Weak Multivariate Subordination of Lévy Processes

Boris Buchmann

Research School of Finance, Actuarial Studies & Statistics
Australian National University
boris.buchmann@anu.edu.au

Joint Work with

Kevin Lu (ANU) & Dilip B. Madan (Uni of Maryland)

2017

introduction

- traditional subordination
- variance gamma vs. variance alpha gamma
- problems of subordination
- weak subordination
- variance generalised gamma convolutions
- weak variance alpha gamma

Lévy process

Let $X = (X_1, \ldots, X_n) \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X}) \in \mathbb{R}^n$ be Lévy process.

Lévy-Khintchine formula: $\Phi_{X(t)}(\boldsymbol{\theta}) = \mathbb{E} e^{\mathrm{i}\langle \boldsymbol{\theta}, X(t)\rangle} = \exp(t\Psi(\boldsymbol{\theta}))$ with

$$\begin{aligned}\Psi(\boldsymbol{\theta}) &= \Psi_X(\boldsymbol{\theta}) = \mathrm{i}\,\langle \boldsymbol{\mu}, \boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\|_\Sigma^2 + \int_{\|\mathbf{x}\|>1} e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1 \; \mathcal{X}(\mathrm{d}\mathbf{x}) \\ &\quad + \int_{0<\|\mathbf{x}\|\le 1} e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1 - \mathrm{i}\,\langle \boldsymbol{\theta}, \mathbf{x}\rangle \;\; \mathcal{X}(\mathrm{d}\mathbf{x})\,, \quad \boldsymbol{\theta} \in \mathbb{R}^n\end{aligned}$$

Lévy-Itô decomposition:

$$X = \boldsymbol{\mu} I + B + \int_0^{\cdot} \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mathbb{J}(\mathrm{d}s, \mathrm{d}\mathbf{x}) + \int_0^{\cdot} \int_{0<\|\mathbf{x}\|\le 1} \mathbf{x}\, \widetilde{\mathbb{J}}(\mathrm{d}s, \mathrm{d}\mathbf{x})$$

traditional subordination

assume T and X are *independent* where

- let $T = (T_1, \ldots, T_n)$ be an n-dimensional subordinator and
- $X = (X_1, \ldots, X_n)$ be another n-dimensional Lévy process, the subordinate,

define another n-dimensional process $X \circ T$ by

$$(X \circ T)(t) := (X_1(T_1(t)), \ldots, X_n(T_n(t))), \qquad t \geq 0$$

$X \circ T$ is again a Lévy process, provided

- $T_1 \equiv T_2 \equiv \cdots \equiv T_n$ are indistinguishable (well-known 'univariate subordination', Rogozin (1965), Zolotarev (Zo58), ..., Sato (1999),...)
- $X_1, \ldots, X_n$ are independent (multivariate subordination in Barndorff-Nielsen, Pedersen, Sato (2001))

Madan-Seneta variance gamma

see Madan & Seneta (1990), ...,Fung & Seneta (2010a,2010b), Seneta (2010).

- take $\boldsymbol{\mu} \in \mathbb{R}^n$, $b > 0$, $\Sigma \in \mathbb{R}^{n \times n}$
- $T \sim \Gamma_S(b, b)$ standard Gamma subordinator
- $B \sim BM^n(\boldsymbol{\mu}, \Sigma)$ Brownian motion with drift, independent of T
- call Y Variance Gamma process, provided

$$Y \stackrel{\mathcal{D}}{=} (B_1(T), \ldots, B_n(T)) = B \circ (T\mathbf{e}) \quad \sim \quad VG^n(b, \boldsymbol{\mu}, \Sigma)$$

- univariate subordination models common time change only

Variance-$\boldsymbol{\alpha}$-Gamma ($V\boldsymbol{\alpha}G$)

Semeraro (2008), Luciano & Semeraro (2010)

