Introduction to quantum invariants of knots

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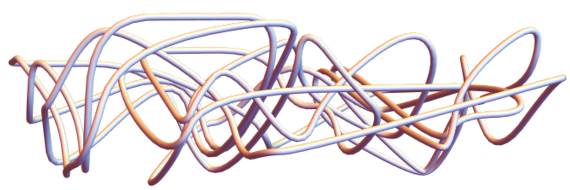
Abstract By introducing a generalized notion of tangles we show how the algebra behind quantum knot invariants comes out naturally. Concrete examples involving finite groups and Jones polynomials are treated, as well as some of the most challenging conjectures in the area. Finally the reader is invited to design his own invariants using the Drinfeld double construction.



Introduction

The purpose of these three lectures is to explain some of the topological motivation behind quantum invariants such as the colored Jones polynomial. In the first lecture we introduce a generalized notion of knots whose topology captures the ribbon Hopf algebra structure that is central to quantum invariants. The second lecture actually defines such algebras and shows how to obtain quantum invariants from them. As an example we discuss the colored Jones polynomial and present a couple of intriguing conjectures related to it. The final lecture is about constructing new examples of such algebras and invariants using the Drinfeld double.

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Although the prerequisites for these lectures are low the reader will probably appreciate the lectures most after having studied some elementary knot theory. Knot diagrams and Reidemeister moves up to the skein-relation definition of the Jones polynomial should be sufficient. Beyond that a basic understanding of the tensor product is useful. Finally if you ever wondered why and how things like the quantum group $U_q\mathfrak{sl}_2$ arise in knot theory then these lectures may be helpful.

To illustrate the nature of the algebras at hand recall that the colored Jones polynomial is closely related to $U_q\mathfrak{sl}_2$, the quantized enveloping algebra of $\mathfrak{sl}_2$. Following [8] $U_q\mathfrak{sl}_2$ is the algebra generated by $1,E,F,H$ defined by the following operations and relations. Our purpose is to demystify and motivate such constructions from a topological/knot theoretical viewpoint.

$$HE-EH=2E \quad \Delta(E)=E\otimes q^{\frac{H}{2}}+1\otimes E \quad S(E)=-Eq^{-\frac{H}{2}} \quad \varepsilon(E)=0$$

$$HF-FH=-2F \quad \Delta(F)=F\otimes 1+q^{-\frac{H}{2}}\otimes F \quad S(F)=-q^{\frac{H}{2}}F \quad \varepsilon(F)=0$$

$$EF-FE=[H] \quad \Delta(H)=H\otimes 1+1\otimes H \quad S(H)=-H \quad \varepsilon(H)=0 \quad [x]=\frac{q^{\frac{x}{2}}-q^{-\frac{x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

$$R=q^{\frac{H\otimes H}{4}}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/4}}{[n]!}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^n(E^n\otimes F^n) \quad \alpha=q^{\frac{H}{2}}$$

The N-colored Jones polynomial arises out of this algebra using the N-dimensional irreducible representation $\rho_N : U_q\mathfrak{sl}_2\to Mat(N\times N)$ defined by the matrices. For a more detailed description see lecture 2.

$$\rho_N(E)=\begin{pmatrix} 0 & [N-1] & 0 & 0 & 0 & \dots \\ 0 & 0 & [N-2] & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & [2] & 0 \\ 0 & \dots & 0 & 0 & 0 & [1] \\ \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_N(F)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ [1] & 0 & 0 & 0 & \dots & 0 \\ 0 & [2] & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & [N-2] & 0 & 0 \\ \dots & 0 & 0 & 0 & [N-1] & 0 \end{pmatrix}$$

$$\rho_N(H) = \begin{pmatrix} (n-1)/2 & 0 & 0 & 0 & 0 & \dots \\ 0 & (n-3)/2 & 0 & 0 & \dots & 0 \\ 0 & 0 & (n-5)/2 & \dots & 0 & 0 \\ 0 & 0 & \dots & -(n-5)/2 & 0 & 0 \\ 0 & \dots & 0 & 0 & -(n-3)/2 & 0 \\ \dots & 0 & 0 & 0 & 0 & -(n-1)/2 \end{pmatrix}$$

We will explain how the topology of knots or rather tangles leads one to consider such algebraic structures, known as ribbon Hopf algebras. Next we will show how one can design ribbon Hopf algebras at will using Drinfeld's double construction. Again emphasizing that the double construction too is directly forced on us by the topological problems we want to solve: i.e. finding invariants of knots.

Our exposition is meant to complement the literature rather than being exhaustive. We chose not to say anything about quantum field theory, which of course is one of the main driving forces of the subject. Our main sources are Ohtsuki [8], Etingof and Schiffmann [1] and Kauffman [6]. Much of this work is inspired by conversations with Dror Bar-Natan.

Lecture 1: Tangles as a ribbon Hopf algebra

The goal of this lecture is to present tangles in a rather non-standard way that will allow us to define many algebraic operations on them. These operations turn out to be the operations that one can do on the tensor algebra of a ribbon Hopf algebra such as $U_q\mathfrak{sl}_2$. To find out what that algebra should do we turn to topology to watch and listen what the tangles have to tell us.

rv-Tangles

Loosley following Kauffman [6] and Bar-Natan we work with a version of rotational virtual tangles, abbreviation rv-tangles. To make sure the algebra comes out unmangled our set up is rather abstract. Instead of relying on diagrams in the plane[1] we prefer working with ribbon graphs with some decorations modulo the usual Reidemeister relations viewed locally. Recall that a ribbon graph is a graph together with a cyclic orientation on the half-edges around each vertex.

Definition 1. An *rv*-ribbon graph is a ribbon graph G with a labelling of both vertices and edges by integers satisfying the following requirements. The degree one vertices (called ends) are required to come in pairs one labelled $+i$ and one $-i<0$, the set of absolute values of end labels is denoted I_G. Each internal vertex is labelled ± 1.

[1] This is the main difference with Kauffman's approach.

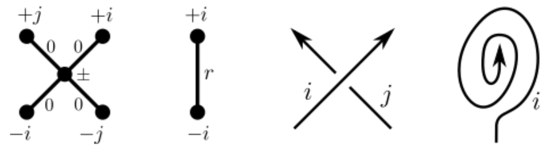

Fig. 1 (left) The two fundamental *rv*-ribbon graphs from which all *rv*-tangles are built by disjoint union and multiplication. The first is the $\pm$ crossing $X^{\pm}_{ij}$ and the second is an edge α^r_i with rotation number r. (right) the usual way of depicting the fundamental graphs in the plane, here the rotation number r is 2.

Two key examples of *rv*-ribbon graphs are the $\pm$ crossing and the edge shown in the figure 1 (left). The two figures on the right show the interpretation in terms of planar diagrams of knots we have in mind. We think of the edge labels as rotation numbers, often arising from taking a braid closure (rotation number ± 1). Edge label 0 will often be omitted for clarity. We define two operations on *rv*-ribbon graphs, disjoint union and multiplication. Together they suffice to build any graph we need. Disjoint union is simply disjoint union of graphs, where we assume the labels of the ends are all distinct. Multiplication is more interesting.

