\alpha^n(U) : \alpha^n(U) \cap U] \tag{3}$$
$$= [\alpha^n(V) : \alpha^n(V) \cap V][\alpha^n(V) \cap V : \alpha^n(U) \cap U]. \tag{4}$$

Note that, since α is an endomorphism on G,

$$[\alpha^n(V) : \alpha^n(U)] \le [V : U]$$

and

$$[\alpha^n(V) \cap V : \alpha^n(U) \cap U] = [\alpha^n(V) \cap V : \alpha^n(V) \cap U][\alpha^n(V) \cap U : \alpha^n(U) \cap U]$$
$$\le [V : U]^2.$$

Note further that all indices are greater than or equal to one. Hence (3) and (4) imply that

$$[V : U]^{-1}[\alpha^n(V) : \alpha^n(V) \cap V] \le \lim_{n\to\infty} [\alpha^n(U) : \alpha^n(U) \cap U]^{\frac{1}{n}}$$
$$\le [V : U]^2[\alpha^n(V) : \alpha^n(V) \cap V]$$

Therefore,

$$\lim_{n\to\infty} [\alpha^n(U) : \alpha^n(V) \cap V]^{\frac{1}{n}} = \lim_{n\to\infty} [\alpha^n(U) : \alpha^n(U) \cap U]^{\frac{1}{n}}.$$

□

Rögnvaldur Möller originally derived this formula from an alternative graph-theoretic characterisation of tidy subgroups and the scale he established in [16].

Automorphisms

No further multiplicativity or submultiplicativity properties hold for the scale in general. More can be said for automorphisms.

Theorem 2. *Suppose that α is an automorphism on G. Then U is minimizing for α if and only if it is minimizing for α^{-1} and $\Delta(\alpha) = s(\alpha)/s(\alpha^{-1})$ where $\Delta : G \to (\mathbb{R}, \times)$ is the modular function.*

Proof. Let U be any compact open subgroup of G and m be the Haar measure on G. Then by definition of the modular function

$$\begin{aligned}\Delta(\alpha) &= \frac{m(\alpha(U))}{m(U)}\\ &= \frac{m(\alpha(U))}{m(\alpha(U)\cap U)} \times \frac{m(\alpha(U)\cap U)}{m(U)}\\ &= \frac{[\alpha(U) : \alpha(U)\cap U]}{[U : \alpha(U)\cap U]}\\ &= \frac{[\alpha(U) : \alpha(U)\cap U]}{[\alpha^{-1}(U) : \alpha^{-1}(U)\cap U]}.\end{aligned}$$

Choosing U minimizing for α gives $\Delta(\alpha) \leq \frac{s(\alpha)}{s(\alpha^{-1})}$, and choosing U minimizing for α^{-1} gives that $\frac{s(\alpha)}{s(\alpha^{-1})} \leq \Delta(\alpha)$. Hence U is minimizing for α if and only if it is minimizing for α^{-1} and

$$\Delta(\alpha) = \frac{s(\alpha)}{s(\alpha^{-1})}.$$

□

The *Braconnier topology* on $\mathsf{Aut}(G)$ is the topology with base elements

$$N_1(K,O) = \{\alpha \in \mathsf{Aut}(G) : \alpha(K) \leq O\}, \text{ and}$$

$$N_2(K,O) = \{\alpha \in \mathsf{Aut}(G) : \alpha^{-1}(K) \leq O\}$$

ranging over all compact subsets K in G and open subsets O in G. This topology is formally stronger than the *compact open topology*, which is the topology with base formed from the sets $N_1(K,O)$, but these two topologies are equal in many cases. The compact open topology is not in general a group topology, because the inverse mapping may fail to be continuous for it, and the Braconnier topology remedies that difficulty.

Examples show that the scale function $s : \mathsf{Aut}(G) \to \mathbb{N}$ need not be continuous with respect to the Bracconier topology on $\mathsf{Aut}(G)$ and discrete topology on $\mathbb{N}$.

Question 1. Is there a topology on $\mathsf{Aut}(G)$, or possibly $\mathsf{End}(G)$, such that the scale is continuous? Is there a topology on $\mathsf{End}(G)$ with $\mathsf{Aut}(G)$ an open subgroup?

The second of these questions is motivated by the fact that the group of invertible operators on a normed space is open in the semigroup of all endomorphisms.

Inner Automorphisms

Each $x \in G$ determines an inner automorphism $\alpha_x : y \mapsto xyx^{-1}$. The homomorphism $G \to \mathsf{Aut}(G)$, $x \mapsto \alpha_x$ induces a function on G called the scale on G, which is also denoted by s.

Theorem 3. *The scale $s : G \to \mathbb{N}$ is continuous with respect to the given topology in G and the discrete topology on $\mathbb{N}$.*

The next theorem gives much more precise information.

