Appell–Lauricella hypergeometric functions over finite fields, and a new cubic transformation formula

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Abstract We define a finite-field version of Appell–Lauricella hypergeometric functions built from period functions in several variables, paralleling the development by Fuselier, et. al [9] in the single variable case. We develop geometric connections between these functions and the family of generalized Picard curves. In our main result, we use finite-field Appell–Lauricella functions to establish a finite-field analogue of Koike and Shiga's cubic transformation [13] for the Appell hypergeometric function F_1, proving a conjecture of Ling Long. We also prove a finite field analogue of Gauss' quadratic arithmetic geometric mean. We use our multivariable period functions to construct formulas for the number of $\mathbb{F}_p$-points on the generalized Picard curves. Lastly, we give some transformation and reduction formulas for the period functions, and consequently for the finite-field Appell–Lauricella functions.

1 Cubic Transformation Formulas

Classical hypergeometric functions are among the most versatile of all special functions. These functions and their finite-field analogues have numerous applications in number theory and geometry. For instance, finite-field hypergeometric functions play a role in proving congruences and supercongruences, they count points modulo p over algebraic varieties and affine hypersurfaces, and in certain instances they provide formulas for the Fourier coefficients of modular forms. We define finite-field hypergeometric functions $\mathbb{F}_D^{(n)}$ in several variables, as an analogue of the classical

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Appell–Lauricella hypergeometric functions of type D. Lauricella's series of type D give a natural generalization of Appell's F_1 functions to n variables and are closely related to generalized Picard curves. Following the literature, we refer to these generalizations as Appell–Lauricella functions. For a comprehensive survey of Appell–Lauricella functions, we refer the reader to the article by Schlosser [19], and to the monograph by Slater [20]. Furthermore, we note that classical hypergeometric functions as well as Appell–Lauricella functions are examples of a more general class called A-hypergeometric functions introduced and studied by Gelfand, Kapranov, and Zelevinsky [10], and further studied by Beukers [4].

We develop the theory of these $\mathbb{F}_D^{(n)}$ finite field hypergeometric functions in several variables, with a focus on their geometric connections to the generalized Picard curves. This parallels the construction (by the second and third authors, et. al.) in [9], categorizing the interplay between classical and finite-field hypergeometric functions in the single-variable setting.

Our results are motivated by a conjecture of Ling Long, related to identities proved by Koike and Shiga [13], [14]. In [13], Koike and Shiga applied Appell's F_1 hypergeometric function in two variables to establish a new three-term arithmetic geometric mean result (AGM), related to Picard modular forms. As a consequence of this cubic AGM, Koike and Shiga proved the following cubic transformation for Appell's F_1-function. Let $x, y \in \mathbb{C}$, and let ω be a primitive cubic root of unity. Then

$$\begin{aligned} &F_1\left[\frac{1}{3};\frac{1}{3},\frac{1}{3};1\,\middle|\,1-x^3,1-y^3\right] \\ &=\frac{3}{1+x+y}\,F_1\left[\frac{1}{3};\frac{1}{3},\frac{1}{3};1\,\middle|\,\left(\frac{1+\omega x+\omega^2 y}{1+x+y}\right)^3,\left(\frac{1+\omega^2 x+\omega y}{1+x+y}\right)^3\right]. \end{aligned} \tag{1}$$

As an application of Appell–Lauricella functions over finite fields, we prove the following finite-field analogue of Koike and Shiga's transformation, as conjectured by Ling Long.

Theorem 1. *Let $p \equiv 1 \pmod 3$ be prime, let ω be a primitive cubic root of unity, and let η_3 be a primitive cubic character in $\widehat{\mathbb{F}_p^\times}$. If $\lambda, \mu \in \mathbb{F}_p$ satisfy $1+\lambda+\mu \neq 0$, then*

$$\begin{aligned} &\mathbb{F}_D^{(2)}\left[\begin{matrix}\eta_3;\ \eta_3\ \eta_3 \\ \varepsilon\end{matrix}\,;\,1-\lambda^3, 1-\mu^3\right] \\ &=\mathbb{F}_D^{(2)}\left[\begin{matrix}\eta_3;\ \eta_3\ \eta_3 \\ \varepsilon\end{matrix}\,;\,\left(\frac{1+\omega\lambda+\omega^2\mu}{1+\lambda+\mu}\right)^3,\left(\frac{1+\omega^2\lambda+\omega\mu}{1+\lambda+\mu}\right)^3\right]. \end{aligned}$$

When $\lambda = \mu$, we have the following corollary.

Corollary 1. *For $p \equiv 1 \pmod 3$ prime, and ω as above, if $\lambda \in \mathbb{F}_p$ satisfies $1+2\lambda \neq 0$, then*

$$ {}_2\mathbb{F}_1\left[\begin{matrix}\eta_3 & \eta_3^2 \\ & \varepsilon\end{matrix}\,;\,1-\lambda^3\right] = {}_2\mathbb{F}_1\left[\begin{matrix}\eta_3 & \eta_3^2 \\ & \varepsilon\end{matrix}\,;\,\left(\frac{1-\lambda}{1+2\lambda}\right)^3\right]. $$

The result of Corollary 1 was first established in [9], using a different method of proof. It is a finite-field version of the cubic transformation

$$ {}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\,;\,1-x^3\right] = \frac{3}{1+2x}\,{}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\,;\,\left(\frac{1-x}{1+2x}\right)^3\right], \tag{2} $$

proved by Borwein and Borwein [5], [6] for $x \in \mathbb{R}$ with $0 < x < 1$, as a cubic analogue of Gauss' quadratic AGM.