Definition 2. For $i,j \in I_G$ and $k \notin I_G$ define $m^{ij}_k(G)$ to be the *rv*-ribbon graph obtained from *rv*-ribbon graph G by merging the edge that ends in $+i$ with the edge that ends on $-j$. The edge label for the new edge is the sum of the labels of the merged edges and the remaining ends $-i,+j$ are renamed $-k,+k$ respectively.

With these definitions in place we can turn to the tangles we are interested in. See figure 3 below to see the multiplication in action to build a diagram of a knot.

Definition 3. An *rv*-tangle is an *rv*-ribbon graph obtained from multiplying finitely many crossings and edges as in figure 1. *rv*-tangles are considered up to the equivalences $R0, R1, R2, R3$ as shown below. $R0$ is relabeling of the ends (multiplication by an edge labelled 0).

Larger tangles are understood to be equivalent if they contain equivalent factors. *rv*-tangles are meant as a language for dealing with diagrams of ordinary knots and tangles more efficiently. To interpret *rv*-tangles we should view the four-valent vertices as crossings, with sign as indicated. The edges are a disjoint union of straight paths in the graph that go from end $-i$ to $+i$ and are directed this way. These straight paths are the components of the tangle, by straight we mean that it takes the second right (straight on a roundabout) at every crossing. The integers on the edges are supposed to represent the rotation number of a tangent vector as it runs from one vertex to the next. Not all *rv*-tangles correspond to usual tangle diagrams as the crossings may be connected in ways that are impossible in the plane. However all usual tangle diagrams without closed components are included. In particular knots can be studied as one component tangles. Also, two usual tangles are equivalent (regular isotopic) if and only if the corresponding *rv*-tangles are.

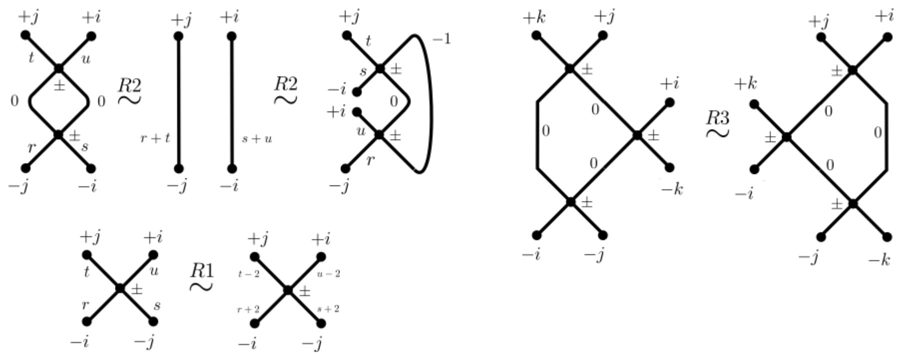

Fig. 2 The equivalence relations (Reidemeister moves).

Theorem 1. *(Kauffman, Bar-Natan)*
Two usual oriented tangles without closed components are regular isotopic if and only if the corresponding rv-tangles are equivalent.

We emphasize that in our set up the crossings in an *rv*-tangle are connected *abstractly*, not necessarily in the plane. This is like Kauffman's rotational virtual tangles [6] but without the need to explicitly discuss 'virtual crossings'. Even if the reader is only interested in usual tangles in the plane, the language of *rv*-tangles is still an elegant and effective way to encode such. As an additional bonus we will see that it brings out the algebra very naturally.

As a first example we write an algebraic description for the trefoil knot viewed as a 1-component tangle (long knot) T by multiplying together three crossings X and one edge containing a negative rotation α^{-1} to take into account the partial closure of the braid. See Figure 3.

$$T = m_1^{17} \circ m_1^{16} \circ m_1^{15} \circ m_1^{14} \circ m_1^{13} \circ m_1^{12}(X^+_{15} \sqcup X^+_{62} \sqcup X^+_{37} \sqcup \alpha_4^{-1})$$

Operations on tangles

The goal of this section is to show that the set of linear combinations of rv-tangles has all the algebraic operations and relations that are valid in the tensor algebra of a ribbon Hopf algebra. Instead of defining ribbon Hopf algebras we will dive right in and list two operations and some natural relations between them. The names of the relations reflect the algebraic structure intended.

The easiest one is tensor product of two *rv*-tangles, it is just another name for disjoint union considered above. We also have already seen multiplication $m_k^{i,j}$. Both these operations satisfy a form of associativity, let's write out what that means for m. Given three components labelled i, j, k it does not matter whether we first connect i

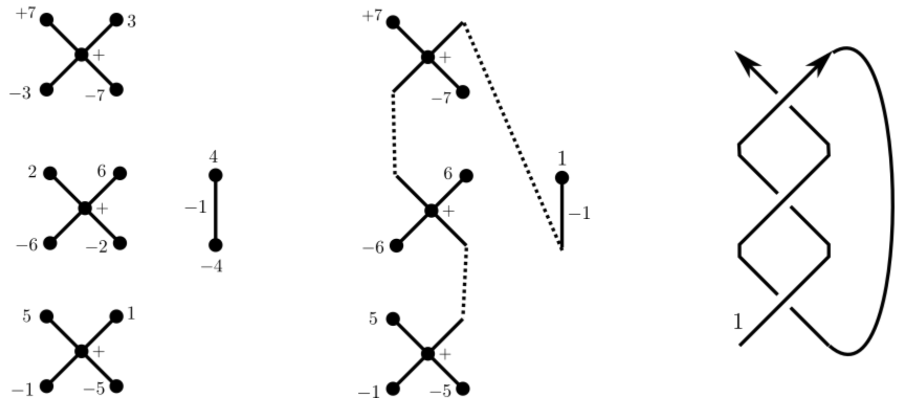

Fig. 3 (Right) The trefoil knot as a long knot (one component usual tangle). (Left) The seven fundamental *rv* tangles that can be assembled to produce the *rv*-tangle corresponding to the usual trefoil on the right. (Middle) An intermediate stage where we already multiplied ends $+1$ with -2 calling everything 1, then multiplied with component 3 and then with 4. The newly made connections are dotted and are abstract (not in the plane!).

to j and the result to k or first connect the j to k and then connect i to the result. In formulas: $m_x^{r,k} \circ m_r^{i,j} = m_x^{i,r} \circ m_r^{j,k}$. Here we called the intermediate result r and the end result x.

Next there is also the unit operation η_i which is disjoint union with a new edge labelled 0 with ends labelled $\pm i$ assuming the label i had not been used before. Dually there is a co-unit operation ε_i that deletes the component i. More interestingly there is the co-multiplication $\Delta^i_{j,k}$ that takes component i and doubles it. By this we mean it replaces component i with two new components j,k running parallel to i (with the same rotation numbers on parallel edges). Finally there is the antipode operation S_i which roughly speaking reverses the orientation of all the arrows on component i.

To give precise definitions of the operations mentioned we show explicitly what they do to the generators and extend them multiplicatively, see figure 4.

By multiplicativity we mean the following. For the co-unit it means $\varepsilon_i(m_i^{ab}) = \varepsilon_b \sqcup \varepsilon_a$. For the co-product it means that first multiplying two components i,j calling the result k and then doubling that component calling the results x,y is the same as first doubling i and j calling the results i', i'' and j', j'' and then multiplying i', j' and i'', j'' calling the results x and y. In formulas $\Delta^k_{x,y} \circ m_k^{i,j} = m_y^{i'',j''} \circ m_x^{i',j'} \circ \Delta^j_{j',j''} \circ \Delta^i_{i',i''}$. The algebra looks complicated but the pictures are really simple!