- let $a, b, \alpha_k > 0$, $b > a\alpha_k$ for $k = 1, \dots, n$ (wlog, $b = 1$), $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, \infty)^n$
- let $S_0, \dots, S_n$ be independent, independent of B, where

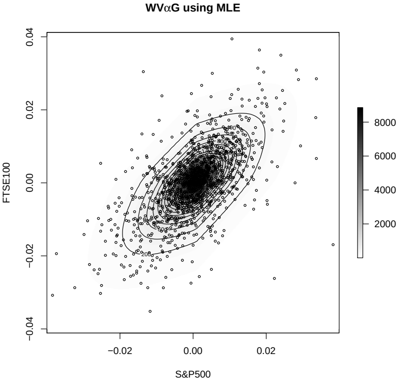
$$S_0 \sim \Gamma_S(a, b), \quad S_k \sim \Gamma_S\left(\frac{b - a\alpha_k}{\alpha_k}, \frac{b}{\alpha_k}\right),$$

- $T = T_{a,b,\boldsymbol{\alpha}}(T_1, \dots, T_n) \stackrel{\mathcal{D}}{=} S_0\boldsymbol{\alpha} + (S_1, \dots, S_n)$ is called an *$\boldsymbol{\alpha}$-Gamma* ($\boldsymbol{\alpha}G$) subordinator
- if $B \sim BM^n(\boldsymbol{\mu}, \Sigma)$ be a Brownian motion, independent of $S_0, \dots, S_n$ with $\Sigma = \mathrm{diag}(\Sigma_{11}, \dots, \Sigma_{nn})$
- $Y \stackrel{\mathcal{D}}{=} B \circ T$ is called $V\boldsymbol{\alpha}G$ process

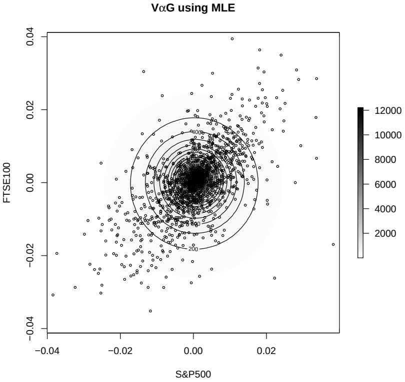
Fitting S&P500-FTSE100 data

fit (strong) $V\alpha G$ and weak $V\alpha G$ process to the S&P500 and FTSE100 index for a 5 year period from 14 February 2011 to 12 February 2016.

MLE Fit



WVαG using MLE
FTSE100
S&P500
0.04
0.02
0.00
−0.02
−0.04
−0.02
0.00
0.02
8000
6000
4000
2000



VαG using MLE
FTSE100
S&P500
0.04
0.02
0.00
−0.02
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−0.04
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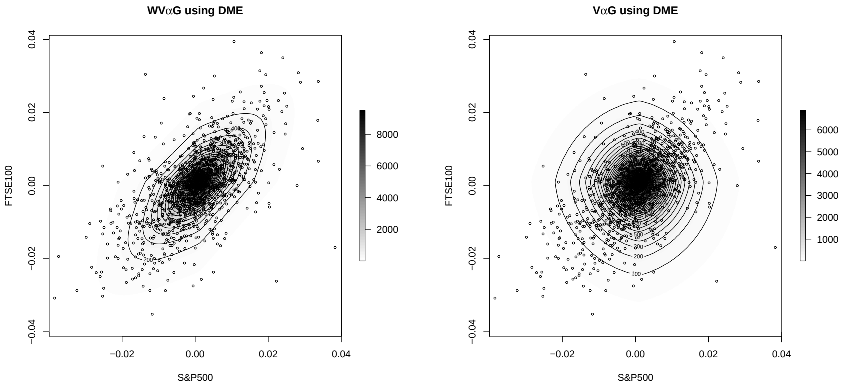
0.00
0.02
12000
10000
8000
6000
4000
2000