Theorem 4. *Suppose that $x \in G$ and let U be a compact open subgroup tidy for x. Then U is tidy for every $y \in UxU$ and $s(y) = s(x)$.*

Proof. In the first instance, let u be an element of U. Consider $y = xu$ and let $u = u_-u_+$ for some $u_- \in U_-$ and $u_+ \in U_+$. Then

$$\begin{aligned}(xu)^2 &= xuxu\\ &= xu_-u_+xu_-u_+\\ &= (xu_-x^{-1})xu_+(xu_-x^{-1})xu_+.\end{aligned}$$

But xu_-x^{-1} is in U_-, since conjugation by x shrinks u_-. Hence,

$$\begin{aligned}(xu)^2 &= u'_-xvxu'_+\\ &= u'_-(xv_-x^{-1})x^2(x^{-1}v_+x)u'_+\\ &= u''_-x^2u''_+\end{aligned}$$

for $u'_-, u''_- \in U_-$ and $u'_+, u''_+ \in U_+$ and some $v = v_-v_+$, where $v_- \in U_-$ and $v_+ \in U_+$. A similar calculation shows that, more generally, if $y = u_1xu_2 \in UxU$ and $n \geq 0$, then$(u_1xu_2)^n = u''_-x^nu''_+$ for some $u''_\pm \in U_\pm$. Hence,

$$\begin{aligned}&[y^nUy^{-n} : y^nUy^{-n} \cap U]\\ &= [(u_1xu_2)^nU(u_1xu_2)^{-n} : (u_1xu_2)^nU(u_1xu_2)^{-n} \cap U]\\ &= [(u_-x^nu_+)U(u_-x^nu_+)^{-1} : (u_-x^nu_+)U(u_-x^nu_+)^{-1} \cap U]\\ &= [u_-x^n(u_+Uu_+^{-1})x^nu_-^{-1} : u_-x^n(u_+Uu_+^{-1})x^nu_-^{-1} \cap U]\\ &= [u_-x^nUx^{-n}u_-^{-1} : u_-x^nUx^{-n}u_-^{-1} \cap U]\\ &= [x^nUx^{-n} : x^nUx^{-n} \cap U]\end{aligned}$$

$$= s(x^n).$$

Hence, by Möller's spectral radius formula,

$$s(y) = \lim_{n\to\infty}[y^nUy^{-n} : y^nUy^{-n} \cap U]^{1/n} = s(x).$$

The $n = 1$ case of the calculation then shows that

$$[yUy^{-1} : yUy^{-1} \cap U] = s(x) = s(y)$$

and so U is minimizing for y. □

2.1 An application of the scale function

Define, for G a non-discrete totally disconnected locally compact group,

$$\mathrm{Per}\,(G) = \{x \in G : \overline{\langle x\rangle} \text{ is compact}\}.$$

Theorem 5. $\mathrm{Per}\,(G)$ *is a closed subset of* G*.*

The theorem answers a question posed by Karl Heinrich Hofmann that was motivated by the following considerations. When G is discrete $\mathrm{Per}\,(G)$ is clearly closed. However $\mathrm{Per}\,(G)$ is not always closed when G is connected. For example, translations can be approximated by rotations in the affine group of the plane.

Lemma 4. *Suppose that* G *is a totally disconnected locally compact group. Let* x *be an element of* $\mathrm{Per}\,(G)$*. Then,*

$$s(x) = 1 = s(x^{-1}).$$

Proof. The quantity $s(\overline{\langle x\rangle})$ is finite because the scale is continuous and the image of a compact set must therefore be finite. But since

$$s(x^n) = s(x)^n,$$

the finiteness of the image implies that $s(x) = 1$. A similar argument shows that $s(x^{-1}) = 1$. □

Now suppose that x is in $\overline{\mathrm{Per}\,(G)}$. Choose U tidy for x. Then $xUx^{-1} = U$ and $x = yu$ for some y in $\mathrm{Per}\,(G)$ and u in U. Hence $yUy^{-1} = U$ and it follows that

$$\langle x\rangle \subseteq \langle y\rangle U \subseteq \overline{\langle y\rangle}U, \text{ which is compact.}$$

□

2.2 The scale and tidy subgroups for homomorphisms

Question 2. Do the concepts of scale and tidy subgroup extend to homomorphisms $\tau : G \to H$?

The answer is 'probably not' – the scale is analogous to the eigenvalues of a linear transformation $T : V \to V$. There is no concept of eigenvalue for linear maps between different vector spaces. Here is a related question.

Question 3. Suppose that $\tau : G \to H$ and $\sigma : H \to G$ are homomorphisms.
Is $s(\tau \circ \sigma) = s(\sigma \circ \tau)$?

The following special case asks for an analogue of singular values. Let G and H be self-dual abelian t.d.l.c. groups, with $\iota_G : \hat{G} \to G$ and $\iota_H : \hat{H} \to H$ isomorphisms. Let $\tau : G \to H$ be a homomorphism and put $\sigma = \iota_G \circ \hat{\tau} \circ \iota_H^{-1}$. Is $s(\tau \circ \sigma) = s(\sigma \circ \tau)$?

3 Subgroups associated with endomorphisms

3.1 Minimising subgroups and their associates

We begin by recalling the following from Section 1.

Theorem 6 (The Structure of Minimising Subgroups). *A compact open subgroup U of a locally compact totally disconnected group is minimizing for an endomorphism α if and only if it is tidy.*

Definition 4. A compact open subgroup U of a locally compact totally disconnected group is *tidy* for an endomorphism α if it satisfies:

(TA) $U = U_+U_-$.
(TB1) $U_{++} = \bigcup_{n\geq 0} \alpha^n(U_+)$ is closed.
(TB2) The sequence of integers $[\alpha^{n+1}(U_+) : \alpha^n(U_+)]$ is constant. (This property is only needed for endomorphisms)

where

$$U_+ = \{x \in U : \exists \{x_n\}_{n\in\mathbb{N}} \subset U \text{ with } x_0 = x \text{ and } \alpha(x_{n+1}) = x_n \ \forall n \in \mathbb{N}$$

and

$$U_- = \{x \in U : \alpha^n(x) \in U \ \forall n \in \mathbb{N}\}.$$

There may be many subgroups that are tidy for a given endomorphism α. For example, if $\alpha \in \mathsf{Aut}(G)$ and U is tidy for α, then so are $\alpha^n(U)$ and $\bigcap_{k=0}^n \alpha^k(U)$ for every integer n. The associated subgroups U_+, U_-, U_{++}, and U_{--} may depend on the choice of tidy subgroup U.

