Ruin probabilities with investments in a risky asset with the price given by a geometric Lévy process

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Abstract

- We study the asymptotic of the ruin probabilities in models where the capital reserve is invested in a risky asset. Our main interest is a model describing the evolution of capital of a venture company selling innovations and investing its reserve into a risky asset with the price given by a geometric Lévy process.
- We find the **exact** asymptotic of the ruin probabilities. Under some natural conditions it decays as a power function. The rate of decay is a positive root of equation determined by characteristics of the price process. When the price follows a gBm the results are reduced to those of our previous works where we used the method of ODEs assuming exponentially distributed jumps.
- Our proofs are based on the theory of distributional equations, in particular, on a recent result by Guivarc'h and Le Page.

Exit probabilities for linear equations

- $R = (R_t)_{t\geq 0}$, $P = (P_t)_{t\geq 0}$ are independent Lévy processes, $P_0 = R_0 = 0$, $\Delta R > -1$.
- $X = X^u$ describes the capital evolution ; it is given by the linear equation $dX = X_- dR + dP$, $X_0 = u$, i.e.

$$X_t = u + X_- \cdot R_t + P_t = u + \int_{]0,t]} X_{s-} dR_s + P_t, \qquad u \in \mathbb{R}_+.$$

- Since $[P, R] = 0$ we have the analog of the Cauchy formula

$$X = \mathcal{E}(R)(u + \mathcal{E}_-^{-1}(R) \cdot P) = \mathcal{E}(R)(u - Y),$$

  where the Doléans exponential $\mathcal{E}(R)$ is the price process, $\mathcal{E}(R) := 1 + \mathcal{E}_-(R) \cdot R$ and $Y := -\mathcal{E}_-^{-1}(R) \cdot P$.
- Ruin time : $\tau^u := \inf\{t : \; X_t^u \leq 0\}$, the exit time from $]0, \infty[$.
- Ruin probability : $\Psi(u) := \mathbf{P}(\tau^u < \infty)$.

Example : price process is a geometric Brownian motion

- $R_t = at + \sigma W_t$, where W is a Wiener process, $\sigma > 0$.
- P as in the Lundberg model, i.e. a compound Poisson with exponentially distributed jumps.
- Let $\beta := 2a/\sigma^2 - 1$.

Theorem

If $\beta > 0$, then $\Psi(u) \sim cu^{-\beta}$, $c > 0$.
If $\beta \le 0$, then $\Psi(u) \equiv 1$.

- Suppose that only a part $\gamma \in]0, 1]$ of the capital is invested in the risky asset. Replacing the parameters a and σ by $a\gamma$ and $\sigma\gamma$ we get that the ruin with probability one will be avoided only if $2a\gamma/(\sigma\gamma)^2 > 1$, i.e. when the share of the risky investment is strictly less than $2a/\sigma^2$.

Method of differential equations

Under some assumptions on the Lévy triplet one can prove that the ruin probability satisfies an integro-differential equation.
Let $R_t = at + \sigma W_t$ (the price process is gBM), $P_t = -ct + x * \mu_t^P$, $\nu^P(dt, dx) = \alpha F(dx)dt$ where F is a probability distribution on $]0, \infty[$, i.e. the Lévy measure $\Pi_P(dx) = \alpha F(dx)$.
Under some assumptions on F one can prove that the function $\Phi(u) = 1 - \Psi(u)$ is a solution of the following equation :

$$\frac{1}{2}\sigma^2 u^2 \Phi''(u) + (au - c)\Phi'(u) - \alpha\Phi(u) + \alpha \int_0^\infty \Phi(u+y)dF(y) = 0.$$

In the case where F is an exponential distribution with parameter μ one can eliminate the integral term and reduce the problem to the study of the ODE of the second order which is, in fact the ODE of the 2nd order for the derivative of Φ, see K&Pergamenshchikov(2015). The main difficulty in this approach is to prove smoothness of Φ. On the other hand it is to prove that Φ (almost) always satisfy the above equation in viscosity sense, see K&Belkina(2016).

