Hypergeometric functions over finite fields

Jenny Fuselier, Ling Long, Ravi Ramakrishna, Holly Swisher, and Fang-Ting Tu

Abstract We discuss recent work of the authors in which we study the translation of classical hypergeometric transformation and evaluation formulas to the finite field setting.

Our approach is motivated by the desire for both an algorithmic type approach that closely parallels the classical case, and an approach that aligns with geometry. In light of these objectives, we focus on period functions in our construction which makes point counting on the corresponding varieties as straightforward as possible.

We are also motivated by previous work joint with Deines, Fuselier, Long, and Tu in which we study generalized Legendre curves using periods to determine a condition for when the endomorphism algebra of the primitive part of the associated Jacobian variety contains a quaternion algebra over $\mathbb{Q}$. In most cases this involves computing Galois representations attached to the Jacobian varieties using Greene's finite field hypergeometric functions.

Jenny Fuselier
High Point University, High Point, NC 27268, USA, e-mail: jfuselie@highpoint.edu

Ling Long
Louisiana State University, Baton Rouge, LA 70803, USA, e-mail: llong@math.lsu.edu

Ravi Ramakrishna
Cornell University, Ithaca, NY 14853, USA, e-mail: ravi@math.cornell.edu

Holly Swisher
Oregon State University, Corvallis, OR 97331, USA, e-mail: swisherh@math.oregonstate.edu

Fang-Ting Tu
Louisiana State University, Baton Rouge, LA 70803, USA, e-mail: tu@math.lsu.edu

1 Motivation

In this talk we discuss recent work of the authors [9], in which we study the translation of classical hypergeometric transformation and evaluation formulas to the finite field setting. The theory of classical hypergeometric functions and hypergeometric functions over finite fields sits inside the broader framework of hypergeometric motives. Hypergeometric functions over finite fields have been developed by several people, including for example Evans [6, 7], Greene [10], Katz [11], and McCarthy [12], and a number of current developments have been discussed at this workshop including recent work of Roberts, Rodriguez-Villegas, and Watkins [13], Doran, et al [8], and Beukers, Cohen, and Mellit [2], for example.

Our approach to translation of classical hypergeometric transformation and evaluation formulas to the finite field setting is motivated by two strong desires:

1. An algorithmic type approach that closely parallels the classical case (and does not require particular ingenuity for each example).
2. An approach that aligns with geometry by explicitly interpreting the finite field hypergeometric functions in terms of Galois representations corresponding to associated algebraic varieties.

In light of these objectives, we focus on period functions in our construction which makes point counting on the corresponding varieties as straightforward as possible.

We are also motivated by previous work joint with Deines, Fuselier, Long, and Tu [3] in which we study generalized Legendre curves $y^N = x^i(1-x)^j(1-\lambda x)^k$, using periods to determine for certain N a condition for when the endomorphism algebra of the primitive part of the associated Jacobian variety contains a quaternion algebra over $\mathbb{Q}$. In most cases this involves computing Galois representations attached to the Jacobian varieties using Greene's finite field hypergeometric functions.

2 Method

From this perspective, the following approach is very natural. We slightly modify the finite field hypergeometric function definition of Greene (or McCarthy) by inductively using the Euler integral representation to construct our analogues. In the classical setting, define the period function ${}_1P_0[a;z] := (1-z)^{-a} = {}_1F_0[a;z]$, and then use the Euler integral formula to define

$$ {}_2P_1[a,b;c;z] := \int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1P_0[a;zt]dt = B(b,c-b){}_2F_1[a,b;c;z], $$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the beta function. To justify calling these period functions we note that Wolfart [15] realized that if the parameters a, b, $c \in \mathbb{Q}$, and $a,b,a-c,b-c \notin \mathbb{Z}$, then the integrals

$$ {}_2P_1\left[\begin{matrix} a\ b \\ c \end{matrix};\lambda\right] \text{ and } (-1)^{c-a-b-1}\lambda^{1-c}\,{}_2P_1\left[\begin{matrix} 1+b-c\ \ 1+a-c \\ 2-c \end{matrix};\lambda\right] $$

are both periods of a generalized Legendre curve $y^N = x^i(1-x)^j(1-\lambda x)^k$, where $N = \mathrm{lcd}(a,b,c)$ (least common denominator), $i = N\cdot(1-b)$, $j = N\cdot(1+b-c)$, and $k = N\cdot a$.