Multiplicativity for the antipode S_i is actually anti-multiplicativity, because if we reverse a component built out of many segments, the order of the segments gets reversed! $S_k \circ m_k^{i,j} = m_k^{j,i} \circ S_j \circ S_i$.

The operations listed above are precisely those of a Hopf algebra and the following relations hold between them. First there is co-associativity: it does not matter if you split component x calling the results i,r and then split r into components j,k or

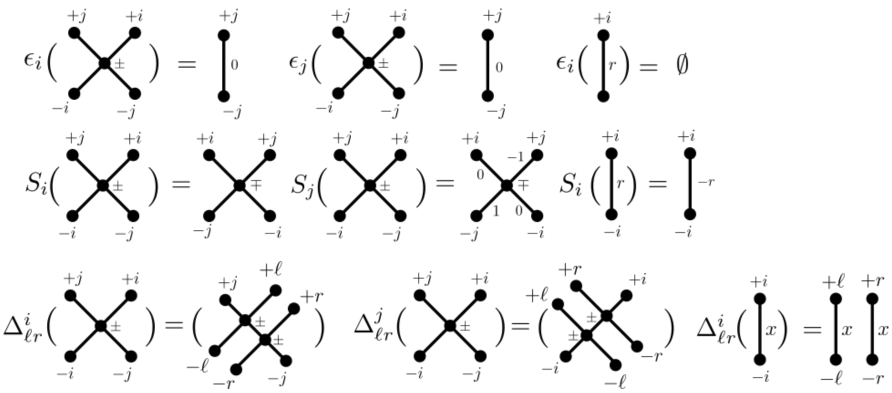

Fig. 4 The operations on the generators.

do it the other way around: first split x into r,k and then split r into i,j. In formulas $\Delta^r_{j,k}\circ\Delta^x_{i,r}=\Delta^r_{i,j}\circ\Delta^x_{r,k}$. Be careful that we really keep track of the order of the two components coming out of Δ.

Even more striking is the the following relation between all the operations we have: Take a component i double it to get components j,k, reverse k and multiply j with k, what do you get? The band spanned by j,k may be retracted and all that is left is a little component, called x, without any internal vertices! This is the same as deleting component i and putting back a single edge called x with label 0. In formulas $m^{j,k}_x\circ S_k\circ\Delta^i_{j,k}=\eta_x\varepsilon_i$.

We should also check that the operations described actually work on equivalence classes of *rv*-tangles. Doing the operation on two *rv*-tangles should always yield the same result (Exercise!).

The following relation is closely related to the $R3$ move. For any tangle T with component i:

$$m^{yj}_w\circ m^{xk}_z X^\pm_{xy}\sqcup\Delta^i_{jk}(T)=m^{jy}_w\circ m^{kx}_z X^\pm_{xy}\sqcup\Delta^i_{kj}(T)$$

It looks complicated but a picture makes it obvious (Exercise!).

The famous Drinfeld element is $U_k=m^{ji}_k\circ S_j(X^+_{ij})$, see figure 5 for a picture. For any tangle T_j it satisfies $m^{ji}_kU_j\sqcup T_i=m^{ij}_kU_j\sqcup S^2_i(T_i)$. The inverse of U_k is $U^{-1}_k=m^{ji}_k\circ S^{-1}_j(X^{-1}_{ij})$. Here inverse means that $m^{ij}_k(U^{-1}_i\sqcup U_j)=\alpha^0_k=m^{ji}_k(U^{-1}_i\sqcup U_j)$

Some more relations are $m^{jr}_xm^{ks}_y\Delta^i_{jk}(U_i)\sqcup m^{bc}_rm^{ad}_sX^+_{ab}\sqcup X^+_{cd}=U_x\sqcup U_y$ The same relation holds when we replace each U_h by $S_h(U_h)$. Also $\varepsilon_j(U_j)=\alpha^0_j$. The element $W_k=m^{ij}_kS(U_i)\sqcup U_j$ commutes with everything and satisfies $S_k(W_k)=W_k$. There exists a square root V_k of W_k, this is called the ribbon element satisfying the same equations as W_k does. How does the ribbon element relate to α_k?

At this point at least some of the symbols in $U_q\mathfrak{sl}_2$ should look more familiar. In the next lectures we will focus more on the algebras and how they lead to invariants

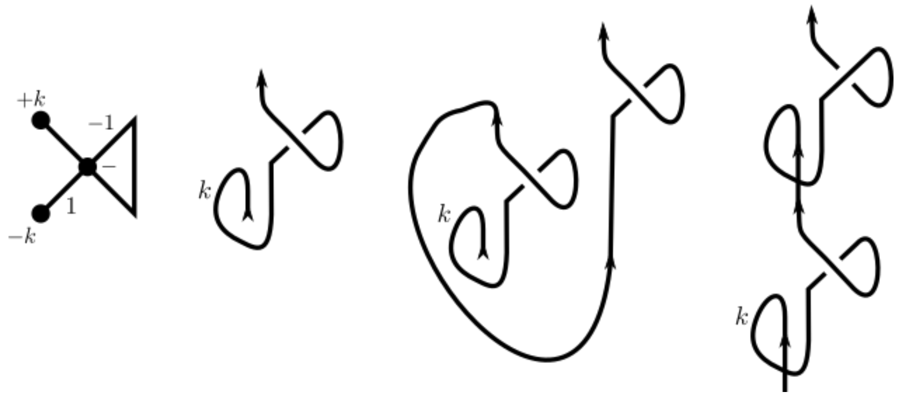

Fig. 5 The *rv*-graph corresponding to the Drinfeld element U_k (left). The second picture shows the same Drinfeld element as a usual tangle drawn in the plane. The third picture shows the square of the Drinfeld element $m_k^{ij} U_i \sqcup U_j$ again drawn as a usual tangle in the plane. Finally the last picture shows a more abstract version of this square where we allow ourselves a more schematic representation of the rotation numbers involved using abstract curls (not crossings!). One of the main points of *rv*-tangles is to not let the plane hold us back and let the algebra and topology mix freely.

of tangles and knots. Looking back we emphasize that although strange looking, our presentation of knot theory is cleaner and more precise than the standard one.

Exercises

Exercise 1: Draw an *rv*-tangle diagram for a figure eight knot 4_1 (viewed as a long knot).

Exercise 2: Apply S_i to the top-left diagram in Figure 2 and show you get the same as the third diagram on the same row of the figure.

Exercise 3: Draw diagrams to interpret the relations at the end of the lecture topologically.

Exercise 4: What happens to the R-matrix of $U_q\mathfrak{sl}_2$ when we set $q = 1, h = 0$? Look up what the universal enveloping algebra of a Lie algebra is. Do you recognise anything?

