# Arithmetic properties of hypergeometric mirror maps and Dwork's congruences

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**Abstract** Mirror maps are power series which occur in Mirror Symmetry as the inverse for composition of $q(z) = \exp(f(z)/g(z))$, called local $q$-coordinates, where $f$ and $g$ are particular solutions of the Picard–Fuchs differential equations associated with certain one-parameter families of Calabi–Yau varieties. In several cases, it has been observed that such power series have integral Taylor coefficients at the origin. In the case of hypergeometric equations, we discuss $p$-adic tools and techniques that enable one to prove a criterion for the integrality of the coefficients of mirror maps. This is a joint work with T. Rivoal and J. Roques. This note is an extended abstract of the talk given by the author in January 2017 at the conference "Hypergeometric motives and Calabi–Yau differential equations" in Creswick, Australia.

## 1 Arithmetic conditions for operators of Calabi–Yau type

An irreducible fourth order differential operator $\mathscr{L}$ in $\mathbb{Q}(z)[d/dz]$ is of Calabi–Yau type if it is of Fuchsian type, self-dual, has 0 as MUM-point and it satisfies certain arithmetic conditions including that

(i) $\mathscr{L}$ has a solution $\omega_1(z) \in 1 + z\mathbb{C}[[z]]$ at $z = 0$ which is $N$-integral[1];
(ii) $\mathscr{L}$ has a linearly independent solution $\omega_2(z) = G(z) + \log(z)\omega_1(z)$ at $z = 0$ with $G(z) \in z\mathbb{C}[[z]]$ and $\exp(\omega_2(z)/\omega_1(z))$ is $N$-integral.

An additional condition is usually considered: the instanton numbers $n_d$ associated with $\mathscr{L}$ belong to $\frac{1}{N}\mathbb{Z}$ for some non-zero integer $N$. As far as we know, a systematic approach to prove the integrality of the $n_d$'s has not yet been developed, even in the case of hypergeometric equations. In this note, we discuss useful $p$-adic tools

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[1] A power series $f(z) \in 1 + z\mathbb{Q}[[z]]$ is $N$-integral if there is $c \in \mathbb{Q}^*$ such that $f(cz) \in \mathbb{Z}[[z]]$.

to prove or disprove Conditions (i) and (ii). A classical example of a differential operator satisfying both (i) and (ii) is

$$\mathscr{L} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4),$$

where $\theta = z\frac{d}{dz}$. Consider the two solutions

$$\omega_1(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} z^n \quad \text{and} \quad \omega_2(z) = G(z) + \log(z)\omega_1(z),$$

with

$$G(z) = \sum_{n=1}^{\infty} \frac{(5n)!}{n!^5}(5H_{5n} - 5H_n)z^n \quad \text{and} \quad H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Then $\omega_1(z)$ has integers coefficients and Lian and Yau proved in [10] that

$$\exp\left(\frac{\omega_2(z)}{\omega_1(z)}\right) \in \mathbb{Z}[[z]].$$

We shall see that hypergeometric techniques presented in this note allow to prove the integrality of the coefficients of $q$-coordinates associated with non-hypergeometric operators. For example, consider the differential operator

$$\mathscr{L} = \theta^3 - z(34\theta^3 + 51\theta^2 + 27\theta + 5) + z^2(\theta+1)^3,$$

whose holomorphic solution is the generating series of the Apéry numbers used by Apéry in its proof of the irrationality of $\zeta(3)$ (see [1]):

$$\omega_1(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n.$$

A second solution is given by the method of Frobenius and reads $\omega_2(z) = G(z) + \log(z)\omega_1(z)$, with

$$G(z) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (2H_{n+k} - 2H_{n-k})z^n.$$

As we will see, a consequence of the results of the author [5] is that

$$\exp\left(\frac{\omega_2(z)}{\omega_1(z)}\right) \in \mathbb{Z}[[z]].$$

First, we present criteria on the integrality of hypergeometric terms.