Other subgroups of G may be associated with a given endomorphism α in a unique way.

Definition 5 (The parabolic and Levi subgroups). Suppose that G is a totally disconnected locally compact group and let α be an endomorphism on G. Define

- the *parabolic subgroup* to be

$$\overrightarrow{\mathsf{par}}(\alpha) = \{x \in G \mid \{\alpha^n(x)\}_{n\in\mathbb{N}} \text{ is pre-compact}\},$$

- the *anti-parabolic subgroup* to be the subgroup

$$\overleftarrow{\mathsf{par}}(\alpha) = \{x \in G \mid \exists\{x_n\}_{n=0}^{\infty} \text{ pre-compact with } x_0 = x \text{ and } \alpha(x_{n+1}) = x_n\}, \text{ and}$$

- the *Levi subgroup* to be the intersection of the parabolic subgroup and the anti-parabolic subgroup:

$$\mathsf{lev}(\alpha) = \overrightarrow{\mathsf{par}}(\alpha) \cap \overleftarrow{\mathsf{par}}(\alpha).$$

It may be checked that these are subgroups of G but an argument using a subgroup tidy for α shows more.

Theorem 7. $\overrightarrow{\mathsf{par}}(\alpha)$*,* $\overleftarrow{\mathsf{par}}(\alpha)$ *and* $\mathsf{lev}(\alpha)$ *are closed subgroups of G.*

Proof (Sketch of the proof that $\overrightarrow{\mathsf{par}}(\alpha)$ *is closed).* Show that $\overrightarrow{\mathsf{par}}(\alpha) \cap U = U_-$, which is closed. A classical lemma of Bourbaki implies that U is closed. □

Definition 6. Suppose that $\alpha \in \mathsf{End}(G)$. Define

- the *contraction subgroup* to be

$$\overrightarrow{\mathsf{con}}(\alpha) = \{x \in G \mid \alpha^n(x) \to 1 \text{ as } n \to \infty\},$$

- the *iterated kernel* to be

$$\mathsf{ker}^{\infty}(\alpha) = \{x \in G \mid \exists n \geq 0 \text{ with } \alpha^n(x) = 1\}, \text{ and}$$

- the *anti-contraction subgroup* to be

$$\overleftarrow{\mathsf{con}}(\alpha) = \{x \in G \mid \exists\{x_n\}_{n=0}^{\infty} \text{ such that } x_n \to 1 \text{ and } \alpha(x_{n+1}) = x_n\}.$$

Clearly, $\mathsf{ker}^{\infty}(\alpha) \leq \overrightarrow{\mathsf{con}}(\alpha)$ and is a normal subgroup of G. Furthermore, $\overrightarrow{\mathsf{con}}(\alpha) \leq V_{--}$ and $\overleftarrow{\mathsf{con}}(\alpha) \leq V_{++}$ for every subgroup V tidy for α. When α is an automorphism, we have that $\overleftarrow{\mathsf{con}}(\alpha) = \overrightarrow{\mathsf{con}}(\alpha^{-1})$. The contraction subgroup for α will be denoted by $\mathsf{con}(\alpha)$ in this case. It is related to the scale of α^{-1}.

Theorem 8 (Baumgartner & W., Jaworski). *Suppose that* α *is an automorphism on G. Then*

$$\overline{\mathsf{con}(\alpha)} = \bigcap\{U_{--} \in \mathrm{COS}(G) \mid U \text{ is tidy for } \alpha\}, \text{ and}$$

$$s(\alpha^{-1}|_{\overline{\mathsf{con}(\alpha)}}) = s(\alpha^{-1}).$$

Remark 5. Theorem 8 was established for metrisable groups in [5] and the metrisability condition removed in [13]. The second part of the theorem implies in particular that, if $s(\alpha^{-1}) > 1$, then the contraction subgroup for α is not trivial.

Question 4. Extend this result to endomorphisms. There will need to be two theorems. One for $\overrightarrow{\mathsf{con}}(\alpha)$ and one for $\overleftarrow{\mathsf{con}}(\alpha)$, e.g.

$$s(\alpha|_{\overrightarrow{\mathsf{con}}(\alpha)}) = s_G(\alpha).$$

Remark 6. Since this lecture was delivered, results about the contraction and anti-contraction subgroups that extend Theorem 8 to endomorphisms have been established by T. Bywaters, H. Glöckner and S. Tillman.

Definition 7. The *nub* subgroup for the endomorphism α on a totally disconnected locally compact group G is

$$\mathsf{nub}(\alpha) = \bigcap \{U \in \mathrm{COS}\,(G) \mid U \text{ is tidy for } \alpha\}\,.$$

The following was established in [5], although the nub terminology was not used.

Theorem 9 (Baumgartner & W.). *Let $\alpha \in \mathsf{Aut}(G)$. Then, the following are equivalent:*

- $\mathsf{nub}(\alpha) = \{1\}$*;*
- $\mathsf{con}(\alpha)$ *is closed; and*
- *if U is tidy above for α, then U is tidy for α.*

Since $\mathsf{nub}(\alpha) = \mathsf{nub}(\alpha^{-1})$, it follows as well from the theorem that $\mathsf{con}(\alpha^{-1})$ is closed whenever $\mathsf{nub}(\alpha)$ is trivial.

