On Risk Aggregation

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Based on 2 studies:
- *Validation of Aggregated Risks Models* with **Michel DACOROGNA** (DEAR-Consulting) & **Laila ELBAHTOURI** (SCOR)
- *Risk concentration under second order MRV* with **Bikramjit DAS** (SUTD Singapore)

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Introduction

▷ Main motivation: study of risk concentration, crucial question in risk management, in order to measure the diversification benefit of a risk portfolio but also in view of capital allocation to each risk of the portfolio.

It can be done with various risk measures and relies mainly on the coherence property, in particular the subadditivity, of the chosen risk measure.

Goal: to study the diversification benefit for a portfolio of dependent risks

- when choosing copula models that are used in practice: we provide analytical results for any threshold of the risk measure and compare them with results obtained via Monte-Carlo method;
- when assuming a multivariate second order regular variation condition on the risks: we provide asymptotic (high threshold) risk concentration results; it allows one to consider examples and results of the literature under a broad umbrella

▷ Definition and notation:

- Popular risk measures:
  variance; Value-at-Risk (VaR) = quantile q;
  Expected Shortfall (ES) =Tail VaR (TVaR)

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 q_\beta(L)\, d\beta \underset{F_L \text{ cont}}{=} \mathrm{E}[L \mid L \geq q_\alpha(L)]$$

- Risk concentration measures - diversification indices :
  Indices have been introduced to quantify and compare the diversification of portfolios, such as the closely related notions of diversification benefit defined by Bürgi et al. (2008) as

$$1 - \frac{\tilde{\rho}(\sum_{i=1}^d X_i)}{\sum_{i=1}^d \tilde{\rho}(X_i)}, \quad \text{with } \tilde{\rho}(\cdot) := \rho(\cdot) - \mathbb{E}(\cdot),$$

  and the diversification index (or risk concentration) defined by Tasche (2008) as,

$$\rho(\sum_{i=1}^d X_i) \,/\, \sum_{i=1}^d \rho(X_i)$$

  for n risks $(X_i, i = 1, \cdots, n)$, ρ denoting a risk measure.

I - Aggregate dependent risks

(Analytical study with M. Dacorogna & L. Elbahtouri (SCOR))

- Aim: to provide explicit expressions for the cdf of aggregated dependent risks, allowing to compute analytically risk measures ρ_α as VaR_α or TVaR_α, and consequently the diversification benefit in order to bypass MC simulations of a large number of dependent risks.

  $\hookrightarrow$ this analytical approach: a way to measure the performance of the MC method, and potentially to replace it in some cases.

- Method based on mixing techniques developed by Oakes-Marshall-Olkin (O.M.O.) (88) to construct multivariate Archimedean copulas.

- Examples developed with Archimedean survival copulas and heavy tailed marginals, to test the numerical convergence against the analytical result.

- Main idea of the O.M.O. method: introduce a latent variable to transform dependent variables into conditionally independent ones, and express the dependence between the variables as an Archimedean survival copula with parameter the latent variable, and obtain the marginal distributions depending on this parameter.
  Theorem by Oakes-Marshall-Olkin
  Let Θ be a positive random variable (rv) with cumulative distribution function (cdf) F_Θ and X_k, $k \geq 1$, be random variables (rv) such that

$$P\left(X_1 > x_1, ..., X_n > x_n \mid \Theta = \theta\right) = \prod_{k=1}^{n} H(x_k)^\theta$$

H being a positive function. This dependence model is a variant of the structure dependence generated by an Archimedean survival copula with generator $\phi = L_\Theta^{-1}$, where L_Θ denotes the Laplace transform of F_Θ:

$$P\left[X_1 > x_1, \ldots, X_n > x_n\right] = L_\Theta\Big(\sum_{i=1}^{n} L_\Theta^{-1}\big(\bar{F}_i(x_i)\big)\Big)$$

where the marginal distributions F_k of X_k, $k = 1, \ldots, n$, are defined by $\overline{F}_k(x) := 1 - F_k(x) = L_\Theta\big(-\ln H(x)\big)$.