# 2 Integrality of hypergeometric terms

## *2.1 Factorial ratios*

Let $\mathbf{e} = (e_1, \ldots, e_u)$ and $\mathbf{f} = (f_1, \ldots, f_v)$ be vectors of positive integers. For every non-negative integer $n$, we set

$$\mathscr{Q}(n) = \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!}$$

and we consider the generating series of $\mathscr{Q}$ :

$$F(z) = \sum_{n=0}^{\infty} \mathscr{Q}(n) z^n,$$

which is a rescaling of a hypergeometric function. We consider the function $\Delta$ of Landau defined for every $x$ in $\mathbb{R}$ by

$$\Delta(x) := \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor.$$

Let $p$ be a prime number. By Legendre's formula, we have

$$v_p(n!) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{p^\ell} \right\rfloor,$$

which yields

$$v_p\big(\mathscr{Q}(n)\big) = \sum_{\ell=1}^{\infty} \Delta\left(\frac{n}{p^\ell}\right).$$

Furthermore, we have $\Delta(x) = \Delta(\{x\}) + (|\mathbf{e}| - |\mathbf{f}|)\lfloor x \rfloor$, where $\{x\}$ is the fractional part of $x$ and $|\mathbf{e}| = e_1 + \cdots + e_u$. Hence the graph of $\Delta$ is essentially determined by its values on $[0,1]$. Landau's function provides a useful criterion for the $N$-integrality of $F(z)$.

**Theorem 2.1 (Landau [9], Bober [2])** *The following assertions are equivalent.*

*(i)* $F(z)$ *is* $N$*-integral ;*
*(ii)* $F(z) \in \mathbb{Z}[[z]]$ *;*
*(iii) For all* $x$ *in* $[0,1]$*, we have* $\Delta(x) \geq 0$.

Landau proved the equivalence of (ii) and (iii) in 1900 while Bober proved in 2009 a result which implies the equivalence with (i). One can easily compute the jumps of $\Delta$ on $[0,1]$ to check Assertion (iii).

The generating series of factorial ratios are rescaling of hypergeometric functions whose parameters have a certain symmetry. Namely, if $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta =$

$(\beta_1,\dots,\beta_s)$ are tuples of parameters in $\mathbb{Q}\cap(0,1]$, then there is $C\in\mathbb{Q}^*$ such that, for every $n\in\mathbb{N}$, we have

$$C^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}=\frac{(e_1n)!\cdots(e_un)!}{(f_1n)!\cdots(f_vn)!},$$

if, and only if

$$\frac{(X-e^{2i\pi\alpha_1})\cdots(X-e^{2i\pi\alpha_r})}{(X-e^{2i\pi\beta_1})\cdots(X-e^{2i\pi\beta_s})}$$

is a ratio of cyclotomic polynomials. We will see that, when this is not the case, we still have a criterion for the $N$-integrality of hypergeometric functions but it involves several Landau's functions: the functions of Christol.

## 2.2 *Generalized hypergeometric functions*

Let $\alpha=(\alpha_1,\dots,\alpha_r)$ and $\beta=(\beta_1,\dots,\beta_s)$ be tuples of elements in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. We set

$$F(z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}z^n.$$

If $\beta_i=1$ for some $i$, then $F(z)$ is annihilated by the hypergeometric differential operator

$$\mathscr{L}=\prod_{i=1}^{s}(\theta+\beta_i-1)-z\prod_{i=1}^{r}(\theta+\alpha_i),$$

which is irreducible if, and only if $\alpha_i\not\equiv\beta_j \mod \mathbb{Z}$. Elementary calculations show that $F(z)$ is $N$-integral if and only if, for almost all primes $p$, we have $F(z)\in\mathbb{Z}_{(p)}[[z]]$, where $\mathbb{Z}_{(p)}$ is the set of the rational numbers whose denominator is not divisible by $p$.

We introduce some definitions to construct useful functions defined by Christol in [3]. If $x$ is a rational number, then we set

$$\langle x\rangle=\begin{cases}\{x\} & \text{if } x\notin\mathbb{Z},\\ 1 & \text{otherwise.}\end{cases}.$$

We write $\preceq$ for the total order on $\mathbb{R}$ defined by

$$x\preceq y\Longleftrightarrow\big(\langle x\rangle<\langle y\rangle \text{ or } (\langle x\rangle=\langle y\rangle \text{ and } x\geq y)\big).$$

Let $d$ be the common multiple of the exact denominators of the $\alpha_i$'s and $\beta_j$'s. For all $a$ coprime to $d$, $1\leq a\leq d$, we set

$$\xi_a(x):=\#\{1\leq i\leq r : a\alpha_i\preceq x\}-\#\{1\leq j\leq s : a\beta_j\preceq x\}.$$

Then we have the following criterion for the $N$-integrality of $F(z)$.

**Theorem 2.2 (Christol [3])** *The following assertions are equivalent.*

*(i)* $F(z)$ *is* $N$*-integral;*
*(ii) For all a coprime to d,* $1 \le a \le d$*, and all x in* $\mathbb{R}$*, we have* $\xi_a(x) \ge 0$.