Exercise 5: (Skein relation). First show that the matrix for $\rho_2(R)$ with respect to the basis $x\otimes x, x\otimes y, y\otimes x, y\otimes y$ of $\mathbb{C}^2\otimes\mathbb{C}^2$ is

$$\rho_2(R) = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{4}} & q^{\frac{1}{4}} - q^{-\frac{3}{4}} & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

Let P be the matrix for the linear transformation that sends $a \otimes b$ to $b \otimes a$. The skein relation is:

$$q^{\frac{1}{4}} PR - q^{-\frac{1}{4}} (PR)^{-1} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) I$$

Lecture 2: Quantum invariants and Hopf algebras

In this lecture we define the algebraic counterpart of the *rv*-tangles and their operations. The rough idea is to have a copy of some algebra correspond to each component of our tangle. This leads us to define quantum invariants of tangles. In particular $U_q\mathfrak{sl}_2$ is such an algebra and we will show how to obtain the colored Jones polynomial from it. Finally some of the most beautiful and challenging conjectures involving the Jones polynomial are mentioned.

Recall that an algebra is a vector space A together with a bilinear, associative multiplication map $m : A \times A \to A$. Good examples of algebras to keep in mind are the group algebra of a finite group $\mathbb{C}G$ and the universal enveloping algebra of a Lie algebra $U(\mathfrak{g})$. Elements in both algebras are defined to be formal linear combinations of products.

Quantum knot invariants

Given an algebra A and a set I define a bigger algebra A_I to be the algebra generated by elements a_i for $a \in A, i \in I$ such that $a_i a'_{i'} = a'_{i'} a_i$ if $i \neq i'$ and satisfy the same relations as $a, a' \in A$ would if $i = i'$. Really A_I is just a tensor product $\bigotimes_{i\in I} A_i$ where all A_i are isomorphic to A. We prefer the subscripts because they are more flexible about the ordering of the tensor factors and we can write the tensor product as a formal product. One should think of the set I as the index set I_G of some *rv*-ribbon graph G. For $I = \emptyset$ we define $A_I = \mathbb{C}$.

To really make the connection to the topology of the last lecture we need to define a multiplication map on A_I. For $i, j, k \notin I$ define $m^{i,j}_k : A_{I\cup\{i,j\}} \to A_{I\cup\{k\}}$ as follows. $m^{i,j}_k(x)$ is the result of moving all factors a_i in x to the left and then replacing all subscripts i, j by k. Notice that by making both subscripts i, j equal to k we are effectively multiplying the elements with subscript i with those of subscript j.

Definition 4. Suppose A is an algebra. By a quantum knot invariant Z we mean a way of assigning to each *rv*-tangle T an element $Z(T) \in A_{I_T}$ where I_T is the set of end-labels of T, in such a way that

$$m_k^{ij} Z(T) = Z(m_k^{ij} T) \quad Z(T \sqcup T') = Z(T)Z(T') \quad T \sim T' \Rightarrow Z(T) = Z(T')$$

Since multiplication can be done either algebraically or topologically, finding a quantum invariant comes down to finding suitable values for the fundamental *rv*-tangles: crossing $Z(X_{ij}^{\pm})$ and edge with rotation $Z(\alpha_i)$. By suitable we mean all the equivalences $R0 - R3$ from figure 2 should be satisfied. Notice that each of these becomes an explicit equation in terms of the values of the fundamental tangles. For example on of the equations implied by $R2$ is

$$m_i^{ik} \circ m_j^{jl} (X_{ij}^{+} \sqcup X_{kl}^{-}) = \alpha_i^0 \sqcup \alpha_j^0$$

It should be noted that the usual quantum Reshetikhin-Turaev quantum invariants such as Jones, HOMFLY, Kauffman etc come from taking our notion of quantum knot invariant and composing it with a representation. However separating the representation from the invariant itself may clarify some issues. For future reference or perhaps as a definition (!) of ribbon Hopf algebra we state the following theorem [8]:

Theorem 2. *Any ribbon Hopf algebra A with R-matrix R and ribbon element α gives rise to a quantum knot invariant sending the crossing to R and the edge with a single rotation to α.*

One goal of these lectures is to introduce the Drinfeld double construction. This is a recipe for constructing a ribbon Hopf algebra and hence a knot invariant starting with a much simpler algebra, a Hopf algebra. In this way we can construct our own invariants instead of only focusing on the well known ones like the Jones polynomials.

For now let's focus on the particular algebra $A = U_q(\mathfrak{sl}_2)$, which happens to be a ribbon Hopf algebra. It yields the Jones polynomial as follows. Define Z by $Z(X_{12}^{+}) = R$ and $Z(\alpha) = \alpha$ referring to the formulas at the very beginning of the first lecture. Here we interpret $E^n \otimes F^n$ as $E_1^n F_2^n$ and do not worry about convergence issues.

Definition 5. If Z is the quantum invariant corresponding to the algebra $A = U_q(\mathfrak{sl}_2)$ as above then the N-colored Jones polynomial of knot K, notation $J_N(K;q)$ is defined as $\frac{1}{N}\mathrm{Tr}\rho_N(Z(K'))$. Here ρ_N is its N-dimensional representation given in the first lecture and K' is 1-component *rv*-tangle whose closure is K.

To get a feel for this construction let's get our hands dirty and make an attempt to compute the 2-colored Jones polynomial of the trefoil knot T using the tangle description from the last lecture:

$$T' = m_1^{15} \circ m_5^{57} \circ m_5^{56} \circ m_1^{14} \circ m_1^{13} \circ m_1^{12} (X_{15}^{+} \sqcup X_{62}^{+} \sqcup X_{37}^{+} \sqcup \alpha_4^{-1})$$

The formula should thus be $J_2(T;q) =$

$$\frac{1}{2}\mathrm{Tr}\rho_2 Z(T') = \mathrm{Tr}\rho_2\, m_1^{17} \circ m_1^{16} \circ m_1^{15} \circ m_1^{14} \circ m_1^{13} \circ m_1^{12} (Z(X_{15})Z(X_{26})Z(X_{37})Z(\alpha_4)^{-1})$$

Now the representation ρ_2 extends to tensor products as $\rho_2(a\otimes b)=\rho_2(a)\otimes\rho_2(b)$ or in other words $\rho_2(a_ib_j)=\rho_2(a)_i\rho_2(b)_j$. Recall that

$$\rho_2(E)=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad \rho_2(F)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad \rho_2(H)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Therefore $\rho_2(\alpha)=\rho_2(q^{H/2})=\begin{pmatrix}q^{\frac{1}{2}}&0\\0&q^{-\frac{1}{2}}\end{pmatrix}$ Also since $\rho_2(E)^2=\rho_2(F)^2=0$ it suffices to only keep the first two terms of the complicated series for $R_{ij}=Z(X^+_{ij})$. What remains is $R_{ij}=q^{\frac{H_iH_j}{4}}(1+(q^{\frac{1}{2}}-q^{-\frac{1}{2}})E_iF_j)$ Before applying the multiplications we find the invariant for the four disjoint tangles to be: $Z(X^+_{15})Z(X^+_{62})Z(X^+_{37})Z(\alpha_4^{-1})=$

$$q^{\frac{H_1H_5}{4}}(1+vE_1F_5)q^{\frac{H_6H_2}{4}}(1+vE_6F_2)q^{\frac{H_3H_7}{4}}(1+vE_3F_7)q^{-\frac{H_4}{2}}=D$$