▷ Example: dependent model with Pareto marginals and Clayton copula

- marginals F_i $(i = 1, \ldots, n)$: Pareto(α, β)-distributed
  $\overline{F_i}(x) := 1 - F_i(x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}$, $x > 0$, with $\alpha > 1$, $\beta > 0$;
- dependence structure: Clayton copula with parameter $\theta > 0$, defined on $[0,1]^n$ via its generator $\varphi_\theta(t) = t^{-\theta} - 1$ ($t \in [0;1]$) by

$$C_\theta\left(u_1, \cdots, u_n\right) = \varphi_\theta^{-1}\left(\sum_{i=1}^n \varphi_\theta(u_i)\right) = \left(\sum_{i=1}^n u_i^{-\theta} - (n-1)\right)^{-1/\theta}$$

Computing directly the pdf f_n of the aggregate risk $S_n = \sum_{i=1}^n X_i$ may be a difficult task, hence the choice of using the mixing technique to work with conditional independence.

- Choose $H(x) = e^{-x}$ $(x > 0)$, $\Theta \sim \Gamma(\alpha; \beta)$, then
  $L_\Theta(t) = \left(1 + \frac{t}{\beta}\right)^{-\alpha}$

  $\Rightarrow \quad \overline{F}_i(x) = L_\Theta(-\ln H(x)) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}$ i.e.survival cdf of Pareto(α, β)

  we get back the marginal distributions of our model

  and $\quad \phi(t) = L_\Theta(t)^{-1} = \beta\left(t^{-1/\alpha} - 1\right),$

  i.e. the generator of a Clayton copula with $\theta = 1/\alpha$

- Apply O.M.O. theorem to obtain: $(X_i \mid \Theta = \theta, i = 1, \cdots, n)$ i.i.d. exponentially distributed with parameter θ
  $\Rightarrow S_n \mid (\Theta = \theta) \sim \Gamma(n, \theta)$

- Use that $\quad f_n(s) = \int_0^\infty f_{S_n|\Theta}(s)\, f_\Theta(\theta)\, d\theta$

Hence the **result**: the pdf f_n of the aggregate risk S_n associated with the model Pareto(α, β)-marginals and Clayton$(1/\alpha)$ copula, is given by

$$f_n(s) = \frac{\beta^\alpha}{B(\alpha, n)} \times \frac{s^{n-1}}{(\beta + s)^{\alpha+n}}, \quad s > 0$$

Result. The TVaR of S_n at threshold κ is given by

$$TVaR_n = \frac{\beta}{(1-\kappa)\,B(\alpha,n)}\,B\left(\left(1+\frac{q_n}{\beta}\right)^{-1};\alpha-1,n+1\right)$$

where $B(x;a,b)$ denotes the incomplete Beta function defined by $B(x;a,b)=\int_0^x u^{a-1}(1-u)^{b-1}$, and $q_n = VaR_\kappa(S_n)$.
In particular,

$$TVaR_2 = \frac{\beta^\alpha}{(1-\kappa)\,(\alpha-1)} \times \frac{\alpha(1+\alpha)q_2^2+2\beta(1+\alpha)q_2+2\beta^2}{(q_2+\beta)^{1+\alpha}}.$$

Consequence. The diversification benefit D_n of the aggregate risk S_n at threshold κ can be expressed, for our model and the risk measure ρ, as:

(i) For ρ=VaR:

$$D_n = 1 - \frac{(q_n - n\,\mathbb{E}(X))}{n\,(q_1 - \mathbb{E}(X))} = 1 - \frac{\frac{1}{n\beta} q_n - \frac{1}{\alpha-1}}{(1-\kappa)^{-1/\alpha} - \frac{\alpha}{\alpha-1}}$$

with $q_1 = \beta\,(1-\kappa)^{-1/\alpha} - 1$.