Viscosity solutions, 1

The model

$$R_t = at + \sigma W_t + \int_0^t \int x(\mu - \nu)(ds, dx) = at + \sigma W_t + x * (\mu - \nu)_t,$$

where $|x| \wedge |x|^2 * \nu_t < \infty$, $\nu(dt, dx) = \Pi(dx)dt$, $\Pi(]-\infty, 0]) = 0$, $P_t = ct - x * \mu_t^P$, $\nu^P(dt, dx) = \alpha F(dx)dt$, $F(]-\infty, 0]) = 0$, i.e. P is a compound Poisson process with the distributions of claims F.

Theorem

The function Φ is a viscosity solution on $]0, \infty[$ of the integro-differential equation $\mathcal{L}\Phi(u) = 0$ where

$$\begin{aligned} \mathcal{L}f(u) \quad := \quad & \frac{1}{2}\sigma^2 u^2 f''(u) + (au + c)f'(u) \\ & + \int [f(u(1+z)) - f(u) - f'(u)uz]\Pi(dz) \\ & + \alpha \int [f(u-z) - f(u)]dF(z) \end{aligned}$$

- The operator $\mathcal{L}$ is well-defined for $f \in C_b^2(u)$, the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ equal to zero on $]-\infty, 0]$ and two times continuously differentiable in the classical sense in a neighborhood of the point $u \in]0, \infty[$.
- A function $\Phi :]0, \infty[\to [0, 1]$ is called *viscosity supersolution* if for every point $u \in]0, \infty[$ and every function $f \in C_b^2(u)$ such that $\Phi(u) = f(u)$ and $\Phi \geq f$ the inequality $\mathcal{L}f(u) \leq 0$ holds.
- A function $\Phi :]0, \infty[\to [0, 1]$ is called *viscosity subsolution* if for every $u \in]0, \infty[$ and every function $f \in C_b^2(u)$ such that $\Phi(u) = f(u)$ and $\Phi \leq f$ the inequality $\mathcal{L}f(u) \geq 0$ holds.
- A function $\Phi :]0, \infty[\to [0, 1]$ is a *viscosity solution* of if Φ is simultaneously a viscosity super- and subsolution.

Theorem

Suppose that the support of $F(dz)$ is $\mathbb{R}_+ \setminus \{0\}$. Let Φ and $\tilde{\Phi}$ be two continuous bounded viscosity solutions with the boundary conditions $\Phi(+0) = \tilde{\Phi}(+0)$ and $\Phi(\infty) = \tilde{\Phi}(\infty)$. Then $\Phi = \tilde{\Phi}$.

Lévy triplets

- (a, σ^2, Π), (a_P, σ_P^2, Π_P) are the Lévy triplets of R, P, i.e.

$$R_t = at + \sigma W_t + h * (\mu - \nu)_t + \bar{h} * \mu_t,$$

compensator $\nu(dt, dx) = dt\Pi(dx)$ with $\Pi(|x|^2 \wedge 1) < \infty$, truncation function $h(x) := xI_{\{|x| \le 1\}}$, $\bar{h}(x) := xI_{\{|x|>1\}}$.
The process P has a similar representation with W^P, μ^P, ν^P.

To exclude trivial or known cases we work under the following

- **Assumption** $\Pi(]-\infty, -1]) = 0$; σ^2 *and* Π *do not vanish simultaneously,* P *is not a subordinator.*

- Stochastic exponential can be written as the usual one :

$$X_t^u = e^{V_t}(u - Y_t),$$

$$Y_t := -\int_{]0,t]} \mathcal{E}_{s-}^{-1}(R)dP_s = -\int_{]0,t]} e^{-V_{s-}}dP_s.$$

Cumulant generating function of log price process

The Lévy process V (the log price) has the form

$$V_t = at - (1/2)\sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1+x) - h) * \mu_t.$$

and has the triplet $(a_V, \sigma^2, \Pi\varphi^{-1})$ where $\varphi : x \mapsto \ln(1+x)$,

$$a_V = a - (1/2)\sigma^2 + \Pi(h(\ln(1+x)) - h).$$

For r.v. V_1 the cumulant generating function (always convex)

$$H(q) := \ln \mathbf{E}\, e^{-qV_1} = -a_V q + \frac{\sigma^2}{2} q^2 + \Pi\big(e^{-q\ln(1+x)} - 1 + qh(\ln(1+x))\big).$$

Put $\underline{q} := \inf\{q \le 0 :\ H(q) < \infty\}$, $\bar{q} := \sup\{q \ge 0 :\ H(q) < \infty\}$.