If $F(z)$ is $N$-integral, then the set of constants $c \in \mathbb{Q}$ such that $F(cz) \in \mathbb{Z}[[z]]$ is $C\mathbb{Z}$ for some $C \in \mathbb{Q}^*$. When $F(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$, then $F(z)$ is $N$-integral and $C$ is called the Eisenstein constant of $F$. Hence we shall also call $C$ the *Eisenstein constant* of $F(z)$. Rivoal, Roques and the author gave in [6] a formula for $C$ when the parameters of the hypergeometric function belong to $(0,1]$.

For every prime $p$, we set

$$\lambda_p = \#\{1 \le i \le r : \alpha_i \in \mathbb{Z}_{(p)}\} - \#\{1 \le j \le s : \beta_j \in \mathbb{Z}_{(p)}\}.$$

If $\alpha$ is a rational number, then we write $\mathrm{den}(\alpha)$ for its exact denominator. As a particular case of Theorem 1 in [6], we have the following formula.

**Theorem 2.3** *If* $\alpha$ *and* $\beta$ *are tuples of elements in* $(0,1]$*,* $r = s$ *and* $F(z)$ *is* $N$*-integral, then the Eisenstein constant of F is*

$$C = \frac{\prod_{i=1}^{r} \mathrm{den}(\alpha_i)}{\prod_{j=1}^{s} \mathrm{den}(\beta_j)} \prod_{p \mid d} p^{-\left\lfloor \frac{\lambda_p}{p-1} \right\rfloor}.$$

In the case of factorial ratios, if $\mathbf{e} = (e_1, \dots, e_u)$ and $\mathbf{f} = (f_1, \dots, f_v)$ are tuples of positive integers, then we have

$$\frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} = \left( \frac{e_1^{e_1} \cdots e_u^{e_u}}{f_1^{f_1} \cdots f_v^{f_v}} \right)^n \frac{\prod_{i=1}^{u} \prod_{r=1}^{e_i} (r/e_i)_n}{\prod_{j=1}^{v} \prod_{r=1}^{f_v} (r/f_j)_n}.$$

If the associated generating series is ($N$-)integral then the Eisenstein constant is indeed

$$C = \frac{e_1^{e_1} \cdots e_u^{e_u}}{f_1^{f_1} \cdots f_v^{f_v}}.$$

### 2.2.1 Landau-like functions

To prove Theorem 2.3, we use Landau-like functions to calculate the $p$-adic valuation of Pochhammer's symbols. To define those functions, we first consider a map $\mathfrak{D}_p$ introduced by Dwork as follows.

Let $p$ be a prime and $\alpha$ in $\mathbb{Z}_{(p)}$. We write $\mathfrak{D}_p(\alpha)$ for the unique element in $\mathbb{Z}_{(p)}$ satisfying $p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}$. We have $\mathfrak{D}_p(1) = 1$ and if $\alpha = r/N$ with $r$ coprime to $N \ge 2$, $1 \le r \le N$, then

$$\mathfrak{D}_p(\alpha) = \frac{s_N\big(\pi_N(p)^{-1}\pi_N(r)\big)}{N},$$

where $s_N$ is the section of the canonical morphism $\pi_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ with values in $\{0,\dots,N-1\}$.

If $p$ does not divide $d$, then, for all positive integers $\ell$, we define the Landau-like function

$$\Delta_{p,\ell}(x) = \sum_{i=1}^{r} \left\lfloor x - \mathfrak{D}_p^{\ell}(\alpha_i) - \frac{\lfloor 1-\alpha_i \rfloor}{p^{\ell}} \right\rfloor - \sum_{j=1}^{s} \left\lfloor x - \mathfrak{D}_p^{\ell}(\beta_j) - \frac{\lfloor 1-\beta_j \rfloor}{p^{\ell}} \right\rfloor + r - s.$$

Christol proved in [3] a Legendre-like formula involving Landau-like function.

**Theorem 2.4** *If $p$ does not divide $d$, then we have*

$$v_p\left(\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right) = \sum_{\ell=1}^{\infty} \Delta_{p,\ell}\left(\frac{n}{p^{\ell}}\right).$$

Our first task to prove Theorem 2.3 was to find a convenient analog of Legenre's formula when $p$ is a divisor of $d$. In this case, Dwork's maps are note defined for every parameters $\alpha_i$ and $\beta_j$. To that end, we proved in [6] an average formula for primes dividing $d$.