Example 1. Let F be a finite group and put $G = F^{\mathbb{Z}}$. Then G is a compact t.d.l.c. group. Define $\alpha \in \mathsf{Aut}(G)$ by

$$\alpha(f)_n = f_{n+1}, \quad (f \in F^{\mathbb{Z}}).$$

Then $\mathsf{nub}(\alpha) = G$.

Note that, in the example,

$$\mathsf{con}(\alpha) = \{f \in G \mid \exists N \in \mathbb{Z} \text{ such that } f_n = 1 \text{ if } n \geq N\}$$

and is dense in G. Moreover, $\mathsf{con}(\alpha) \cap \mathsf{con}(\alpha^{-1})$ is equal to the subgroup of functions with finite support, which is also dense.

It may be shown that $\mathsf{nub}(\alpha)$ is also the largest closed subgroup of G on which the restriction of α is ergodic. This fact extends a result due to Aoki in the 1980's who proved for t.d.l.c. groups a conjecture of Halmos that any locally compact group for which there is an ergodic automorphism must be compact. The method of tidy subgroups allows this to be proved in a few lines.

Another characterisation of the nub is that

$$\begin{aligned}\mathsf{nub}(\alpha) &= \overline{(\mathsf{con}(\alpha)\cap\mathsf{par}(\alpha^{-1}))}\\ &= \{x\in G : \alpha^n(x)\to 1 \text{ as } n\to\infty, \{\alpha^{-n}(x)\}_{n=0}^{\infty} \text{ is precompact}\}.\end{aligned}$$

The structure of nub subgroups may be described in some detail, see [27, 28]. Among the results is that $\mathsf{con}(\alpha|_{\mathsf{nub}(\alpha)})$ is dense in $\mathsf{nub}(\alpha)$, and $\mathsf{con}(\alpha|_{\mathsf{nub}(\alpha^{-1})})$ is dense in $\mathsf{nub}(\alpha^{-1})$. Their intersection may fail to be dense however.

It may happen that $\mathsf{con}(\alpha)$ is closed. That is the case when G is a p-adic Lie group for example, see [5]. A key example is a *restricted product with shift* which, for some given finite group F, is the group

$$G = \{f = (f_n)_{n\in\mathbb{Z}} : f_n \in F,\ \exists N \text{ such that } f_n = 1 \text{ for all } n\geq N\}$$

with the *shift automorphism* on G defined by

$$\alpha(f)_n = f_{n+1}, \qquad (n\in\mathbb{Z}).$$

The shift automorphisms satisfies $\mathsf{con}(\alpha) = G$, which is closed. The structure of general closed contraction groups may be described, see [10].

Theorem 10 (Glöckner & W.). *Let G be a t.d.l.c. group and $\alpha\in\mathsf{Aut}(G)$. Suppose that $\mathsf{con}(\alpha) = G$. Then*

- *$G = N\times T$, where N and T are α-invariant, N is a divisible subgroup of G and T a torsion subgroup;*
- *N is isomorphic to the direct sum of a finite number of nilpotent p-adic Lie groups for the primes p dividing $s(\alpha^{-1})$; and*
- *T has a composition series*

$$T_0 = \{1\}\lhd T_1\lhd\cdots\lhd T_j\lhd\cdots\lhd T_{r-1}\lhd T_r = T$$

of closed α-invariant subgroups such that T_{j+1}/T_j is isomorphic to a restricted product with shift.

4 Flat groups of automorphisms

Analogies with linear algebra are suggested by, or have motivated, several of the ideas seen so far. The 'spectral radius' formula is one of the ideas suggesting an analogy between the scale and eigenvalues of a linear transformation; and the fact that the methods of linear algebra apply to all linear transformations and not just invertible ones was one reason for thinking that the characterisation of subgroups minimising for automorphisms would extend to endomorphisms.

How the method of tidy subgroups might extend to more than one automorphism simultaneously is also suggested by this analogy. Finding a subgroup tidy for an endomorphism is the analogue of finding a Jordan basis for a linear transformation, which essentially can only be done when the linear transformations commute. On the other hand, when two linear transformations do share a common Jordan basis, they commute modulo upper triangular matrices. The following results were suggested by these observations and established in [26].

Theorem 11. *Let* $\{\alpha_1,\ldots,\alpha_k\} \subset \mathsf{Aut}(G)$ *be a commuting set of automorphisms of the t.d.l.c. group G. Then there is* $U \in \mathrm{COS}(G)$ *that is tidy for every* α *in* $\langle\alpha_1,\ldots,\alpha_k\rangle$.

Theorem 12. *Suppose, for some* $\alpha,\beta \in \mathsf{Aut}(G)$*, that there is* $U \in \mathrm{COS}(G)$ *that is tidy for every* $\gamma \in \langle\alpha,\beta\rangle$*. Then* $s([\alpha,\beta]) = 1$*, that is,* α *and* β *commute modulo the* uniscalar *elements in* $\langle\alpha,\beta\rangle$.

Remark 7. It is important that these results refer to *groups* of automorphisms. It is not automatically the case that, if α and β share a common tidy subgroup U, then U is tidy for every $\gamma \in \langle\alpha,\beta\rangle$. This complicates the proof of the first theorem and means that the hypothesis of the second theorem needs to be strictly stronger than that α and β should share a common tidy subgroup.