2 Quadratic Transformations: Revisiting Gauss' Quadratic AGM

In [9], the authors give a dictionary for the correspondence between results on classical hypergeometric functions and and finite-field hypergeometric functions. Given a transformation for classical hypergeometric functions, this dictionary can be used to predict the form of the analogous transformation for finite-field hypergeometric functions. They also use a calculus-style method of converting the proofs of classical identities to the finite-field setting, provided the classical identity satisfies the following condition: It can be proved using only the binomial theorem, the reflection and multiplication formulas [for the gamma function], or their corollaries (such as the Pfaff-Saalschütz formula) [9].

We illustrate this calculus-style method of translating classical results by proving the following theorem.

Theorem 2. *The quadratic arithmetic-geometric mean of Gauss, given for $x \in \mathbb{C}$ by*

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix}\middle|\,1-x^2\right] = \frac{2}{1+x}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix}\middle|\left(\frac{1-x}{1+x}\right)^2\right], \tag{3} $$

can be proved using only the binomial theorem, the reflection and duplication formulas for the gamma function, and special evaluations of the ${}_3F_2$ and ${}_2F_1$ functions, using the Pfaff- Saalschütz formula and Gauss' formula, respectively.

As a consequence, translating this alternate proof of Gauss' quadratic AGM and analyzing the associated error terms on the finite-field side, we also obtain the following corollary.

Corollary 2. *Let $p \equiv 1 \pmod 4$ be prime, let ϕ be the quadratic character in $\widehat{\mathbb{F}_p^\times}$, and let ε be the trivial character in $\widehat{\mathbb{F}_p^\times}$. If $\lambda \in \mathbb{F}_p$ satisfies $1+\lambda \neq 0$, then*

$$ {}_2\mathbb{F}_1\left[\begin{matrix}\phi & \phi \\ & \varepsilon\end{matrix}\middle| 1-\lambda^2\right] = {}_2\mathbb{F}_1\left[\begin{matrix}\phi & \phi \\ & \varepsilon\end{matrix}\middle| \left(\frac{1-\lambda}{1+\lambda}\right)^2\right], \tag{4} $$

3 Connections to Picard Curves

Taking the approach used in [9], our finite-field Appell–Lauricella hypergeometric functions are defined as normalizations of finite-field period functions $\mathbb{P}_D^{(n)}$, which we also define. These period functions are naturally related to periods of the generalized Picard curves

$$ C_\lambda^{[N;i,j,\mathbf{k}]}: \; y^N = x^i(1-x)^j(1-\lambda_1 x)^{k_1}\cdots(1-\lambda_n x)^{k_n}, \tag{5} $$

defined for distinct complex numbers $\lambda_1,\ldots,\lambda_n \neq 0,1$ and positive integers $N,i,j,k_1,\ldots,k_n$ that satisfy the conditions $\gcd(N,i,j,k_1,\ldots,k_n)=1$ and $N \nmid i+j+k_1+\cdots+k_n$. As a consequence, the $\mathbb{P}_D^{(n)}$ functions are ideally suited for counting $\mathbb{F}_p$-points on Picard curves. We prove a theorem which gives the number of $\mathbb{F}_p$-points on the generalized Picard curves in a simple, elegant formula. This is analogous to the point-counting result for the generalized Legendre curves that was established by the second and third authors, et. al. in [7]. We also compute the genus of the generalized Picard curves $C_\lambda^{[N;i,j,\mathbf{k}]}$, following methods of Archinard [1].

4 Transformation and Reduction Formulas

Transformation and reduction formulas for classical hypergeometric functions have been successfully translated to the finite-field setting, first by Greene and also by authors such as McCarthy, and Fuselier et. al. (See [11], [17], [9] for details.) Transformation formulas for classical Appell–Lauricella hypergeometric functions, many of which can be found in the monograph by Slater [20] or the survey paper of Schlosser [19], may be translated into the finite-field setting using the same methods. We carry out this process, proving several identities for the period functions $\mathbb{P}_D^{(n)}$ and hypergeometric functions $\mathbb{F}_D^{(n)}$. Among other things, these include a finite-field analogue of the Pfaff-Kummer transformation,

$$ F_1\left[a;b_1,b_2;c \,\middle|\, x,y\right] = (1-x)^{-b_1}(1-y)^{-b_2} F_1\left[c-a;b_1,b_2;c \,\middle|\, \frac{x}{x-1},\frac{y}{y-1}\right], $$

and Euler's transformation,

$$ F_1\left[a;b_1,b_2;c \,\middle|\, x,y\right] = (1-x)^{c-a-b_1}(1-y)^{-b_2} F_1\left[c-a;c-b_1-b_2,b_2;c \,\middle|\, x,\frac{x-y}{1-y}\right], $$

which hold for all $a, b_1, b_2, c \in \mathbb{C}$ and all x, y for which the series are defined.

We note that another version of finite-field Appell–Lauricella functions is independently defined by He [12] and Li, et. al. [16], which closely follows Greene's definition. For their version, they establish several degree 1 transformation and reduction formulas, including some that are analogous to the identities we prove.

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