Main result

Theorem

if H has a root $\beta > 0$, $\beta \notin \partial \mathrm{dom}\, H$, and $\Pi_P(|\bar{h}|^\beta) < \infty$, then

$$0 < \liminf_{u \to \infty} u^\beta \Psi(u) \le \limsup_{u \to \infty} u^\beta \Psi(u) < \infty. \qquad (1)$$

If, moreover, $\Pi_P(]-\infty, 0[) = 0$ and the law $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$, then $\Psi(u) \sim C_\infty u^{-\beta}$, $C_\infty > 0$.

Thus, for the model with upward jumps we have an exact asymptotic if the distribution of the increment of log-price process is non-arithmetic, i.e. is not concentrated on the set $\mathbb{Z}d := \{\pm nd, n = 0, 1, \ldots\}$, $d > 0$.

Reduction

So, $X^u = e^V(u - Y)$. Obviously, $\tau^u = \inf\{t \geq 0 : \; Y_t \geq u\}$. Put $G(u) := \mathbf{P}(Y_\infty > u)$.

Lemma

If $Y_t \to Y_\infty$ a.s. where Y_∞ is finite and unbounded from above, then for all $u > 0$

$$G(u) \leq \; \Psi(u) = \frac{G(u)}{\mathbf{E}\left(G(X_{\tau^u}) \,|\, \tau^u < \infty\right)} \leq \frac{G(u)}{G(0)}.$$

In particular, if $\Pi_P(]-\infty, 0]) = 0$, then $\Psi(u) = G(u)/G(0)$.

Proof. Let τ be a stopping time, ξ be a $\mathcal{F}_\tau$-measurable r.v.,

$$Y_{\tau,\infty} := \begin{cases} -\lim_{N\to\infty} \int_{]\tau,\tau+N]} e^{-(V_{t-} - V_\tau)} dP_t, & \tau < \infty, \\ 0, & \tau = \infty. \end{cases}$$

is well defined.

On the set $\{\tau < \infty\}$

$$Y_{\tau,\infty} = e^{V_\tau}(Y_\infty - Y_\tau) = X_\tau + e^{V_\tau}(Y_\infty - u).$$

The Lévy process Y starts afresh at τ, i.e. the conditional law of $Y_{\tau,\infty}$ given $(\tau,\xi) = (t,x) \in \mathbb{R}_+ \times \mathbb{R}$ coincides with $\mathcal{L}(Y_\infty)$ and

$$\mathbf{P}\left(Y_{\tau,\infty} > \xi,\ \tau < \infty\right) = \mathbf{E}\, G(\xi)\, I_{\{\tau<\infty\}}.$$

Thus, if $\mathbf{P}(\tau < \infty) > 0$, then

$$\mathbf{P}\left(Y_{\tau,\infty} > \xi,\ \tau < \infty\right) = \mathbf{E}\left(G(\xi)\,|\,\tau < \infty\right)\, \mathbf{P}(\tau < \infty)\,.$$

Noting that $\Psi(u) := \mathbf{P}(\tau^u < \infty) \geq \mathbf{P}(Y_\infty > u) > 0$, we get that

$$\begin{aligned} G(u) &= \mathbf{P}\left(Y_\infty > u,\ \tau^u < \infty\right) = \mathbf{P}\left(Y_{\tau^u,\infty} > X_{\tau^u},\ \tau^u < \infty\right) \\ &= \mathbf{E}\left(G(X_{\tau^u})\,|\,\tau^u < \infty\right)\, \mathbf{P}(\tau^u < \infty). \end{aligned}$$

The result follows since $X_{\tau^u} \leq 0$ on the set $\{\tau^u < \infty\}$. In the case where $\Pi_P(]-\infty,0]) = 0$, the process X^u crosses zero in a continuous way, i.e. $X_{\tau^u} = 0$ on this set.