Example 2. Let $G = \mathbb{Q}_p^d$ and define $\alpha,\beta \in \mathsf{Aut}(G)$ by

$$\begin{aligned} \alpha(x_1,\ldots,x_d) &= p(x_1,\ldots,x_d) \\ \text{and}\ \ \beta(x_1,x_2,\ldots,x_d) &= (px_1,p^2x_2,\ldots,p^dx_d). \end{aligned}$$

Then $U = \mathbb{Z}_p^d$ is tidy for every $\gamma \in \langle\alpha,\beta\rangle$. However,

$$V = \left\{(z_1,\ldots,z_d) \in U \mid z_i \equiv z_j \pmod{p},\ i,j \in \{1,\ldots,d\}\right\}$$

is tidy for α and β, and indeed for every γ in the semigroup generated by α and β, but not for every $\gamma \in \langle\alpha,\beta\rangle$.

Proof (d=2). In this case,

$$V = \{(z_1,z_2) \in \mathbb{Z}_p^2 : z_1 \equiv z_2 \pmod{p}\}$$

and

$$\alpha(V) = \left\{(pz_1,pz_2) \in \mathbb{Z}_p^2 : z_1 \equiv z_2 \pmod{p}\right\} \leq V$$

because $pz_1 \equiv pz_2 \equiv 0 \pmod{p}$. We also have

$$\beta(v) = \{(pz_1,p^2z_2) : z_1 \equiv z_2 \pmod{p})\} \leq V.$$

Hence, V is tidy for α and β, $V_+ = \{0\}$, $V_- = V$ and $V_{--} = G$ for both α and β. We also have that $s(\alpha) = 1 = s(\beta)$, $s(\alpha^{-1}) = p^2$ and $s(\beta^{-1}) = p^3$. Hence V is tidy for α and β. Moreover, if $\gamma = \alpha^m\beta^n$, $m,n \geq 0$, then $\gamma(V) \subseteq V$, so that V is tidy.

The subgroup V is not minimizing for $\alpha\beta^{-1}$ however. Calculation shows that

$$V \cap \alpha\beta^{-1}(V) = \{(w_1, w_2) \in \mathbb{Z}_p^2 : w_i \equiv 0 \pmod{p}\} = p\mathbb{Z}_p^2.$$

$$[\alpha\beta^{-1}(U) : p\mathbb{Z}^2] = p^2,$$

which is larger than the corresponding index found for U,

$$[\alpha\beta^{-1}(\mathbb{Z}_p^2) : \alpha\beta^{-1}(\mathbb{Z}_p^2) \cap \mathbb{Z}_p^2] = [\alpha\beta^{-1}(\mathbb{Z}_p^2) : \mathbb{Z}_p^2] = p.$$

It may also be seen that V is not tidy for $\alpha\beta^{-1}$, for

$$V_+ = \bigcap_{k\geq 0}^{\infty} (\alpha\beta^{-1})^k(V) = p\mathbb{Z}_p^2 \text{ and}$$
$$V_- = \bigcap_{k\geq 0}^{\infty} (\alpha\beta^{-1})^{-k}(V) = p\mathbb{Z}_p \oplus \{0\}, \text{ so that}$$
$$V_+V_- = p\mathbb{Z}_p^2 \neq V \text{ and } V \text{ is not tidy above.}$$

□

4.1 Flat groups and the flat-rank

Definition 8. A subgroup $\mathscr{H} \leq \mathrm{Aut}(G)$ is *flat* if there is $U \in \mathrm{COS}(G)$ that is tidy for every $\alpha \in \mathscr{H}$. The *uniscalar subgroup* of the flat group $\mathscr{H}$ is

$$\mathscr{H}_1 = \left\{\alpha \in \mathscr{H} \mid s(\alpha) = 1 = s(\alpha^{-1})\right\}.$$

$\mathscr{H}_1$ is a subgroup because $\alpha \in \mathscr{H}_1$ if and only if $\alpha(U) = U$ for any, and hence all, subgroups tidy for $\mathscr{H}$.

Theorem 13. *Suppose that $\mathscr{H} \leq \mathsf{Aut}(G)$ is finitely generated and flat, and let U be tidy for $\mathscr{H}$. Then $\mathscr{H}_1 \triangleleft \mathscr{H}$ and there is $r \in \mathbb{N}$ such that*

$$\mathscr{H}/\mathscr{H}_1 \cong \mathbb{Z}^r.$$

1. *There is $d \in \mathbb{N}$ such that*
$$U = U_0U_1 \ldots U_d,$$
where for every $\alpha \in \mathscr{H}$: $\alpha(U_0) = U_0$ and,
for every $j \in \{1, 2, \ldots, d\}$, either $\alpha(U_j) \leq U_j$ or $\alpha(U_j) \geq U_j$.

2. For each $j \in \{1, 2, \ldots, d\}$ there is a homomorphism $\rho_j : \mathscr{H} \to \mathbb{Z}$ and a positive integer s_j such that
$$[\alpha(U_j) : U_j] = s_j^{\rho_j(\alpha)},$$
where

$$[\alpha(U_j):U_j]=\begin{cases}[\alpha(U_j):U_j], & \text{if } U_j\le\alpha(U_j),\\ [U_j:\alpha(U_j)]^{-1}, & \text{if } U_j\ge\alpha(U_j).\end{cases}$$

3. For each $j\in\{1,2,\dots,d\}$,
$$\widetilde{U}_j:=\bigcup_{\alpha\in\mathscr{H}}\alpha(U_j)$$
is a closed subgroup of G.