Digital Moment Estimation Fit



problems of traditional subordination

- let $B \sim BM^1(0,1)$ be univariate standard BM, and
  $t \mapsto (B(t), B(t))$ is Lévy process.
  $t \mapsto (t, 2t)$ is subordinator, but $\mathbb{E}[B(s)B(2t)] = s \wedge (2t) \neq \sigma^2(s \wedge t)$
  $\Rightarrow t \mapsto (B(t), B(2t))$ is *not* Lévy process

- N be a Poisson process with unit rate independent of B
  $t \mapsto (t, N(t))$ is subordinator, but
  $\mathbb{E}[B(t)B(N(t))] = \mathbb{E}[t \wedge N(t)] = t(1 - e^{-t})$, $0 \leq t \leq 1$,
  nonlinear $\Rightarrow t \mapsto (B(t), B(N(t))$ is not Lévy process

proposition assume $B = (B_1, B_2) \sim BM^2(\boldsymbol{\mu}, \Sigma = (\Sigma_{kl}))$,
$T = (T_1, T_2) \sim S^2(\mathbf{d}, \mathcal{T})$ are independent and

- $\boldsymbol{\mu} = \mathbf{0}$ or $\mathbb{E}[\|T(1)\|] < \infty$ and
- neither $T_1 \equiv 0$ nor $T_2 \equiv 0$ and
- $\Sigma_{12} \neq 0$

if, in addition, $B \circ T$ is Lévy process then $T_1 \equiv T_2$

multi-parameter time

proposition assume $\mathbf{t} = (t_1, \ldots, t_n) \in [0, \infty)^n$ and $X \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X}) = L^n(\Psi)$.
Then $X(\mathbf{t}) = (X_1(t_1), \ldots, X_n(t_n)) \in \mathbb{R}^n$ is infinitely divisible with

$$\Phi_{X(\mathbf{t})}(\boldsymbol{\theta}) = \mathbb{E} \exp(\mathrm{i} \langle \boldsymbol{\theta}, X(\mathbf{t}) \rangle) = \exp((\mathbf{t} \diamond \Psi)(\boldsymbol{\theta})) , \quad \boldsymbol{\theta} \in \mathbb{R}^n.$$

here

$$\begin{aligned} (\mathbf{t} \diamond \Psi)(\boldsymbol{\theta}) &:= \sum_{k=1}^{n} (t_{(k)} - t_{(k-1)}) \Psi(\boldsymbol{\pi}_{\{(k),\ldots,(n)\}}(\boldsymbol{\theta})) \\ &= \mathrm{i} \langle \mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c}(\mathbf{t}, \mathcal{X}), \boldsymbol{\theta} \rangle - \frac{1}{2} \|\boldsymbol{\theta}\|^2_{\mathbf{t} \diamond \Sigma} \\ &\quad + \int_{\mathbb{R}^n_*} \left(e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x} \rangle} - 1 - \mathrm{i} \langle \boldsymbol{\theta}, \mathbf{x} \rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}) \right) \mathbf{t} \diamond \mathcal{X}(\mathrm{d}\mathbf{x}) \end{aligned}$$

outer product

let $\mathbf{t} = (t_1, \dots, t_n) \in [0, \infty)^n$

- if $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$

$$\mathbf{t} \diamond \boldsymbol{\mu} = (t_1\mu_1, \dots, t_n\mu_n)$$

- if $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n \times n}$

$$\mathbf{t} \diamond \Sigma = ((t_k \wedge t_l)\Sigma_{k,l})_{k,l}$$

- if $\mathcal{X}$ Lévy measure then

$$\mathbf{t} \diamond \mathcal{X} := \sum_{k=1}^{n} (t_{(k)} - t_{(k-1)}) \mathcal{X}_{\{(k),\dots,(n)\}} \qquad (t_{(0)} := 0)\,,$$

where $(t_{(1)} \leq \dots \leq t_{(n)})$ is the order statistic of $\mathbf{t} = (t_1, \dots, t_n)$

definition

let $T \sim S^n(\mathbf{d}, \mathcal{T})$ and $X \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$.