Here we set $v=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$ and only include the terms that are non-zero when applying ρ_2. The variables mostly commute because they have different subscripts (are on different components), this will change once we start multiplying (joining components). Already now we should be careful that $H_1E_1=E_1H_1=2E_1$. All but the last multiplication are really easy. For example m_1^{12} means we should move all subscripts 1 to be left of the subscripts 2 and then change all 1 or 2 subscripts to 1. The same works for the next two m_1^{13} and m_1^{14}, we obtain:

$$W=m_5^{57}\circ m_5^{56}\circ m_1^{14}\circ m_1^{13}\circ m_1^{12}(D)=$$

$$q^{\frac{H_1H_5}{4}}(1+vE_1F_5)q^{\frac{H_5H_1}{4}}(1+vE_5F_1)q^{\frac{H_1H_5}{4}}(1+vE_1F_5)q^{-\frac{H_1}{2}}$$

To be able to carry out the last step m_1^{15} we have to move all subscripts 5 to the right of the subscripts 1. The powers of q can be moved using the relations $q^{cH}E=q^{2c}Eq^{cH}$ and $q^{cH}F=q^{-2c}Fq^{cH}$, (Exercise!). For example $E_5F_1q^{\frac{H_5H_1}{4}}=q^{\frac{H_5H_1}{4}-\frac{H_5}{2}+\frac{H_1}{2}+1}E_5F_1$.

For clarity let us write out $W=\sum_{j=1}^{8}W^j$ into eight terms W^j and compute m_1^{15} for each term individually. The first term is $W^1=q^{\frac{H_1H_5}{4}}q^{\frac{H_5H_1}{4}}q^{\frac{H_1H_5}{4}}q^{-\frac{H_1}{2}}$. Since only H is involved we may move the first term to the far right without cost and then set H_5 to H_1 to get $m_1^{15}W^1=q^{\frac{3H_1^2}{4}-\frac{H_1}{2}}$. Taking the trace in the ρ_2 representation our term gives

$$\mathrm{Tr}\rho_2(m_1^{15}W^1)=(1+q)q^{\frac{1}{4}}$$

Next we work with $W^2=q^{\frac{H_1H_5}{4}}vE_1F_5q^{\frac{H_5H_1}{4}}q^{\frac{H_1H_5}{4}}q^{-\frac{H_1}{2}}$. It suffices to bring the q-powers to the middle, the E_1,F_1 to the left and the E_5,F_5 to the right. We find

$$\begin{aligned}\mathrm{Tr}\rho_2 m_1^{15} W^2 &= \mathrm{Tr}\rho_2 v E_1 q^{\frac{H_1H_5}{4}+\frac{H_5}{2}} q^{\frac{H_5H_1}{2}+H_1} q^{-\frac{H_1}{2}} F_5|_{1=5} \\ &= \mathrm{Tr}\rho_2 v E_1 q^{\frac{H_1^2}{4}+H_1} F_1 = v q^{\frac{3}{4}}(q+q^{-1})\end{aligned}$$

Applying $\mathrm{Tr}\rho_2 m_1^{15}$ to the remaining six terms $W^3,\dots W^8$ and summing should yield the 2-colored Jones polynomial of the trefoil knot, i.e the ordinary Jones polynomial. There are easier ways of getting the same result but those tend to hide what is going on, making them of less use for serious applications such as the ones below. The technique illustrated here can be carried out even for Z itself without applying any representation.

Conjectures on the colored Jones polynomial

We pause our account of quantum invariants to illustrate the depth and lure of the subject by stating a few famous conjectures on the asymptotics of the colored Jones polynomial: The modularity conjecture, the AJ-conjecture and the slope conjecture. Each connects the Jones polynomial to an apparently completely different field. There may be more natural perspectives on these conjectures from quantum field theory but our purpose here is mainly to state some challenging problems in a concise way.

Recall our notation for the N-colored Jones polynomial of knot K is $J_N(K;q)$. It is always a Laurent polynomial in $q^{1/2}$.

The Modularity Conjecture is a radical generalization of the volume conjecture [5] connecting the Jones polynomial to hyperbolic geometry. Or perhaps more fundamentally to $SL(2,\mathbb{C})$ Chern-Simons theory. We take for granted the amazing facts that many knots allow a unique hyperbolic (finite volume complete) metric on their complement in the three-sphere. Such knots are called hyperbolic. By uniqueness (Mostow Rigidity) any property of the metric is a topological invariant of our knot K. Denote by Vol_K the volume of the complement with respect to that metric. Also denote the field generated by the traces of the holonomy representation of the knot group into $PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$ by F_K. It is known as the trace field of K and is always a number field.

Modularity Conjecture [9]
Consider a hyperbolic knot K. Define $J:\mathbb{Q}/\mathbb{Z}\to\mathbb{C}$ by $J(\frac{r}{s}) = J_s(K;e^{\frac{2\pi}{s}})$. For any $a,b,c,d\in\mathbb{Z}$ with $ad-bc=1$ and any sequence $(X_n)\in\mathbb{Q}$ going to infinity with bounded denominators, there exist $\Delta(\frac{a}{c}), A_j(\frac{a}{c})\in\mathbb{C}$ such that

$$\frac{J(\frac{aX_n+b}{cX_n+d})}{J(X_n)} \sim_{n\to\infty} \left(\frac{2\pi}{h_n}\right)^{\frac{3}{2}} e^{\frac{\mathrm{Vol}_K}{h_n}} \sum_{j=0}^{\infty} A_j\left(\frac{a}{c}\right) h_n^j$$

where $h_n = \frac{2\pi}{X_n + \frac{d}{c}}$ and $A_j(\frac{a}{c}), \Delta^{2c}(\frac{a}{c}) \in F_K(e^{\frac{2\pi a}{c}})$.

Another interesting conjecture is the AJ-conjecture. Define the function $J_\cdot(K;q)$: $\mathbb{N} \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by sending N to $J_N(K;q)$. It was shown that $J_\cdot(K;q)$ satisfies a q-difference equation (recursion) in the following sense. Define operators $\hat{M}$ and $\hat{L}$ on functions on $\mathbb{N}$ by $(\hat{L}f)(N) = f(N+1)$ and $(\hat{M}f)(N) = q^N f(N)$. Then there is a non-commutative polynomial $\hat{A}(\hat{M}, \hat{L}, q)$ in $\hat{M}, \hat{L}$ with coefficients in $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ such that $\hat{A}J_\cdot(K;q) = 0$ as a function. Up to some unimportant factors this polynomial $\hat{A}$ is unique so the following conjecture makes sense:

AJ Conjecture [2, 3]
Setting $q = 1$ in $\hat{A}$ yields the $SL(2,\mathbb{C})$ A-polynomial of the knot K.

Roughly speaking the A-polynomial is a plane curve in $\mathbb{C}^2$ specifying which values M, L for eigenvalues of the peripheral subgroup of the knot group can be extended to a representation of the knot group into $SL(2,\mathbb{C})$.

Finally the slope conjecture makes a connection with an apparently different field of low-dimensional topology: essential orientable surfaces in the knot complement. Viewed in the knot exterior an essential surface Σ may end on the boundary of the knot in a certain homology class $a\mu + b\lambda$. Note that the surface may intersect the torus boundary in several disjoint components, it does not have to be a Seifert surface. In that case we say that Σ has slope $\frac{a}{b}$. According to the slope conjecture some slopes of surfaces are detected by the degree of the colored Jones polynomial. More precisely it is known that for sufficiently large N there is a $p \in \mathbb{N}$ and quadratic polynomials $Q_0, \ldots, Q_{p-1}$ (all dependent on the knot) such that for all $0 \leq r \leq p-1$ we have $\deg J_N(K;q) = Q_r(N)$ whenever $N = r \mod p$.