4. The natural numbers r and d, the homomorphisms $\rho_j:\mathscr{H}\to\mathbb{Z}$ and positive integers s_j are independent of the subgroup U tidy for α.

Remark 8. The numbers $s_j^{\rho_j(\alpha)}$ are analogues of absolute values of eigenvalues for α and the subgroups $\widetilde{U}_j=\bigcup_{\alpha\in\mathscr{H}}\alpha(U_j)$ are the analogues of common eigenspaces for the automorphisms in $\mathscr{H}$.

Example 3 (A). Take $G=\mathbb{Q}_p^d$, and α,β as before. Take $U=\mathbb{Z}_p^d$ as a tidy subgroup. The number of factors will be d. How do we obtain the factors? Note that

$$U_{\alpha+},U_{\beta+}=\{0\}$$
$$U_{\alpha-},U_{\beta-}=U_-.$$

Choose $\alpha\beta^{-1}$, then

$$\alpha\beta^{-1}(z_1,\cdots,z_d)=(z_1,p^{-1}z_2,\dots,p^{1-d}z_d).$$

Calculating the factoring of U determined by $\alpha\beta^{-1}$ we obtain

$$U_{\alpha\beta^{-1}{}_+}=U,\ \ U_{\alpha\beta^{-1}{}_-}=\mathbb{Z}_p\oplus\{0\},$$

which identifies one factor but not the others. To separate out these factors choose, for each $i\in\{1,\dots,d\}$ the element $\gamma=\alpha^i\beta^{-1}$. Then

$$U_{\gamma_+}=\{0\}\oplus(\mathbb{Z}_p)^{d-i+1},\ \ U_{\gamma_-}=(\mathbb{Z}_p)^i\oplus\{0\}\text{ and }U_{\gamma_+}\cap U_{\gamma_-}=\{0\}\oplus\underbrace{\mathbb{Z}_p}_{i}\oplus\{0\}.$$

We see that generators alone are insufficient to separate all the factors in Theorem 13.1 but they can be separated by using additional elements. That is the strategy of the proof of Theorem 13.

Example 4. Let $G=SL(n,\mathbb{Q}_p)$ and let H be subgroup of the diagonal matrices in G. Let $\alpha_h(x)=hxh^{-1}$. Then:

- $r=n-1$;
- $d=n(n-1)$;
- ρ_j are roots of H; and
- $\widetilde{U}_j$ are root subgroups of G.

Definition 9. The number r appearing in the theorem is the *rank* of the flat group $\mathcal{H}$. The maximum rank of any flat group of inner automorphisms of the t.d.l.c. group G is the *flat-rank* of G.

Example 5.
- Let $G = \mathbb{Q}_p^d \rtimes \langle \alpha, \beta \rangle$. Then G has flat-rank 2.
- Let $G = SL(n, \mathbb{Q}_p)$. Then G has flat-rank $n-1$.
- Let $G = \mathrm{AAut}(T)$ be the group of almost automorphisms of the regular tree T. Then G has infinite flat-rank.

4.2 Further results about flat groups

The theorem on finitely generated flat groups can be applied to show that a flat group of automorphisms contains uniscalar elements when the group is not abelian. The following is established in [19].

Theorem 14. *Let G be a t.d.l.c. group. Then,*

- *Every finitely generated nilpotent subgroup of* $\mathsf{Aut}(G)$ *is flat.*
- *Every polycyclic group subgroup of* $\mathsf{Aut}(G)$ *is virtually flat, that is, has a flat subgroup of finite index.*

The subgroup of upper triangular matrices in $SL(n, \mathbb{Z})$ is nilpotent and is non-abelian when $n \geq 3$. Hence, if $n \geq 3$ and $\rho : SL(n, \mathbb{Z}) \to G$ is a homomorphism with G a t.d.l.c. group, then there is an upper triangular T such that $\rho(T)$ is uniscalar. Further work using deep theorems about $SL(n, \mathbb{Z})$ and a theorem about groups that commensurate in bounded fashion deduce from this that $\rho(SL(n, \mathbb{Z}))$ normalises a compact open subgroup of G, see Remark 10 below. For details and additional references see [19].

5 T.d.l.c. groups and geometry

In this section we only consider automorphisms of t.d.l.c. groups. The aim is to survey actions of t.d.l.c. groups that may be viewed as geometric.

5.1 Symmetric spaces modulo a compact open subgroup

Many groups have geometric representations that aid understanding of the group. Semi-simple real Lie groups, for example, act on a real symmetric space [11] and semi-simple Lie groups over a totally disconnected locally compact field may be represented as acting on a simplicial complex called an *affine building* and also on a related simplicial complex called a *spherical building* [22, 9]. In the case when the group has rank 1, *e.g.*, $SL(2,\mathbb{Q}_p)$, the affine building is a tree and the spherical building is the set of ends of the tree. Other examples of t.d.l.c. groups, such as Kac-Moody groups [23], also act on buildings and on the boundary of the building. Automorphism groups of buildings are themselves t.d.l.c. groups and which then come with their own natural geometric representation. They, and their closed subgroups, are a rich source of examples of t.d.l.c. groups.

The so-called 1-skeleton of an affine building is a graph and path length then defines a metric on the set of vertices of this graph. As a metric space, it contains geometric 'flats', which are subsets quasi-isometric to $\mathbb{Z}^r$ for some r. This number r is the *geometric rank* of the building. In many cases of groups acting on a building, such as Kac-Moody groups, the group also has an algebraic rank. Under certain hypotheses, it may be shown that the geometric rank of the building, the algebraic rank of the group and the flat-rank are all equal, see [3].