- *T subordinates X in the weak sense, $Y \stackrel{\mathcal{D}}{=} (T, X \odot T)$*

:iff $Y = (Y_1, Y_2) \sim L^{2n}(\Psi_Y)$ is Lévy process with for $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$, $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, we have characteristic exponent

$$\Psi_Y(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \mathrm{i}\,\langle \boldsymbol{\theta}_1, \mathbf{d} \rangle + (\mathbf{d} \diamond \Psi_X)(\boldsymbol{\theta}_2) + \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t}, X(\mathbf{t}))}(\boldsymbol{\theta}) - 1)\mathcal{T}(\mathrm{d}\mathbf{t})$$

- *T subordinates X in semi-strong sense*

:iff $Y = (Y_1, Y_2)$ is as above and $Y_1 \equiv T$

characterisation

proposition let $T \sim S^n(\mathbf{d}, \mathcal{T})$, $X \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$
Then $Y \overset{\mathcal{D}}{=} (T, X \odot T)$ in the weak sense
iff $Y = (Y_1, Y_2) \sim L^{2n}(\boldsymbol{\mu} = (\mathbf{m}_1, \mathbf{m}_2), \Theta, \mathcal{Y})$ with

$$\Theta = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \diamond \Sigma \end{pmatrix},$$

$$\begin{aligned}
\mathcal{Y}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) &= (\delta_{\mathbf{0}} \otimes (\mathbf{d} \diamond \mathcal{X}))(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) + \mathbb{P}(X(\mathbf{t}) \in \mathrm{d}\mathbf{x})\, \mathcal{T}(\mathrm{d}\mathbf{t}), \\
\mathbf{m}_1 &= \mathbf{d} + \int_{[0,\infty)^n_*} \mathbf{t}\, \mathbb{P}\big((\mathbf{t}, X(\mathbf{t})) \in \mathbb{D}\big)\, \mathcal{T}(\mathrm{d}\mathbf{t}), \\
\mathbf{m}_2 &= \mathbf{c}(\mathbf{d}, \mathcal{X}) + \mathbf{d} \diamond \boldsymbol{\mu} + \\
&\quad + \int_{[0,\infty)^n_*} \mathbb{E}[X(\mathbf{t}) \mathbb{1}_{\mathbb{D}}(\mathbf{t}, X(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t})
\end{aligned}$$

examples

- in the semi-strong sense

$$((I, 2I), (B, B) \odot (I, 2I)) \stackrel{\mathcal{D}}{=} ((I, 2I), Y))$$

for $Y \sim B^2((0,0), [(1,1), (1,2)])$
indeed:

$$(1,2) \diamond [(1,1), (1,1)] = [(1,1), (1,2)]$$

- let B, B^*, N be independent, $B \stackrel{\mathcal{D}}{=} B^*$, N Pois pro with unit rate; in the semi-strong sense:

$$((I, N), (B, B) \odot (I, N)) \stackrel{\mathcal{D}}{=} ((I, N), (B^*, B \circ N))$$

existence theorem

if X and T are as above, then

• there exists $Y \stackrel{\mathcal{D}}{=} (T, X \odot T)$ in the weak sense

• on augmentation of underlying prob.' space, where T lives, we can construct a Lévy process Y_2 such that

$$(T, Y_2) \stackrel{\mathcal{D}}{=} (T, X \odot T) \quad \text{in the semi-strong sense}$$

marginal projection

let $T \sim S^n(\mathbf{d}, \mathcal{T})$, $X \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$