**Theorem 2.5** *Assume that $\alpha$ and $\beta$ are tuples of $r$ elements in $(0,1]$ such that $F(z)$ is $N$-integral. Let $p$ be a prime divisor of $d$ and write $d = p^f D$ where $D$ is coprime to $p$.*

*For every $a$ coprime to $p$, $1 \le a \le p^f$, and all positive integers $\ell$, we choose a prime $p_{a,\ell}$ satisfying $p_{a,\ell} \equiv p^{\ell} \mod D$ and $p_{a,\ell} \equiv a \mod p^f$. Then*

$$v_p\left(C^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right) = \frac{1}{\varphi(p^f)} \sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p^f} \sum_{\ell=1}^{\infty} \Delta_{p_{a,\ell},1}\left(\frac{n}{p^{\ell}}\right) + n\left\{\frac{\lambda_p}{p-1}\right\}.$$

In the case of factorial ratios, we have again

$$\frac{(e_1 n)!\cdots(e_u n)!}{(f_1 n)!\cdots(f_v n)!} = C^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}.$$

$\alpha$ and $\beta$ are tuples of elements in $(0,1]$. For every $p$ not dividing $d$ and every $\ell$, the map $\mathfrak{D}_p^{\ell}$ induces a permutation on $\alpha$ and $\beta$. Hence we have

$$
\begin{aligned}
\Delta_{p,\ell}(x) &= \sum_{i=1}^{r}\left\lfloor x-\mathfrak{D}_p^{\ell}(\alpha_i)-\frac{\lfloor 1-\alpha_i\rfloor}{p^{\ell}}\right\rfloor-\sum_{j=1}^{s}\left\lfloor x-\mathfrak{D}_p^{\ell}(\beta_j)-\frac{\lfloor 1-\beta_j\rfloor}{p^{\ell}}\right\rfloor+r-s\\
&= \sum_{i=1}^{r}\left\lfloor x-\mathfrak{D}_p^{\ell}(\alpha_i)\right\rfloor-\sum_{j=1}^{s}\left\lfloor x-\mathfrak{D}_p^{\ell}(\beta_j)\right\rfloor+r-s\\
&= \sum_{i=1}^{r}\lfloor x-\alpha_i\rfloor-\sum_{j=1}^{s}\lfloor x-\beta_j\rfloor+r-s\\
&= \sum_{i=1}^{u}\lfloor e_i x\rfloor-\sum_{j=1}^{v}\lfloor f_j x\rfloor\\
&= \Delta(x).
\end{aligned}
$$

Hence, in both cases, the formulas of Theorems 2.4 and 2.5 reduce to Legendre's one.

# 3 Integrality of the coefficients of $q$-coordinates

## 3.1 A glimpse of Dwork's result

Consider the power series

$$F(z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}z^n,$$

$$G(z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\left(\sum_{i=1}^{r}H_{\alpha_i}(n)-\sum_{j=1}^{s}H_{\beta_j}(n)\right)z^n,$$

where, for $n\in\mathbb{N}$ and $x\in\mathbb{Q}\setminus\mathbb{Z}_{\le 0}$, we set $H_x(n)=\sum_{k=0}^{n-1}\frac{1}{x+k}$.

Then $G(z)+\log(z)F(z)$ is annihilated by the hypergeometric operator $\mathscr{L}$ if there are at least two 1's in $\beta$. The $q$-coordinate is

$$q(z)=\exp\left(\frac{G(z)+\log(z)F(z)}{F(z)}\right)=z\exp\left(\frac{G(z)}{F(z)}\right).$$

A consequence of a lemma of Dieudonné and Dwork is that, for every prime $p$, we have

$$q(z)\in\mathbb{Z}_p[[z]]\Longleftrightarrow\frac{G}{F}(z^p)-p\frac{G}{F}(z)\in p\mathbb{Z}_p[[z]].$$

Let $p$ be a prime not dividing $d$ and write $F_1(z)$ (resp. $G_1(z)$) for $F(z)$ (resp. $G(z)$) with the substitutions

$$\alpha \leftrightarrow (\mathfrak{D}_p(\alpha_1),\ldots,\mathfrak{D}_p(\alpha_r)) \quad \text{and} \quad \beta \leftrightarrow (\mathfrak{D}_p(\beta_1),\ldots,\mathfrak{D}_p(\beta_s)).$$