Example 6. The group $SL(2,\mathbb{Q}_p)$ has flat-rank 1 and acts on the regular tree with valency $p+1$. Trees have geometric rank equal to 1.

Vertex stabilisers for this action are maximal compact subgroups of $SL(2,\mathbb{Q}_p)$. Indeed, there is $v \in V(T)$ such that $\mathrm{stab}_G(v) = SL(2,\mathbb{Z}_p)$, which is one of the maximal compact subgroups of $SL(2,\mathbb{Z}_p)$. The homogeneous space $SL(2,\mathbb{Q}_p)/SL(2,\mathbb{Z}_p)$ may thus be identified with the $SL(2,\mathbb{Q}_p)$-orbit of v, which is one of two such orbits in $V(T)$. Note that any compact subgroup of a t.d.l.c. group is contained in an open compact subgroup, so that these maximal compact subgroups are open and the quotient topology on $SL(2,\mathbb{Q}_p)/SL(2,\mathbb{Z}_p)$ is discrete.

5.2 Cayley-Abels graphs

Suppose that G is a compactly generated t.d.l.c. group and let $U \in \mathrm{COS}(G)$. A graph, $\Gamma(K,U)$, may be defined by choosing a compact, symmetric generating set, K, for G and setting

$$V(\Gamma) = G/U \text{ and } E(\Gamma) = \left\{(gU,hU) \in V(\Gamma)^2 \mid h^{-1}g \in UKU\right\}.$$

Then $\Gamma(K,U)$ is a locally finite graph and the translation action of G on Γ is by graph automorphisms. This action is transitive and vertex stabilisers are all conjugates of U.

Any graph on which G acts vertex-transitively and with compact open vertex stabilisers is called a *Cayley-Abels graph* for G. Hence $\Gamma(K,U)$ is a Cayley-Abels graph. The graphs $\Gamma(K,U)$ are not unique and depend on the choices of K and U. All Cayley-Abels graphs for G are quasi-isometric however, see [1, 15, 17].

The Cayley-Abels graph guarantees that every compactly generated t.d.l.c. group acts on a locally finite connected graph. This graph is not canonical however because there may be many non-isomorphic Cayley-Abels graphs. On the other hand, the automorphism group of any locally finite connected graph Γ is totally disconnected when equipped with the topology of uniform convergence on compact sets. The vertex stabilisers will be compact open subgroups of the automorphism group and so Γ is a Cayley-Abels graph for its automorphism group. There is thus an equivalence between compactly generated t.d.l.c. groups and closed subgroups of automorphism groups of connected locally finite graphs.

Example 7.
- Let Γ be a regular tree. Then $G = \mathsf{Aut}(\Gamma)$ is a t.d.l.c. group and Γ is a Cayley-Abels graph for G.
- The group $PSL(n, \mathbb{Q}_p)$ acts on a Bruhat-Tits building of rank $n-1$. The 1-skeleton of this building is not a Cayley-Abels graph for the group because the action is not transitive. However, the building is quasi-isometric to a Cayley-Abels graph because vertex stabilisers are compact and there are only finitely many orbits for the G-action.

Remark 9. In many examples, the orbit $H.v \subset \Gamma(K,U)$ of a flat subgroup $H \leq G$ is quasi-isometric to $\mathbb{Z}^r$, where r is the flat-rank of H. However, this only holds when H_1, the uniscalar subgroup in H, is compact.

5.3 Actions on sets of subgroups of G

The set COS (G) is a discrete metric space with the metric defined by

$$d(U,V) = \log\left([U : U \cap V]\,[V : U \cap V]\right).$$

For each $\alpha \in \mathsf{Aut}(G)$ the map $U \mapsto \alpha(U)$ is an isometry of COS (G) and the map $\mathsf{Aut}(G) \to \mathrm{Isom}(\mathrm{COS}\,(G))$ is a homomorphism.

Subgroups tidy for α may be characterised as those whose α-orbit is a straight line in COS (G), see [6].

Proposition 1.

$$\begin{aligned} &U \textit{ is tidy for } \alpha \in \mathsf{Aut}(G)\\ \iff &d(\alpha^m(U), \alpha^n(U)) = |m-n|\, d(U, \alpha(U)) \textit{ for every } m \geq 0. \end{aligned}$$

The relationship between Cayley-Abels graphs and the metric G-space COS (G) is seen in the following.

Proposition 2. *The function* $\psi : \Gamma(K,U) \to \mathrm{COS}\,(G)$ *defined by*

$$\psi(xU) = xUx^{-1}, \quad xU \in V(\Gamma(K,U)),$$

is bounded with respect to the geodesic distance on $\Gamma(K,U)$ *and is injective if and only if* $N_G(U) = U$.

Remark 9 points out that if $H \leq G$ is flat with rank r, then there is an H-orbit in $\Gamma(K,U)$ that is quasi-isometric to $\mathbb{Z}^r$ if and only if the uniscalar subgroup of H is compact. On the other hand, H-orbits in COS $((G))$ are always quasi-isometric to $\mathbb{Z}^r$ if H is flank with rank r. The following, proved in [4], goes in the opposite direction.