Slope Conjecture [4]
For any knot K the leading coefficient of Q_r is the slope of an essential surface in the complement of K.

Exercises

Exercise 1: Let A be the algebra of complex valued 2x2 matrices, $\mathrm{End}(C^2)$. Write out all the equations one needs to solve for a 4x4 matrix $Z(X_{ij})$ and a 2x2 matrix $Z(\alpha_i)$ to obtain a quantum invariant in A.

Exercise 2: Compute the 2-colored Jones polynomial of the trefoil using the formulas at the beginning of lecture 1 and the topological description of the trefoil.

Exercise 3: The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is defined to be the vector space spanned by formal non-commutative products of Lie algebra elements modulo the relations $[X,Y] = XY - YX$ for any $X,Y \in \mathfrak{g}$. What operations make the universal enveloping algebra into a Hopf algebra?

Exercise 4: Prove the identities $q^{cH}E = q^{2c}Eq^{cH}$ and $q^{cH}F = q^{-2c}Fq^{cH}$.

Exercise 5: Find out how the volume conjecture is a special case of the modularity conjecture.

Lecture 3: Drinfeld double

The goal of the last lecture is to show how new quantum invariants may be constructed from Hopf algebras. Hopefully this will inspire the reader to look for interesting and useful invariants beyond the usual ones. We will illustrate the technique by working out the case of the group algebra of a finite group $\mathbb{C}G$ carefully. The construction is known as the Drinfeld double construction. It may seem foreign at first but is actually very natural in the sense that its algebraic structure is forced on us by topology. Not the other way around as it often appears.

Hopf algebras

An important example of a Hopf algebra is the group algebra of a finite group $\mathbb{C}G$ and another example is its dual, the functions on G, say $\mathrm{Fun}(G)$. One instance of the Drinfeld double is $D(G) = \mathbb{C}C \rtimes Fun(G)$. In the final lecture we will see that the resulting quantum invariant counts the number of representations of the fundamental group of the knot complement into G. If one hopes to understand invariants like colored Jones that relate to representations of the fundamental group into $G = SL(2,\mathbb{C})$ it is a good idea to first understand similar invariants for G finite.

A definition of a Hopf algebra is given below. Notice that in the context of tangles we also had multiplication m, a unit component η, doubling of a component Δ, reversal S of a component and deletion of a component ε. They satisfied certain natural relations and those are precisely the axioms for Hopf algebras. Notice however that no notion of crossing is present here.

Definition 6. A Hopf algebra is an algebra H together with for any set I algebra morphisms $\varepsilon_i : H_{I\sqcup\{i\}} \to H_I$ and $\Delta^i_{jk} : H_I \to H_{I\sqcup\{j,k\}}$, and an anti-algebra morphism $S : H_I \to H_I$. Satisfying the following axioms:

a. $\Delta^i_{ij} \circ \Delta^i_{ik} = \Delta^k_{jk} \circ \Delta^i_{ik}$
b. $\varepsilon_i \circ \Delta^i_{ij} = \varepsilon_j \circ \Delta^i_{ij} = \mathrm{id}_i$
c. $m^{ij}_i \circ S_i \circ \Delta^i_{ij} = m^{ij}_i \circ S_j \circ \Delta^i_{ij} = 1_i\varepsilon_i$

In our running example of the group algebra (which is the vector space with basis the group elements), the multiplication is multiplication in the group extended linearly to the whole space. The unit is the unit in the group and $\Delta^i_{jk}(g_i) = g_j g_k$ for all $g \in G$. Again this definition is extended linearly to the whole of $\mathbb{C}G$. The co-unit is defined for $g \in G$ by $\varepsilon_i(g_i) = 0$ if $g \neq 1$ and $g_i(1_i) = 1$. The antipode S_i is defined by $S_i(g_i) = g_i^{-1}$. When $i \neq z$ we set $\Delta^i_{jk}(g_z) = g_z$ and $\varepsilon_i(g_z) = g_z$ and $S_i(g_z) = g_z$. In this case all the axioms listed are easy to check (Exercise!).

The complex valued functions on G, with pointwise multiplication also form a Hopf algebra called $\mathrm{Fun}(G)$ (Exercise!). The Hopf algebra structure can be described conveniently in terms of the basis of delta functions. For each $g \in G$ define the delta function $\delta^g \in Fun(G)$ by $\delta^g(h) = 0$ if $g \neq h$ and 1 if $g = h$. The co-unit is defined by $\varepsilon_i(\delta^g_i) = \delta^g(1)$. The co-product is defined by $\Delta^i_{jk}(\delta^g_i) = \sum_{g=ab} \delta^a_j \delta^b_k$ and $S_i(\delta^g_i) = \delta_i^{g^{-1}}$. As before, when $i \neq z$ the functions Δ^i_{jk} and ε_i and S_i send δ^g_z to itself and are extended linearly.

Alternatively we can describe $\mathrm{Fun}(G)$ as the dual space $\mathbb{C}G^*$. The basis dual to the basis $\{g\}_{g\in G}$ of $\mathbb{C}G$ is the basis $\{\delta^g\}_{g\in G}$ of delta functions. The co-multiplication in $\mathrm{Fun}(G)$ is just the transpose of the multiplication in $\mathbb{C}G$ and the dual of the multiplication in $\mathrm{Fun}(G)$ is the comultiplication in $\mathbb{C}G$.

The group algebra itself is a little too simple to accommodate our *rv*-tangle language. In particular there is no natural candidate element in $\mathbb{C}G_{\{12\}}$ for the crossing X_{12} to map to. In the final lecture we will combine $\mathbb{C}G$ with its dual $\mathrm{Fun}(G)$ to make a bigger algebra where we can represent crossings and all the other properties of *rv*-tangles. This construction works for any Hopf algebra and is known as the Drinfeld double construction. The more involved quantum group $U_q\mathfrak{sl}_2$ also comes out of this construction in a natural way. Perhaps more importantly it allows you to design your own knot invariant!

Drinfeld double

Before introducing the Drinfeld double construction let us recall two crucial properties of *rv*-tangles. First and foremost there is a notion of crossing, the fundamental tangle $X^{\pm}_{ij}$ satisfying a couple of natural algebraic properties. First we know what happens when we double one of the components, this was included in our definition of Δ^i_{jk}. In formulas (draw the pictures!)

$$\begin{aligned} \Delta^i_{xy} X^+_{ij} &= m^{jz}_j X^+_{y,j} \sqcup X^+_{xz} \\ \Delta^j_{xy} X^+_{ij} &= m^{iz}_i X^+_{i,x} \sqcup X^+_{zy} \\ m^{yj}_w \circ m^{xk}_z X^{\pm}_{xy} \sqcup \Delta^i_{jk}(T) &= m^{jy}_w \circ m^{kx}_z X^{\pm}_{xy} \sqcup \Delta^i_{kj}(T) \end{aligned} \tag{1}$$

The last line is not included in the definition of Δ but is a rather simple compatibility between the crossing and the doubling of a component. A direct algebraic consequence of these three relations is the Reidemeister $R3$ relation or Yang-Baxter equation (Exercise!)

$$m_i^{i,x} m_j^{j,y} m_k^{k,z} X_{ij}^+ \sqcup X_{xk}^+ \sqcup X_{yz}^+ = m_k^{k,x} m_j^{j,z} m_i^{i,y} X_{jk}^+ \sqcup X_{ix}^+ \sqcup X_{yz}^+ \tag{2}$$

To find a knot invariant Z we start with an unknown algebra A and assume Z already intertwines $\sqcup$ and m_k^{ij}. Applying Z to the above equations then yields algebraic equations we would like to solve.