Inductively we define higher period functions ${}_{n+1}P_n$, gathering additional beta function terms in front of the ${}_{n+1}F_n$. We can then use the following "dictionary" which is well-known to experts to translate these period functions to the finite field setting. Here $q = p^e$ for prime p, and $\widehat{\mathbb{F}_q^\times}$ denotes the group of multiplicative characters on $\mathbb{F}_q^\times$, where each character A is extended to $\mathbb{F}_q$ by defining $A(0)=0$. Furthermore $N\in\mathbb{N}$, $a,b\in\mathbb{Q}$ with common denominator N, $\eta_N \in \widehat{\mathbb{F}_q^\times}$ has order N, $\overline{A}$ denotes the complex conjugate of A, and ζ_p is a primitive pth root of unity.

$$ \begin{array}{ccc} a=\frac{i}{N},\ b=\frac{j}{N} & \leftrightarrow & A,B\in\widehat{\mathbb{F}_q^\times},\ A=\eta_N^i,\ B=\eta_N^j \\ x^a & \leftrightarrow & A(x) \\ -a & \leftrightarrow & \overline{A} \\ \int_0^1\ \mathrm{d}x & \leftrightarrow & \sum_{x\in\mathbb{F}} \\ \Gamma(a) & \leftrightarrow & g(A)=\sum_{x\in\mathbb{F}_q^\times}\zeta_p^{x+x^p+x^{p^2}+\cdots+x^{p^{e-1}}} \\ B(a,b) & \leftrightarrow & J(A,B)=\sum_{x\in\mathbb{F}_q}A(x)B(1-x) \end{array} $$

Thus we correspondingly define ${}_1\mathbb{P}_0[A;\lambda] := \overline{A}(1-\lambda)$,

$$ {}_2\mathbb{P}_1[A,B;C;\lambda] := \sum_{y\in\mathbb{F}_q} B(y)\overline{B}C(1-y)\,{}_1\mathbb{P}_0[A;\lambda y], $$

and define ${}_{n+1}\mathbb{P}_n$ inductively. We obtain a nice point counting formula for the hypergeometric variety $X_\lambda : y^N = x_1^{i_1}\cdots x_n^{i_n}(1-x_1)^{j_1}\cdots(1-x_n)^{j_n}(1-\lambda x_1\cdots x_n)^k$. In particular, we have for $q\equiv 1 \pmod N$,

$$ \#X_\lambda(\mathbb{F}_q) = 1 + q^n + \sum_{m=1}^{N-1} {}_{n+1}\mathbb{P}_n\left[\begin{matrix} \eta_N^{-mk} & \eta_N^{mi_n} & \cdots & \eta_N^{mi_1} \\ & \eta_N^{mi_n+mj_n} & \cdots & \eta_N^{mi_1+mj_1} \end{matrix};\lambda\right]. $$

Normalizing the period functions ${}_{n+1}\mathbb{P}_n$ by dividing by the appropriate Jacobi sums using the dictionary, gives our finite field analogues to the classical hypergeometric functions which we denote by ${}_{n+1}\mathbb{F}_n$. For example when $n=1$,

$$ {}_2\mathbb{F}_1[A,B;C;\lambda] := \frac{1}{J(B,C\overline{B})}\,{}_2\mathbb{P}_1[A,B;C;\lambda]. $$

If none of the "top" parameters are the trivial character, or match with one of the "bottom" parameters, we call the ${}_{n+1}\mathbb{P}_n$ or ${}_{n+1}\mathbb{F}_n$ *primitive*. Our definition of ${}_{n+1}\mathbb{F}_n$

has two nice properties that match the classical case: it is 1 when evaluated at 0, and in the primitive case it is symmetric in both the top or bottom parameters.

We note that key properties such as the reflection and multiplication formulas for the Gamma function translate using the dictionary to properties of the Gauss sum. Our method allows that any classical formula proved using these properties, as well as their corollaries such as the Pfaff-Saalchütz formula, can be translated directly (introducing error terms as needed) and thus we can indeed use an algorithmic type approach to translation that closely parallels proofs in the classical case. However, this approach does not work for everything in the classical setting; for example proofs involving a derivative structure can not be translated in this way.