Reduction to distributional equations

Put

$$M_n := e^{-(V_n - V_{n-1})}, \qquad Q_n := -\int_{]n-1,n]} e^{-(V_t - V_{n-1})} dP_t.$$

Clearly, $\mathcal{L}(M_1, Q_1) = \mathcal{L}(M_n, Q_n)$ and

$$Y_n = -e^{-V_{n-1}} \int_{]0,n]} e^{-(V_t - V_{n-1})} dP_t = Q_1 + M_1 Q_2 + M_1 M_2 Q_3 + \dots$$

Thus, $Y_n = Q_1 + M_1 Y_{n-1,1}$ where

$$Y_{n-1,1} := Q_2 + M_2 Q_3 + \dots M_2 \dots M_{n-1} Q_n.$$

Suppose that $Y_n \to Y_\infty$. Then $Y_\infty = Q_1 + M_1 Y_{\infty,1}$ a.s. Hence,

$$Y_\infty \overset{d}{=} Q + M\, Y_\infty\,, \quad Y_\infty \text{ independent of } (M, Q).$$

Implicit renewal theory

Let $M > 0$ be such that $\mathcal{L}(\ln M)$ is non-arithmetic and

$$\mathbf{E}\, M^\beta = 1\,, \quad \mathbf{E}\, M^\beta\, (\ln M)^+ < \infty \qquad \text{for some } \beta > 0.$$

Then $\ln \mathbf{E} M \in [-\infty, 0[$ and $\kappa := \mathbf{E} M^\beta \ln M \in]0, \infty[$.

Lemma (Goldie, 1991)

Let M satisfies the conditions above, $\mathbf{E}\,|Q|^\beta < \infty$. Then the distributional equation $Y_\infty \overset{d}{=} Q + M\, Y_\infty$, Y_∞ independent of (M, Q) has a unique solution Y_∞ and

$$\lim_{u\to\infty} u^\beta\, \mathbf{P}(Y_\infty > u) = C_+ := \frac{1}{\beta\,\kappa}\, \mathbf{E}\, \left(((Q + MY_\infty)^+)^\beta - ((MY_\infty)^+)^\beta\right).$$

Lemma (Guivarc'h, Le Page, 2015 ; Buraczewski, Damek, 2017)

$$C_+ > 0 \;\Leftrightarrow\; Y_\infty \;\; \textit{unbounded from above.}$$

Moments of the maximal function $Y_1^* := \sup_{t \le 1} |Y_t|$

Lemma

If $\Pi_P(|\bar{h}|^p) + \mathbf{E} \sup_{t \le 1} e^{-pV_t} < \infty$ *for* $p > 0$*, then* $\mathbf{E}\, Y_1^{*p} < \infty$*.*

The proof follows from the Alex Novikov inequalities for the integral $I = g * (\mu^P - \nu^P)$ where $g^2 * \nu_1^P < \infty$. In dependence of the parameter $\alpha \in [1,2]$ they have the following form :

$$\mathbf{E} I_1^{*p} \le C_{p,\alpha} \begin{cases} \mathbf{E}\left(|g|^\alpha * \nu_1^P\right)^{p/\alpha}, & \forall\, p \in]0,\alpha], \\ \mathbf{E}\left(|g|^\alpha * \nu_1^P\right)^{p/\alpha} + \mathbf{E}\,|g|^p * \nu_1^P, & \forall\, p \in [\alpha, \infty[. \end{cases}$$

If $H(q) < \infty$, then the process $m_t(q) := e^{-qV_t - tH(q)}$ is a martingale and $\mathbf{E}\, e^{-qV_t} = e^{tH(q)}$, $t \in [0,1]$. From this it is easy to deduce that

$$\mathbf{E} \sup_{t \le 1} e^{-pV_t} < \infty \qquad \forall\, p \in]\underline{q}, \bar{q}[.$$

Convergence of Y

The convergence Y_t as $t \to \infty$ can be easily established under very weak assumptions.