The main idea of the Drinfeld double construction is to start with equations 1 and 2 and a candidate solution and *find the algebra in which that candidate solution actually solves the equation*. In this way we really let the topology decide what the algebra should be and make sure the answer $R_{ij} = Z(X_{ij}^+)$ is nice to begin with.

Drinfeld's idea is to start with any Hopf algebra H and form $D(H) = H^* \otimes H$.

Definition 7. Define the Drinfeld Double of a Hopf algebra H to be the vector space $H^* \otimes H$ with the following properties: Writing elements $\phi \otimes h \in D(H)$ as ϕh we assume that the Hopf algebra rules from H or H^* are still valid when either $\psi = 1$ or $h = 1$. Define a coproduct and counit as follows:

$$\Delta^i_{jk}(\psi_i h_i) = \Delta^i_{jk}(\psi_i)\Delta^i_{jk}(h_i) \quad \varepsilon_i(\psi_i h_i) = \varepsilon_i(\psi_i)\varepsilon_i(h_i)$$

The book [1] is a useful reference for the following theorem.

Theorem 3. *Let $\{h^n\}$ be a basis for H and $\{\phi^n\}$ the dual basis of H^*. If $R_{ij} = \sum_n \phi_i^n h_j^n \in D(H)_{\{ij\}}$ satisfies equation (1) then the multiplication in $D(H)$* **must** *be defined as follows:*

$$\phi h \psi g = \sum_{n,m} (\psi_1^{1,n}(S_1 h_1^{1,m})\psi_3^{3,n}(h_3^{3,m}))\phi \psi^{2,n} h^{2,m} g$$

where $\Delta^2_{23}\Delta^1_{12}(x_1) = \sum_n x_1^{1,n} x_2^{2,n} x_3^{3,n}$ for any x. Also the antipode must be $S(\phi h) = S(h)S^{-1}(\phi)$

Let us see what the Drinfeld double $D(\mathbb{C}G)$ of the group algebra is. It is the vector space $\mathrm{Fun}(G) \otimes \mathbb{C}G$. Elements in this space will be written as sums of formal products ϕg where $\phi \in \mathrm{Fun}(G)$ and $g \in G$. To write down the product rule explicitly we first compute $\Delta^2_{23}\Delta^1_{12}(g_1) = g_1 g_2 g_3$ and $\Delta^2_{23}\Delta^1_{12}(\delta_1^a) = \sum_{rst=a} \delta_1^r \delta_2^s \delta_3^t$. Since $S(g) = g^{-1}$ the product rule becomes

$$\delta^a h \delta^b g = \sum_{rst=b} (\delta^r(h^{-1})\delta^t(h))\delta^a \delta^s hg = \delta^a \delta^{hbh^{-1}} hg$$

because we can solve $s = r^{-1}bt^{-1}$ and the delta functions tell us that $r = h^{-1}$ and $t = h$.

As an illustration of the theorem we derive the above product rule directly from imposing the Yang-Baxter relation. Write $R_{ij} = \sum_{g\in G} \delta_i^g g_j$ then equation (2) reads:

$$\sum_{f,g,h} \delta_1^f \delta_1^g f_2 \delta_2^h g_3 h_3 = \sum_{a,b,c} \delta_1^b \delta_1^c \delta_2^a c_2 a_3 b_3$$

Since $\delta^x \delta^y = \delta^x(y)\delta^x$ the equation simplifies to

$$\sum_{f,h} \delta_1^f f_2 \delta_2^h f_3 h_3 = \sum_{a,c} \delta_1^c \delta_2^a c_2 a_3 c_3$$

Comparing terms in the first and third component we must have $f = c$ and $fh = ac$ and hence $a = fhf^{-1}$. From the second component we find $f\delta^h = \delta^{fhf^{-1}} f$ exactly the product rule prescribed by the Drinfeld double.

To get a full quantum invariant we must also find the value of the rotation α. Referring to the first lecture we start by computing the Drinfeld element $U = \sum_g g^{-1}\delta^g = \sum_g \delta^g g^{-1}$. We also need to compute $S(U) = \sum_g \delta^{g^{-1}} g = U$. We have seen that $\alpha^2 = US(U)$ so we may take $\alpha = U$.

We now consider a representation $\rho : D(\mathbb{C}G) \to \mathrm{End}(\mathbb{C}G)$ defined by $\rho(\phi h)(a) = \phi(hah^{-1})hah^{-1}$ for any $a \in \mathbb{C}G$. The reader should check that $\rho(xy) = \rho(x)\rho(y)$ (Exercise!). In this representation the crossing is sent to the following map $\rho(R_{ij}) \in End(\mathbb{C}G_{\{ij\}})$.

$$\rho(R_{ij})(a_i b_j) = \sum_g \rho(\delta^g)_i \rho(g)_j (a_i b_j) = \sum_g \delta^g(a) a_i (gbg^{-1})_j = a_i g_j b_j g_j^{-1}$$

Presenting a knot as the closure of a braid $\beta = \prod \sigma$ we get a map $\rho(\beta) \in \mathrm{End}(\mathbb{C}G_{\{1,2,\dots,n\}})$. Using the standard basis $\{g^i\}$ any basis element $g_1^1 g_2^2 \dots g_n^n$ of $\mathbb{C}G_{\{1,2,\dots,n\}}$ gets sent by $\rho(\beta)$ to some other element in such a way that we can mark the arcs of the braid diagram with group elements such that the basis element is below and the image is on top and at each crossing the incoming under-arc gets conjugated by the upper arc to produce the outgoing under-arc. Taking the trace of $\rho(\beta)$ sums over the initial basis elements and forces the output at the top to be equal to the input. This is precisely the setup for the Wirtinger presentation of the knot group. Hence we see that $\mathrm{Tr}\rho(\beta) = \#\{\text{representations of } \pi_1(S^3 - K) \text{ into } G\}$.

$U_q\mathfrak{sl}_2$ *as a Drinfeld Double*

Coming back to our initial object of interest $U_q\mathfrak{sl}_2$ we would like to show how what we learned so far helps to demystify the formulas we started with. So far we studied finite groups, to make the connection to Lie groups and their algebras we should replace the group ring $\mathbb{C}G$ by the universal enveloping algebra $U(\mathfrak{g})$. This is the algebra of formal monomials of Lie algebra elements modulo the relations $[X,Y] = XY - YX$. Roughly the idea is that every element of a Lie group is generated by elements $\exp(X)$, where $X \in \mathfrak{g}$. Such exponentials naturally produce sums of Lie algebra elements that we may interpret in $U(\mathfrak{g})$.