3 Galois Interpretation

We can interpret the ${}_{n+1}\mathbb{P}_n$ or ${}_{n+1}\mathbb{F}_n$ functions as traces of Galois representations at Frobenius elements via the corresponding hypergeometric algebraic varieties. In the $n=1$ case we make this explicit in the following theorem.

For a given number field K, denote its ring of integers by $\mathscr{O}_K$, its algebraic closure by $\overline{K}$, and set $G_K := \mathrm{Gal}(\overline{K}/K)$. We call a prime ideal $\mathfrak{p}$ of $\mathscr{O}_K$ unramified if it is coprime to the discriminant of K. Fix $\lambda \in \overline{\mathbb{Q}}$. Given a rational number of the form $\frac{i}{m}$, a number field K containing $\mathbb{Q}(\zeta_m, \lambda)$ and a prime ideal $\mathfrak{p}$ of K coprime to the discriminant of K, one can assign a multiplicative character $\iota_{\mathfrak{p}}(\frac{i}{m})$ to the residue field $\mathscr{O}_K/\mathfrak{p}$ of $\mathfrak{p}$ with size $q(\mathfrak{p}) = |\mathscr{O}_K/\mathfrak{p}|$. This assignment, based on the mth power residue symbol, is compatible with the Galois perspective when $\mathfrak{p}$ varies. It is also compatible with field extensions of K. Thus the finite field analogues of classical period (or hypergeometric) functions are viewed as the converted functions over the finite residue fields, unless otherwise specified. We show the following.

Theorem 1 *Let $a,b,c \in \mathbb{Q}$ with least common denominator N such that a, b, $a-c$, $b-c \notin \mathbb{Z}$ and $\lambda \in \overline{\mathbb{Q}} \setminus \{0,1\}$. Let K be the Galois closure of $\mathbb{Q}(\lambda, \zeta_N)$ with the ring of integers $\mathscr{O}_K$, and ℓ any prime. Then there is a 2-dimensional representation $\sigma_{\lambda,\ell}$ of $G_K := Gal(\overline{K}/K)$ over $\mathbb{Q}_\ell(\zeta_N)$, depending on a,b,c, such that for each unramified prime ideal $\mathfrak{p}$ of $\mathscr{O}_K$ for which λ and $1-\lambda$ can be mapped to nonzero elements in the residue field, $\sigma_{\lambda,\ell}$ evaluated at the arithmetic Frobenius conjugacy class $Frob_{\mathfrak{p}}$ at $\mathfrak{p}$ is an algebraic integer (independent of the choice of ℓ), satisfying*

$$Tr\,\sigma_{\lambda,\ell}(Frob_{\mathfrak{p}}) = -{}_2\mathbb{P}_1\left[\begin{matrix}\iota_{\mathfrak{p}}(a) & \iota_{\mathfrak{p}}(b)\\ & \iota_{\mathfrak{p}}(c)\end{matrix}\,;\,\lambda; q(\mathfrak{p})\right].$$

As a corollary to this theorem, given a primitive ${}_2\mathbb{P}_1$ and a prime ℓ we can compute the L-function of the corresponding 2-dimensional Galois representation (ℓ-adic) as a product over good primes of terms involving ${}_2\mathbb{P}_1$ and $({}_2\mathbb{P}_1)^2$.

4 Examples of Translated Identities

As examples of our techniques we use both our algorithmic type approach as well as the Galois perspective to translate several classical hypergeometric formulas to the finite field setting, including transformations of degree $1,2,3$, algebraic identities, and evaluation formulas.

For example, we translate a Clausen identity between the square of a ${}_2F_1$ and a ${}_3F_2$. From a differential equations perspective, this identity is indicating that the symmetric square of the 2-dimensional solution space to the corresponding hypergeometric differential equation (HDE) corresponding to the ${}_2F_1$ is the 3-dimensional solution space of the HDE corresponding to the ${}_3F_2$. Translating this identity to the finite field setting yields a finite field hypergeometric transformation due to Greene and Evans [4] which in our notation more closely matches the classical identity. With the representation theoretic perspective, it indicates the fact that the tensor square of a 2-dimensional representation (associated to the ${}_2\mathbb{F}_1$) is its symmetric square (which is a 3-dimensional representation associated to the ${}_3\mathbb{F}_2$) plus its alternating square (which is a linear representation).