Theorem 15 (Baumgartner, Schlichting, W.). *Suppose that all balls in the metric space* $\mathrm{COS}(G)$ *are finite. Let* $\mathscr{H} \leq \mathsf{Aut}(G)$ *be such that the the* $\mathscr{H}$*-orbit* $\{\alpha(U) \mid \alpha \in \mathscr{H}\}$ *is quasi-isometric to* $\mathbb{Z}^r$*. Then* $\mathscr{H}$ *is virtually flat.*

Question 5. Does the conclusion of Theorem 15 hold for all t.d.l.c. groups, rather than just those for which all balls in $\mathrm{COS}(G)$ are finite?

Remark 10. The answer to this question is 'yes' in the flat-rank 0 case, see [18, 7]. In other words, if $\left\{d(U, hUh^{-1}) : h \in H\right\}$ is bounded for some $U \in \mathrm{COS}(G)$, then there is $V \in \mathrm{COS}(G)$ such that $hVh^{-1} = V$ for every $h \in H$. This is one of the additional theorems used in [19] that was referred to in the comments following Theorem 14.

The space of directions

Definition 10. The *ray* generated by $\alpha \in \mathsf{Aut}(G)$ and based at $U \in \mathrm{COS}(G)$ is the sequence $\{\alpha^n(U)\}_{n=0}^{\infty}$. An automorphism α on G *moves towards infinity* if for any pair $V \leq W \in \mathrm{COS}(G)$ there is $n \geq 1$ such that $\alpha^n(V) \not\leq W$.

It may be seen that α moves towards infinity if and only if $s(\alpha) > 1$. A pseudo-metric may be defined on all rays $\{\alpha^n(U)\}_{n=0}^{\infty}$ such that α moves towards infinity. Identifying rays that are distance 0 apart and completing with respect to the metric yields the *space of directions*, see [6].

Example 8. Let $G = \mathsf{Aut}(T)$, with T a regular tree. Then the space of directions is the set of ends of the tree with all distinct points being distance 2 apart.

The space of directions is computed in a number of cases in [6] and is seen to be a familiar space in many well-known examples. However, it is also seen that it can be quite complicated. An example is given in [6] where the space of directions is isometric to the set of Borel subsets on $[0,1]$ with the metric $d(A,B) = m(A \triangle B)$, where m is the Lebesgue measure and two sets are identified if they differ on a set of measure 0. It might be of interest to know the space of directions for groups whose geometric structure is not well understood.

Question 6. What is the space of directions for Neretin's group?

The Chabauty space

Definition 11. Let G be a locally compact group. The *Chabauty space*, $\mathrm{SUB}(G)$, is the set of all closed subgroups of G equipped with the topology generated by the subsets

$$\mathscr{N}_{K,O}(C) = \{D \in \mathrm{SUB}(G) \mid D \cap K \subset CO,\ C \cap K \subset DO\},$$

where $K \subset G$ is compact and $O \subset G$ is a neighbourhood of 1.

The set $\mathrm{SUB}(G)$ is a compact topological space and the action of $\mathsf{Aut}(G)$ on G induces a natural action on $\mathrm{SUB}(G)$ by homeomorphisms.

We use the following lemma in the proof of the next theorem.

Lemma 5. *Let O be a neighbourhood of 1. Let $\alpha \in \mathsf{Aut}(G)$, suppose that U is tidy for α and let $U_0 = U_+ \cap U_- = \bigcap_{j\in\mathbb{Z}} \alpha^j(U)$ be the largest α-invariant subgroup of U. Then there is a non-negative integer N greater than or equal to zero such that $\alpha^n(U_-) \subset U_0 O$ for all natural numbers $n \geq N$.*

Proof. The sequence $\{\alpha^n(U_-)\}_{n\geq 0}$ of compact subgroups of U_- decreases U_0. Since U_0O is an open neighbourhood of U_0, there is N such that $\alpha^N(U_-) \subset U_0O$. □

Proposition 3. *Let $\alpha \in \mathsf{Aut}(G)$ and suppose that $U \in \mathrm{COS}(G) \subset \mathrm{SUB}(G)$ is tidy for α. Then $\alpha^n(U) \to U_{++}$ with respect to the Chabauty topology in $\mathrm{SUB}(G)$.*

Proof. Let $K \subset G$ be compact and let $O \subset G$ be an open neighbourhood of 1. Consider $\mathcal{N}_{K,O}(U_{++})$. We will find a natural number M such that $\alpha^n(U) \in \mathcal{N}_{K,O}(U_{++})$ whenever $n \geq M$.

Now, $K \cap U_{++}$ is a compact subgroup of U_{++}. We have $U_{++} = \bigcup_{k\geq 0} \alpha^k(U_+)$ and U_+ is relatively open in U_{++}. Hence there is a k such that $\alpha^k(U_+) \geq K \cap U_{++}$. Hence,

$$K \cap U_{++} \subset \alpha^k(U_+) \subset \alpha^n(U) \text{ for all } n \geq k. \tag{5}$$

Choose N as in Lemma 5, so that $\alpha^n(U_-) \subset U_0O$ for every $n \geq N$. Then we have

$$\begin{aligned} n \geq N \implies \alpha^n(U) &= \alpha^n(U_+)\alpha^n(U_-) \\ &\subset \alpha^n(U_+)O \subset U_{++}O. \end{aligned} \tag{6}$$

Equations (5) and (6) imply that, if $n \geq \max(k, M)$, then

$$K \cap U_{++} \subset \alpha^n(U)O \text{ and } K \cap \alpha^n(U) \subset U_{++}O,$$

i.e. $\alpha^n(U) \in \mathcal{N}_{K,O}(U_{++})$ and so $\alpha^n(U) \to U_{++}$ as $n \to \infty$. □

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