Focusing on $\mathfrak{g} = \mathfrak{sl}_2$ with generators E, F, H its universal enveloping algebra $U(\mathfrak{sl}_2)$ is the algebra with 1 generated by E, F, H subject to the relations $HE - EH = 2E$ and $HF - FH = -2F$ and $EF - FE = H$ (compare to the formulas in lecture 1). The natural way to turn this into a Hopf algebra is to set the coproduct to be $\Delta^i jk(X_i) = X_j + X_k$ and $\varepsilon(X) = 0$ and $S(X) = -X$ for $X = E, F$ or H.

To understand how $U_q\mathfrak{sl}_2$ arises from this simple setup we restrict ourselves to the Lie subalgebra $\mathfrak{b}$ generated by E,H only. By the same formulas the universal enveloping algebra $U(\mathfrak{b})$ is still a Hopf algebra. We claim its Drinfeld double $D(U(\mathfrak{b}))$ is almost isomorphic to $U(\mathfrak{sl}_2)$. The only difference is that in $D(U(\mathfrak{b}))$ one gets generators H,E and H^* and E^*. It is natural to identify F with E^* but to get $U(\mathfrak{sl}_2)$ one has to quotient out by the additional relation $H^* = H$.

Already this Drinfeld double gives an interesting knot invariant, it produces the Alexander polynomial of a knot (Exercise!).

To get the quantized enveloping algebra $U_q\mathfrak{sl}_2$ we follow the same procedure, but we use a modified version of $U(\mathfrak{b})$ called $U_q(\mathfrak{b})$. Its relations are $HE-EH=2E$ as before but with a modified coproduct $\Delta(H) = H_1+H_2$ and $\Delta(E) = E_1q^{\frac{H_2}{2}}+E_2$, with $q=e^h$. This may seem arbitrary but there are not many possibilities if one wants a co-associative Δ that equals the usual one setting $q=1$.

Applying the double construction to this modified $U_q(\mathfrak{b})$ yields $U_q\mathfrak{sl}_2$ after setting $H^* = H$. The form of the R-matrix R should now be recognizable as consisting of dualbasis F^n and basis E^n. The coefficients are there to normalize properly [1].

The dual of $U_q\mathfrak{sl}_2$

In our finite group examples the dual of the group algebra $\mathbb{C}G$ was $\mathrm{Fun}(G)$. In the context of Lie groups and algebras it still makes sense to talk about the functions on the group but it is easier if one only allows nice functions. For $G = SL(2)$ we consider the polynomial functions on G. These are all generated by the matrix elements. In this way we see that $\mathrm{Fun}(SL(2)) = \mathbb{C}[a,b,c,d]/(ad-bc=1)$.

$\mathbb{C}G^*$ is supposed to be isomorphic to $\mathrm{Fun}(G)$ and in our present example there is at least a way to pair the equivalent $U(\mathfrak{sl}_2)$ in a non-degenerate way with $\mathrm{Fun}(SL(2))$. Given $X \in \mathfrak{sl}_2$ and $f \in \mathrm{Fun}(G)$ we consider $f\rho_2(X)$, i.e. evaluation of f on the representation of X as a 2 by 2 matrix.

Instead of deforming the Lie algebra we consider deforming the dual matrix group. This gives a different perspective on $U_q\mathfrak{sl}_2$ or rather it dual. To find a natural way of deforming $\mathrm{Fun}(SL(2))$ we call on one of the essential properties of $SL(2)$ namely its action on the plane. This sends a vector with coordinates (x,y) to a new vector in the plane with coordinates $(ax+by, cx+dy)$. Manin [7] proposed to first deform the functions on the plane, that is $\mathbb{C}[x,y]$. His idea was that any such deformation will naturally lead to a deformation of the symmetries of the plane $SL(2)$. All we need to do is insist that the deformed $SL(2)$ still acts as symmetries of the deformed plane.

Calling on quantum mechanics a natural way to view a pair of coordinates is as position z and momentum p of a single particle on a line. These coordinates naturally deform to a pair of non-commuting variables Z,P satisfying the Heisenberg commutation relation $ZP-PZ=-ih$. Setting $x=e^Z$ and $y=e^P$ we find (Exercise!) that $yx = qxy$ with $q=e^{-ih}$ and absorbing the factor $-i$ into h we find our desired quantum plane, or rather the functions on it form the following algebra:

$\mathrm{Fun}_q(\mathbb{C}^2) = \mathbb{C}\langle x, y\rangle/(yx = qxy)$. Here the angled brackets mean non-commuting polynomials in x, y.

With this preparation we can carry out Manin's proposal and ask what commutation relations a, b, c, d need to satisfy so that the quantum plane is preserved. Since (x, y) satisfy $yx = qxy$ we require $(ax + by, cx + dy)$ to satisfy the same relation: $(cx + dy)(ax + by) = q(ax + by)(cx + dy)$. Assuming a, b, c, d commute with x, y and q is a scalar we can compare coefficients of x^2 on the left and right hand side to find $ca = qac$.

To get more commutation relations we also require the transposed action (on row vectors) to be preserved so (x, y) gets sent to $(ax + cy, cx + dy)$. This yields the additional relation $(bx + dy)(ax + cy) = q(ax + cy)(bx + dy)$. The reader should check that these requirements yield the following six relations: $ba = qab$, $ca = qac$, $dc = qcd$, $db = qbd$ and $cb + qda = qad + q^2bc$ and $bc + qda = qad + q^2cb$. Assuming $q^2 \neq 1$ the last two are equivalent to $bc = cb$ and $ad - q^{-1}bc = da - qbc$. We recognize this last equation as statement about the deformed determinant. It expresses the fact that $ad - q^{-1}bc$ is a central element so we may quotient out by the relation $ad - q^{-1}bc = 1$ to obtain $\mathrm{Fun}_q(SL(2)) =$

$$\langle a, b, c, d\rangle/(ba = qab, dc = qcd, db = qbd, dc = qcd, bc = cb, ad - q^{-1}bc = 1)$$

Viewing the a, b, c, d as deformed matrix elements we may still use the same duality pairing with ρ_2 of $U_q\mathfrak{sl}_2$ to see that the two deformations are compatible and actually dual. Our arguments with the quantum plane are just another way of describing and motivating the Drinfeld double construction. In the end it all comes down to the same non-commutative geometry, whose applications to low-dimensional topology and other fields are endless.

Exercises

Exercise 1: Prove that both the group algebra of a finite group and its dual are Hopf algebras.

Exercise 2: Compute the Alexander polynomial of the trefoil by looking at the quantum invariant coming from the Drinfeld double of $U(\mathfrak{b})$ with the usual Hopf algebra structure.

Exercise 3: Show that the q-determinant $ad - q^{-1}bc$ in the last section is indeed central. What is the right notion of q-trace here?

Exercise 4: Find an interesting Hopf algebra and use the Drinfeld double construction to produce your own quantum invariant!

Epilogue

The purpose of these notes was to introduce quantum invariants, show that they connect to interesting parts of mathematics and convince the reader to construct their own invariants. Hopefully our discussion of $U_q\mathfrak{sl}_2$ makes the generators and relations we started with look less arbitrary. Much more can be said for example in terms of deformation quantization of Poisson-Lie groups but it is also important to note that the field of quantum invariants is still young. The simple things always come last and we're not there yet.

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