Then Dwork proved in [7] the following. Assume that $r=s$, for all $\ell \in \mathbb{N}$, $\mathfrak{D}_p^{\ell}(\beta_i) \in \mathbb{Z}_p^{\times}$, plus some fundamental but hard to read interlacing conditions (depending on $p$) on elements of $\alpha$ and $\beta$. Then we have

$$\frac{G_1}{F_1}(z^p) - p\frac{G}{F}(z) \in p\mathbb{Z}_p[[z]].$$

In particular, if $\mathfrak{D}_p$ induces a permutation on $\alpha$ and $\beta$, which is the case for factorial ratios, then $F_1 = F$, $G_1 = G$ and Dwork's result yields

$$\frac{G}{F}(z^p) - p\frac{G}{F}(z) \in p\mathbb{Z}_p[[z]],$$

so that $q(z) \in \mathbb{Z}_p[[z]]$.

## 3.2 Factorial ratios

If the interlacing conditions hold for every (explicitly) large enough primes $p$, then $q(z)$ is $N$-integral. Methods for the remaining primes were developed by Lian-Yau (1998, [10]), Zudilin (2002, [11]), Krattenthaler-Rivoal (2009, [8]) for infinite families of factorial ratios, yielding proofs of $q(Cz) \in \mathbb{Z}[[z]]$ where $C$ is the Eisenstein constant of $F(z)$.

In the case of factorial ratios, we have

$$G(z) = \sum_{n=0}^{\infty} \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} \left( \sum_{i=1}^{u} e_i H_{e_i n} - \sum_{j=1}^{v} f_j H_{f_j n} \right) z^n$$

and

$$\Delta(x) = \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor.$$

We gave a criterion for the integrality of the taylor coefficients of $q(z)$ in 2012 (see [4]).

**Theorem 3.1** *If $F(z)$ is $N$-integral with Eisenstein constant $C$, then the following assertions are equivalent.*

*(i)* *$q(z)$ is $N$-integral ;*
*(ii)* *$q(Cz) \in \mathbb{Z}[[z]]$ ;*
*(iii)* *we have $|\mathbf{e}| = |\mathbf{f}|$ and, for all $x \in [1/M, 1)$, we have $\Delta(x) \geq 1$, where $M$ is the largest element in $\mathbf{e}$ and $\mathbf{f}$.*

The proof of (iii) $\Rightarrow$ (i) is essentially a consequence of Dwork's results. Legendre's formula and Landau's functions play an important role in the proof of Theorem 3.1. When $F(z)$ is the generating series of multisums of binomial coefficients (such as Apéry numbers), it seems impossible to apply an analog of the proof of Theorem 3.1. To prove the integrality of the coefficients of the associated $q$-coordinate, we prove a generalization of Theorem 3.1 to several variables and then we specialize the multivariate $q$-coordinates.

### 3.3 Factorial ratios of linear forms

Let $e = (\mathbf{e}_1, \ldots, \mathbf{e}_u)$ and $f = (\mathbf{f}_1, \ldots, \mathbf{f}_v)$ be tuples of nonzero vectors in $\mathbb{N}^d$. Consider

$$F(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_u\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_v\cdot\mathbf{n})!}\mathbf{z}^{\mathbf{n}}.$$

For every $k \in \{1, \ldots, d\}$, write

$$G_k(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_u\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_v\cdot\mathbf{n})!}\left(\sum_{i=1}^{u}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i\cdot\mathbf{n}} - \sum_{j=1}^{v}\mathbf{f}_j^{(k)}H_{\mathbf{f}_j\cdot\mathbf{n}}\right)\mathbf{z}^{\mathbf{n}},$$

where $\mathbf{e}_i^{(k)}$ is the $k$-th component of $\mathbf{e}_i$. The $q$-coordinates are

$$q_k(\mathbf{z}) = z_k \exp\left(\frac{G_k(\mathbf{z})}{F(\mathbf{z})}\right), \quad 1 \le k \le n.$$

The associated Landau function is

$$\Delta(\mathbf{x}) = \sum_{i=1}^{u}\lfloor \mathbf{e}_i\cdot\mathbf{x}\rfloor - \sum_{j=1}^{v}\lfloor \mathbf{f}_j\cdot\mathbf{x}\rfloor, \quad (\mathbf{x}\in\mathbb{R}^d).$$

The non-trivial zone for $\Delta$ is defined by

$$\mathscr{D} := \left\{\mathbf{x}\in[0,1)^d : \text{there is } \mathbf{d} \text{ in } \mathbf{e} \text{ or } \mathbf{f} \text{ such that } \mathbf{d}\cdot\mathbf{x}\ge 1\right\}.$$

Observe that if $\mathbf{x}$ belongs to $[0,1)^d \setminus \mathscr{D}$, then we have $\Delta(\mathbf{x}) = 0$. We proved in [5] the following criterion.