Proposition

If there is $p > 0$ such that $H(p) < 0$, and $\Pi_P(|\bar{h}|^p) < \infty$, then Y_t converge a.s. to a finite r.v. Y_∞ unbounded from above and solving the distributional equation

$$Y_\infty \overset{d}{=} Y_1 + M_1 \, Y_\infty \, , \quad Y_\infty \text{ independent of } (M_1, Y_1).$$

Proof. We assume wlg that $p < 1$ and $H(p+) \neq \infty$. For $j \geq 2$

$$Y_j - Y_{j-1} = M_1 \dots M_{j-1} Q_j, \qquad .$$

Since $\rho := \mathbf{E} M_1^p = e^{H(p)} < 1$ and $\mathbf{E} M_1 \dots M_{j-1} |Q_j| = \rho^j \mathbf{E} |Y_1|^p$, we have $\mathbf{E} \sum_{j\geq 1} |Y_j - Y_{j-1}|^p < \infty$. Hence, $\sum_{j\geq 1} |Y_j - Y_{j-1}|^p < \infty$ a.s. But then $\sum_{j\geq 1} |Y_j - Y_{j-1}| < \infty$ a.s. and the sequence Y_n converges to some finite random variable Y_∞.

Put

$$\Delta_n := \sup_{n-1\le v\le n} \left| \int_{]n-1,v]} e^{-V_{s-}}\, dP_s \right|, \qquad n \ge 1.$$

Note that

$$\mathbf{E}\,\Delta_n^p = \mathbf{E} \prod_{j=1}^{n-1} M_j^p \sup_{n-1\le v\le n} \left| \int_{]n-1,v]} e^{-(V_{s-}-V_{n-1})}\, dP_s \right|^p = \rho^{n-1}\, \mathbf{E}\, Y_1^{*p} < \infty.$$

For any $\varepsilon > 0$ we get using the Chebyshev inequality that

$$\sum_{n\ge 1} \mathbf{P}(\Delta_n > \varepsilon) \le \varepsilon^{-p} \mathbf{E}\, Y_1^{*p} \sum_{n\ge 1} \rho^{n-1} < \infty.$$

By the Borel–Cantelli lemma $\Delta_n(\omega) \le \varepsilon$ for all $n \ge n_0(\omega)$ for each ω except a null-set. This implies the convergence $Y_t \to Y_\infty$ a.s.

Let us consider the sequence

$$Y_{1,n} := Q_2 + M_2 Q_3 + \cdots + M_2 \dots M_n Q_{n+1}$$

converging a.s. to a random variable $Y_{1,\infty}$ distributed as Y_∞. Passing to the limit in the obvious identity $Y_n = Q_1 + M_1 Y_{1,n-1}$ we obtain that $Y_\infty = Q_1 + M_1 Y_{1,\infty}$. For finite n the random variables $Y_{1,n}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,n}) = \mathcal{L}(Y_n)$. Hence, $Y_{1,\infty}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,\infty}) = \mathcal{L}(Y_\infty)$ and $\mathcal{L}(Y_\infty) = \mathcal{L}(Q_1 + M_1 Y_{1,\infty})$.

It remains to check that Y_∞ is unbounded from above.

Ruin with probability one

Proposition

Suppose that $\mathbf{E}M_1^{-\delta} < 1$ *and* $\mathbf{E}M_1^{-\delta}|Q_1|^\delta < \infty$ *for some* $\delta \in]0,1[$ *and* Q_1 *is unbounded from above. Then* $\Psi(u) \equiv 1$.

More specific conditions of the ruin almost surely in terms of triplets :

Theorem

Suppose that $0 \in \operatorname{int}\operatorname{dom} H$ *and* $\Pi_P(|\bar{h}|^\varepsilon) < \infty$ *for some* $\varepsilon > 0$. *If* $a_V + \Pi(\bar{h}(\ln(1+x))) \le 0$, *then* $\Psi(u) \equiv 1$.

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