- if $Y = (Y_1, Y_2) \stackrel{\mathcal{D}}{=} (T, X \odot T)$ holds in the weak sense,

then $Y_1 \stackrel{\mathcal{D}}{=} T$ and $Y_2 \sim L^n(\mathbf{m}_2, \Theta_2, \mathcal{Y}_2)$ with

$$\begin{aligned} \mathbf{m}_2 &= \mathbf{c}(\mathbf{d}, \mathcal{X}) + \mathbf{d} \diamond \boldsymbol{\mu} + \int_{[0,\infty)^n_*} \mathbb{E}[X(\mathbf{t}) \mathbf{1}_{\mathbb{D}}(X(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ \Theta_2 &= \mathbf{d} \diamond \Sigma \\ \mathcal{Y}_2(\mathrm{d}\mathbf{x}) &= \mathbf{d} \diamond \mathcal{X}(\mathrm{d}\mathbf{x}) + \int_{[0,\infty)^n_*} \mathbb{P}(X(\mathbf{t}) \in \mathrm{d}\mathbf{x})\, \mathcal{T}(\mathrm{d}\mathbf{t}) \end{aligned}$$

properties

assume independent X and T

weak subordination extends traditional one: if $T_1 = \ldots = T_n$ or $X_1, \ldots, X_n$ are independent then $(T, X \circ T) \stackrel{\mathcal{D}}{=} (T, X \odot T)$

marginal consistency: if X and T are independent then $(T_k, (X \odot T)_k) \stackrel{\mathcal{D}}{=} (T_k, X_k \circ T_k)$, $1 \leq k \leq n$

weak linearity: if $T^{(1)}, \ldots, T^{(m)}$ are independent with $T^{(j)} \sim S^n(\mathbf{0}, \mathcal{T}^{(j)})$, then $T := \sum_{j=1}^m T^{(j)} \sim S^n(\mathbf{0}, \sum_{j=1}^m \mathcal{T}^{(j)})$ and $(T, X \odot T) \stackrel{\mathcal{D}}{=} \sum_{j=1}^m Z^{(j)}$, where $Z^{(1)}, \ldots, Z^{(m)}$ are independent with $Z^{(j)} \stackrel{\mathcal{D}}{=} (T^{(j)}, X \odot T^{(j)})$

properties

assume independent X and T

monotone case: if $Y = (T, X \odot T)$ in the weak sense and $T_1 \leq \ldots \leq T_n$ then $Y(t) \stackrel{\mathcal{D}}{=} (T(t), X(T(t)), \ t \geq 0$

ray subordination: if $T = R\boldsymbol{\tau}$ for $R \sim S^1$ and deterministic, $\boldsymbol{\tau} \in [0, \infty)^n$, then $(T, X \odot T) \stackrel{\mathcal{D}}{=} (I^{\otimes n} \diamond \boldsymbol{\tau}, \boldsymbol{\tau} \diamond X) \circ (R\mathbf{e})$ for $\mathbf{e} = (1, \ldots, 1)$ and $\boldsymbol{\tau} \diamond X \sim L^n(\boldsymbol{\tau} \diamond \Psi_X)$ independent of T

Thorin's generalised gamma convolutions subordinator

- $n = 1$: Thorin (1977),.., James, Roynette & Yor (2008), Schilling, Song & Vondracek (2010),...; $n \geq 1$: Barndorff-Nielsen, Maejima & Sato (2006), Bondesohn (2009), Pérez-Abreu & Stelzer (2014)
- call T a *Thorin subordinators*, $T \sim GGC_S^n(\mathbf{d}, \mathcal{U})$, whenever, simultaneously, $\mathbf{d} \in [0,\infty)^n$ and $\mathcal{T}$ is a Thorin measure concentrated in $[0,\infty)^n_*$ and T has Laplace exponent, for $t \geq 0, \boldsymbol{\lambda} \in [0,\infty)^n$

$$-\ln \mathbb{E} \exp\{-\langle \boldsymbol{\lambda}, T(t)\rangle\} = t\langle \mathbf{d}, \boldsymbol{\lambda}\rangle + t\int_{[0,\infty)^n_*} \ln \frac{\|\mathbf{t}\|^2 + \langle \boldsymbol{\lambda}, \mathbf{t}\rangle}{\|\mathbf{t}\|^2} \; \mathcal{U}(\mathrm{d}\mathbf{t})$$