For an example of an algebraic type identity, consider the following identity in Slater [14, (1.5.20)] which gives that

$${}_2F_1\left[\begin{matrix} a\ a-\frac{1}{2} \\ 2a \end{matrix}\, ; z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}.$$

To see its finite field analogue, it is tempting to translate the right hand side into a corresponding character evaluated at $\frac{1+\sqrt{1-z}}{2}$ using the dictionary. However, Theorem 1 implies that one character is insufficient as the corresponding Galois representations should be 2-dimensional. Instead, our translated identity becomes the following. For $\mathbb{F}_q$ of odd characteristic, ϕ the quadratic character, $A \in \widehat{\mathbb{F}_q^\times}$ having order at least 3, and $z \in \mathbb{F}_q$,

$${}_2\mathbb{F}_1\left[\begin{matrix} A\ A\phi \\ A^2 \end{matrix}\, ; z\right] = \left(\frac{1+\phi(1-z)}{2}\right)\left(\overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right)\right).$$

The proof is quite straightforward using only translations of Kummer 24 relations as well as the reflection and duplication formulas. This example highlights that the Galois perspective allows us to predict analogues beyond the dictionary alone.

As another example, we use our dictionary technique to translate a quadratic ${}_2F_1$ transformation of Kummer [1, Thm. 3.1.1] which we first show can be proved using only the multiplication and reflection formulas with the Pfaff-Saalschütz identity. The finite field version we obtain is equivalent to a quadratic formula of Greene in [10], but holds for all values in $\mathbb{F}_q$. Our proof (although it might appear technical on the surface) is very straightforward. In comparison, the approaches of Evans and Greene to higher order transformation formulas (such as [5, 10]) often involve clever changes of variables. This example also demonstrates that our method has the capacity to produce finite field analogues that are satisfied by all values in $\mathbb{F}_q$.

As an explicit application of finite field formulas in computing the arithmetic invariants of hypergeometric varieties, we use the finite field quadratic transformation from the previous example to obtain the decomposition of a generically 4-dimensional abelian variety arising naturally from the generalized Legendre curve $y^{12} = x^9(1-x)^5(1-\lambda x)$.

Acknowledgements Many thanks to the International Mathematical Research Institute MATRIX in Australia for hosting the workshop on Hypergeometric Motives and Calabi–Yau Differential Equations where this talk was presented.

References

1. George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.
2. Frits Beukers, Henri Cohen, and Anton Mellit. Finite hypergeometric functions. arXiv: 1505.02900.
3. Alyson Deines, Jenny G. Fuselier, Ling Long, Holly Swisher, and Fang Ting Tu. Generalized Legendre Curves and Quaternionic Multiplication. *J. Number Theory* 161, 175–203, 2016.
4. Ron J. Evans and John Greene. Clausen's theorem and hypergeometric functions over finite fields. *Finite Fields Appl.*, 15(1):97–109, 2009.
5. Ron J. Evans and John Greene. A quadratic hypergeometric 2F1 transformation over finite fields. *Proc. Amer. Math. Soc.*, 145, 1071–1076, 2017.
6. Ronald J. Evans. Identities for products of Gauss sums over finite fields. *Enseign. Math. (2)*, 27(3-4):197–209, 1982.
7. Ronald J. Evans. Character sums over finite fields. In *Finite fields, coding theory, and advances in communications and computing (Las Vegas, NV, 1991)*, volume 141 of *Lecture Notes in Pure and Appl. Math.*, pages 57–73. Dekker, New York, 1993.
8. Charles F. Doran, Tyler L. Kelly, Adriana Salerno, Steven Sperber, John Voight, and Ursula Whitcher. Zeta functions of alternate mirror Calabi–Yau families arXiv:1612.09249
9. Jenny G. Fuselier, Ling Long, Ravi Ramakrishna, Holly Swisher, and Fang Ting Tu. Hypergeometric Functions over Finite Fields. arXiv:1510.02575
10. John Greene. Hypergeometric functions over finite fields. *Trans. Amer. Math. Soc.*, 301(1):77–101, 1987.
11. Nicholas M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.
12. Dermot McCarthy. Transformations of well-poised hypergeometric functions over finite fields. *Finite Fields Appl.*, 18(6):1133–1147, 2012.
13. David Roberts, Fernando Rodriguez-Villegas, and Mark Watkins. Motivic supercongruences.
14. Lucy Joan Slater. *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966.
15. Jürgen Wolfart. Werte hypergeometrischer Funktionen. *Invent. Math.*, 92(1):187–216, 1988.