**Theorem 3.2** *Assume that $F(\mathbf{z}) \in \mathbb{Z}[[\mathbf{z}]]$. Then the following assertions are equivalent:*

*(i) For every $k$, we have $q_k(\mathbf{z}) \in \mathbb{Z}[[z]]$ ;*
*(ii) we have $|e| = |f|$ and, for every $\mathbf{x}\in\mathscr{D}$, $\Delta(\mathbf{x}) \ge 1$.*

To apply Theorem 3.2 to the case of Apéry numbers (associated with $\zeta(3)$), we consider the bivariate power series

$$F(x,y)=\sum_{n_1,n_2\geq 0}\frac{(2n_1+n_2)!^2}{n_1!^4n_2!^2}x^{n_1}y^{n_2}$$

and

$$G_2(x,y)=\sum_{n_1,n_2\geq 0}\frac{(2n_1+n_2)!^2}{n_1!^4n_2!^2}(2H_{2n_1+n_2}-2H_{n_2})x^{n_1}y^{n_2}.$$

In this case, we have

$$\Delta(x,y)=2\lfloor 2x+y\rfloor-4\lfloor x\rfloor-2\lfloor y\rfloor.$$

We have

$$\mathscr{D}=\big\{(x,y)\in[0,1)^2\,:\,2x+y\geq 1\big\}$$

and if $\mathbf{x}\in\mathscr{D}$, then $\Delta(\mathbf{x})\geq 2$. Hence we have $q_2(x,y)\in\mathbb{Z}[[x,y]]$ by Theorem 3.2. Taking $x=y$ yields

$$q_2(x,x)=\exp\left(\frac{G_2(x,x)}{F(x,x)}\right)\in\mathbb{Z}[[x]],$$

where

$$F(x,x)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^n$$

and

$$G_2(x,x)=\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(2H_{n+k}-2H_{n-k})x^n,$$

as expected.

## 3.4 Generalized hypergeometric $q$-coordinates

In this section, we briefly comment analog results in the (univariate) general case.

Write $m(a)$ for the smallest element in $(\{a\alpha_1,\dots,a\alpha_r,a\beta_1,\dots,a\beta_s\},\preceq)$. We consider the following assertion, denoted $H$: For all $a$ coprime to $d$, $1\leq a\leq d$, for all $x\in\mathbb{R}$ satisfying $m(a)\preceq x\prec a$, we have $\xi_a(x)\geq 1$. We consider a product of $q$-coordinates whose $N$-integrality is strongly related to the one of $q(z)$:

$$\tilde{q}(z)=\prod_{a=1,\gcd(a,d)=1}^{d}q_{\langle a\alpha\rangle,\langle a\beta\rangle}(z).$$

Then we proved in [6] the following criterion.

**Theorem 3.3** *Assume that $\mathscr{L}$ is irreducible and that $F(z)$ is $N$-integral. Then*

(i) *if $r=s$ and Assertion $H$ holds, then $\tilde{q}(z)$ is $N$-integral.*

*Furthermore, the following assertions are equivalent:*

(ii) $q(z)$ *is* $N$*-integral ;*
(iii) $\tilde{q}(z)$ *is* $N$*-integral and* $\tilde{q}(z) = q(z)^{\varphi(d)}$.

## *3.5 A brief overview of the $p$-adic strategy*

The first step is to reduce the problem for each prime by the following classical result: if $x \in \mathbb{Q}$, then $x \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$ for all primes $p$.

Then we get ride of the exponential by applying the lemma of Dieudonné and Dwork.

**Lemma 3.4**

$$z \exp\left(\frac{G(z)}{F(z)}\right) \in \mathbb{Z}_p[[z]] \Longleftrightarrow \frac{G}{F}(z^p) - p\frac{G}{F}(z) \in pz\mathbb{Z}_p[[z]].$$

Then, in all proofs, one has to generalize a theorem on formal congruences of Dwork to prove that

$$F_{s-1}(z^p)F(z) \equiv F(z^p)F_s(z) \mod p^s\mathbb{Z}_p[[z]],\ (\forall s \geq 1),$$

where $F_s(z) := \sum_{n=0}^{p^s-1} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} a_n z^n$.

The last main step is to prove congruences for harmonic numbers $H_\alpha(n)$ and the $p$-adic Gamma function.

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