- if $n = 1$, $d = 0$ and $\mathcal{U} = \alpha\delta_\beta$ for some $\alpha, \beta > 0$, T is Gamma
- any α-*Gamma* subordinator is Thorin
- BKMS (2016) summarises econometrical literature:

Semeraro (2008), Luciano & Semeraro (2010), Guillaume (2013), ...

variance generalised gamma convolution

- let $\mathcal{T}$ be n-dimensional Thorin measure, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\mathbf{d} \in [0,\infty)^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$, $B \sim BM^n(\boldsymbol{\mu},\Sigma)$, $T \sim GGC^n_S(\mathbf{d},\mathcal{T})$
  call Y *Variance Generalised Gamma Convolution*-process with parameters $\mathbf{d},\boldsymbol{\mu},\Sigma,\mathcal{T}$

$$Y \overset{\mathcal{D}}{=} B \odot T \quad \sim VGG^n(\mathbf{d},\boldsymbol{\mu},\Sigma,\mathcal{U}) := BM^n(\boldsymbol{\mu},\Sigma) \odot GGC^n_S(\mathbf{d},\mathcal{U})$$

$n = 1$. Grigelionis (2007) subordinates Brownian motion with *univariate* Thorin subordinator$=VGG^{n,1}$-class in BKMS
example: variance gamma

$n \geq 1$. BKMS subordinate independent-component Brownian motion with multivariate Thorin subordinator$=VGG^{n,n}$-class in BKMS
example: variance-α-gamma

- $VGG^{n,1} \cup VGG^{n,n} \subseteq VGG^n$

characteristics

let $Y \sim VGG^n(\mathbf{d} = \mathbf{0}(wlog), \boldsymbol{\mu}, \Sigma, \mathcal{U})$

- for $\boldsymbol{\theta} \in \mathbb{R}^n$, we have

$$\Psi_Y(\boldsymbol{\theta}) = -\int_{[0,\infty)_*^n} \ln \frac{\|\mathbf{t}\|^2 - \mathrm{i}\,\langle \mathbf{t} \diamond \boldsymbol{\mu}, \boldsymbol{\theta} \rangle + \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\mathbf{t}\diamond\Sigma}}{\|\mathbf{t}\|^2}\, \mathcal{U}(\mathrm{d}\mathbf{t})$$

- $Y \sim L^n(\mathbf{m}_2, \mathbf{d} \diamond \Sigma, \mathcal{Y})$, $\mathbf{m}_2 = \mathbf{d} \diamond \boldsymbol{\mu} + \int_{\mathbb{D}_*} \mathbf{y}\, \mathcal{Y}(\mathrm{d}\mathbf{y})$ and

$$\mathcal{Y} = \left(\frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2} \otimes \mathcal{V}_{\|\mathbf{u}\|^2, \mathbf{u}\diamond\boldsymbol{\mu}, \mathbf{u}\diamond\Sigma}(\mathrm{d}\mathbf{y}) \right) \circ \left((\mathbf{u}, \mathbf{y}) \mapsto \mathbf{y} \right)^{-1}$$

here $\mathcal{V}_{b,\boldsymbol{\mu},\Sigma}$ is the Lévy measure of $VG^n(b, \boldsymbol{\mu}, \Sigma)$

self-decomposability

- a Lévy process Y is called *self-decomposable* :iff for any $0 < b < 1$ for some $Z_b \in \mathbb{R}^n$ independent of Y

$$Y(1) \stackrel{\mathcal{D}}{=} Z_b + bY(1)$$

complement Halgreen (1978), Grigelionis (2007)

theorem let $Y \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$

- if $n = 1$ or $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$ then Y is self-decomposable
- if $n \geq 2$, Σ is invertible, $\boldsymbol{\mu} \neq \mathbf{0}$ and, in addition,

$$\int_{(0,\infty)^n} (1 + \|\mathbf{u}\|^{1/2}) \frac{\|\mathbf{u}\|^{n/2} \mathcal{U}(\mathrm{d}\mathbf{u})}{(\prod \mathbf{u})^{1/2}} < \infty$$

then Y is cannot self-decomposable

- directly applicable to VG^n and $V\alpha G^n$

weak $V\alpha G$ processes

if $B \sim BM^n(\boldsymbol{\mu}, \Sigma)$ with an arbitrary covariance matrix and T be an $\boldsymbol{\alpha}G$ subordinator,

$Y \stackrel{\mathcal{D}}{=} B \odot T$ is called a *weak $V\boldsymbol{\alpha}G$-process*.

weak $V\boldsymbol{\alpha}G$ processes shares properties with its strong counterpart

- each component has both common and/or idiosyncratic time changes.
- it has VG marginals $(B \odot T)_k \stackrel{\mathcal{D}}{=} B_k \circ T_k \sim VG^1(1/\alpha_k, \mu_k, \Sigma_{kk})$.
- jump measure has full support.

but

- the weak $V\boldsymbol{\alpha}G$ process has a wider range of dependence structure.

decomposition of the weak $V\alpha G$ process

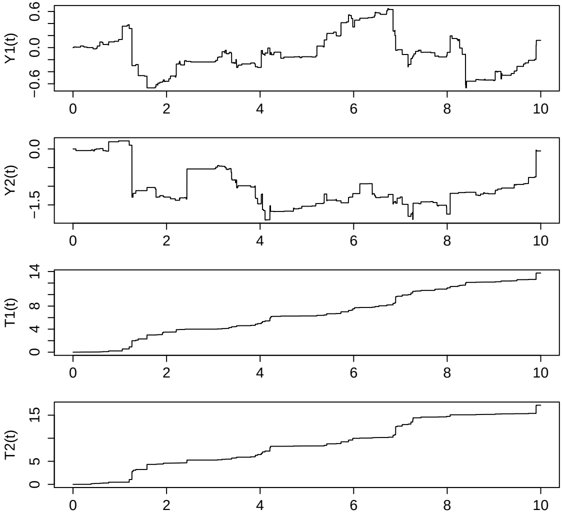
Let $B \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $B^{(k)}$ be independent copies of B, and $\boldsymbol{\alpha} \diamond B \sim BM^n(\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \boldsymbol{\alpha} \diamond \Sigma)$, then

$$
\begin{aligned}
B \odot T &\overset{\mathcal{D}}{=} B \odot (S_0 \boldsymbol{\alpha}) + \sum_{k=1}^{n} B^{(k)} \odot (S_k \mathbf{e}_k) \\
&\overset{\mathcal{D}}{=} (\boldsymbol{\alpha} \diamond B) \circ (S_0 \mathbf{e}) + \sum_{k=1}^{n} (B_k^{(k)} \circ S_k) \mathbf{e}_k \\
&\overset{\mathcal{D}}{=} V_0 + \sum_{k=1}^{n} V_k \mathbf{e}_k \, .
\end{aligned}
$$

Each term in the last line is independent with

$$
\begin{aligned}
V_0 &\sim VG^n\left(a, \frac{a}{b}\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \frac{a}{b}\boldsymbol{\alpha} \diamond \Sigma\right), \\
V_k &\sim VG^1\left(\frac{b - a\alpha_k}{\alpha_k}, \frac{b - a\alpha_k}{b}\mu_k, \frac{b - a\alpha_k}{b}\Sigma_{kk}\right).
\end{aligned}
$$

weak $V\alpha G$ processes



characteristic function and moments

using the decomposition, the weak $V\boldsymbol{\alpha}G$ process has characteristic exponent

$$\Psi_{B\odot T}(\boldsymbol{\theta}) = -a\ \ln\left(1 - \mathrm{i}\frac{\langle \boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \boldsymbol{\theta}\rangle}{b} + \frac{\|\boldsymbol{\theta}\|^2_{\boldsymbol{\alpha}\diamond\Sigma}}{2b}\right) - \sum_{k=1}^{n} \beta_k \ln\left(1 - \mathrm{i}\frac{\alpha_k \mu_k \theta_k}{b} + \frac{\alpha_k \theta_k^2 \Sigma_{kk}}{2b}\right).$$

marginal moments (=univariate VG^1 distribution)

$$\mathbb{E}((B \odot T)_k(1)) = \mu_k, \quad \mathrm{Var}((B \odot T)_k(1)) = (b\Sigma_{kk} + \mu_k^2 \alpha_k)/b\,,$$

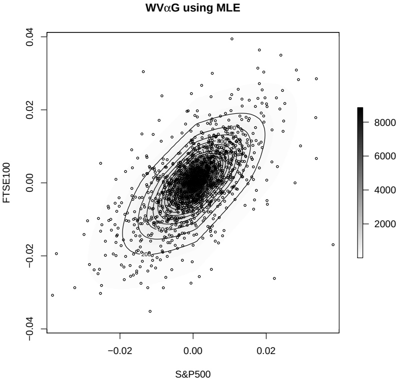
for $1 \leq k \neq l \leq n$, the covariance is

$$\mathrm{Cov}((B\odot T)_k(1), (B\odot T)_l(1)) = (ab(\alpha_k \wedge \alpha_l)\,\Sigma_{kl} + a\alpha_k\alpha_l\mu_k\mu_l)/b^2\,.$$

Fitting S&P500-FTSE100 data

revisited: fit (strong) $V\alpha G$ and weak $V\alpha G$ process to the S&P500 and FTSE100 index for a 5 year period from 14 February 2011 to 12 February 2016.

MLE Fit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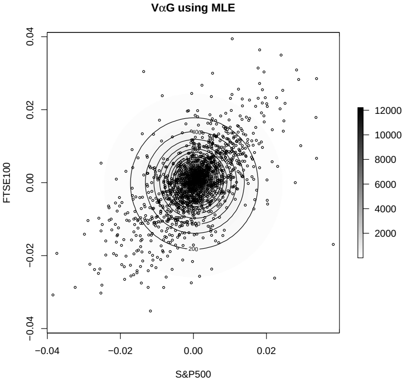
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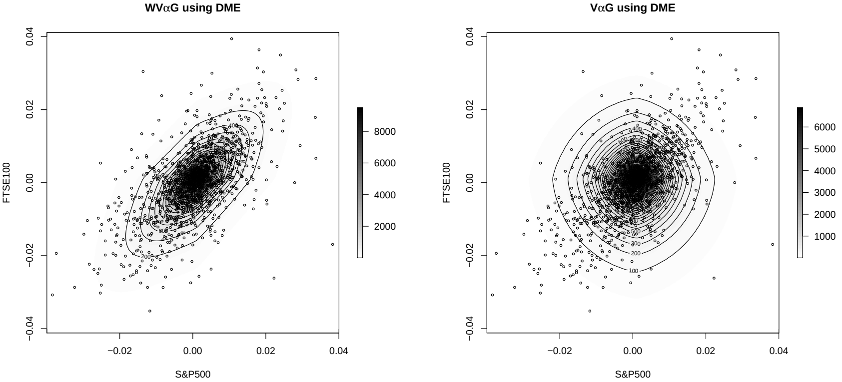
Digital Moment Estimation Fit



THANK YOU FOR YOUR ATTENTION!

talk based on: •B., Lu, K., Madan, D. (2017) Weak Subordination of Multivariate Lévy Processes. To appear in *Bernoulli*.

further literature in & discussion in:

- B.B., Kaehler, B.D., Maller, R.A. & Szimayer, A. (2016). Multivariate Subordination using Generalised Gamma Convolutions with Applications to Variance Gamma Processes & Option Pricing. *Stochastic Process. Appl.* **127**, 2208-2242

- Michaelsen, M. & Szimayer, A. (2017). Marginal consistent dependence modeling using weak subordination for Brownian motions. *Preprint.* Universität Hamburg.