(ii) For ρ=TVaR:

$$D_n = 1 - \frac{\frac{(\alpha-1)}{n(1-\kappa)B(\alpha,n)} B\left(\left(1+\frac{q_n}{\beta}\right)^{-1}; \alpha-1, n+1\right) - 1}{\alpha\left((1-\kappa)^{-1/\alpha} - 1\right)}$$

which simplifies, for $n = 2$, to:

$$D_2 = 1 - \frac{\frac{\beta^{\alpha-1}}{2(1-\kappa)} \times \frac{\alpha(1+\alpha)\,q_2^2 + 2\beta(1+\alpha)\,q_2 + 2\beta^2}{(q_2+\beta)^{1+\alpha}} - 1}{\alpha\left((1-\kappa)^{-1/\alpha} - 1\right)}.$$

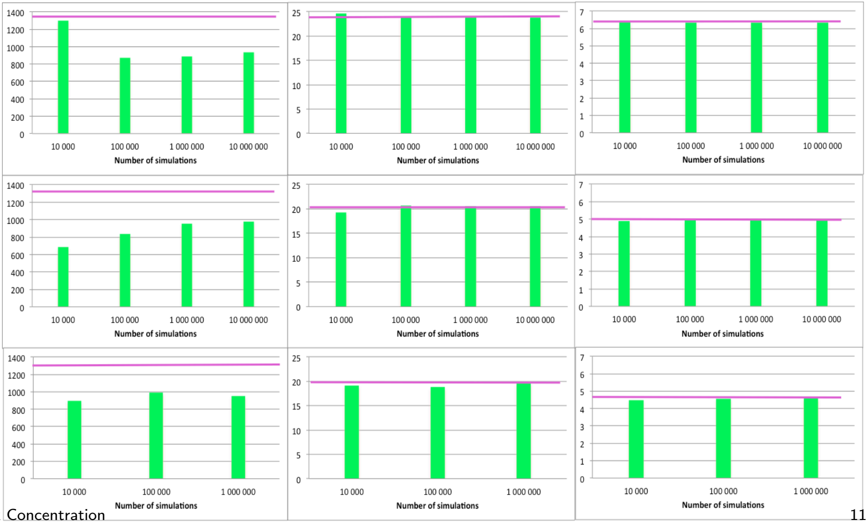
▷ Numerical application and discussion

Cases : $(\alpha = 3, \theta = 1/3)$ i.e. moderate tail and dependence
$(\alpha = 2, \theta = 1/2)$ heavier tail
$(\alpha = 1, \theta \approx 0.91)$ i.e. very heavy tail (e.g. earthquake distrib
and a relatively strong dependence (θ)

We run 10 sets of simulations (changing the seed of the random generator) for each case, varying the number of simulations per run from 10'000 to 10 million.

We report here the average value over the 10 sets of simulations. It is worth noticing that the standard deviation of those sets decreases, as expected, with the number of simulations.

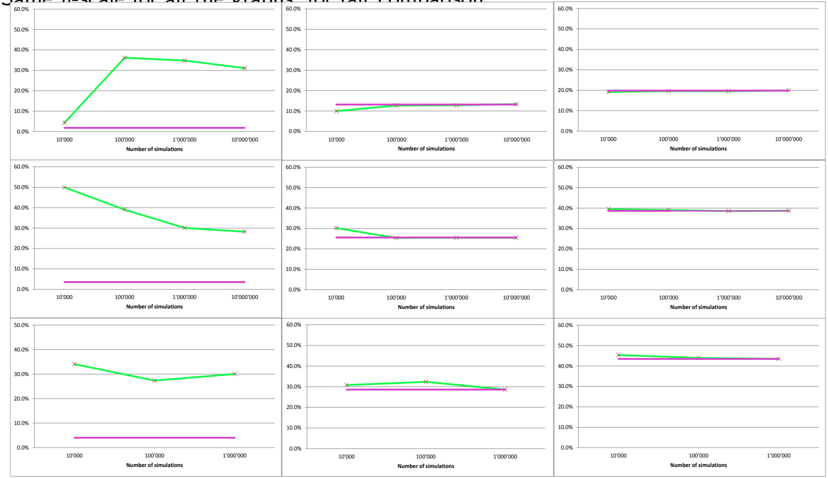
Convergence of the TVaR of S_n at 99.5% for $\alpha = 1.1,\ 2,\ 3$ from left to right, for an aggregation factor $n = 2,\ 10,\ 100$ from up to down. The purple line corresponds to the analytical value; the green plots are the average values obtained from the MC simulations. The y-scale gives the normalized TVaR_n/n (the same for each column)



▷ Comments

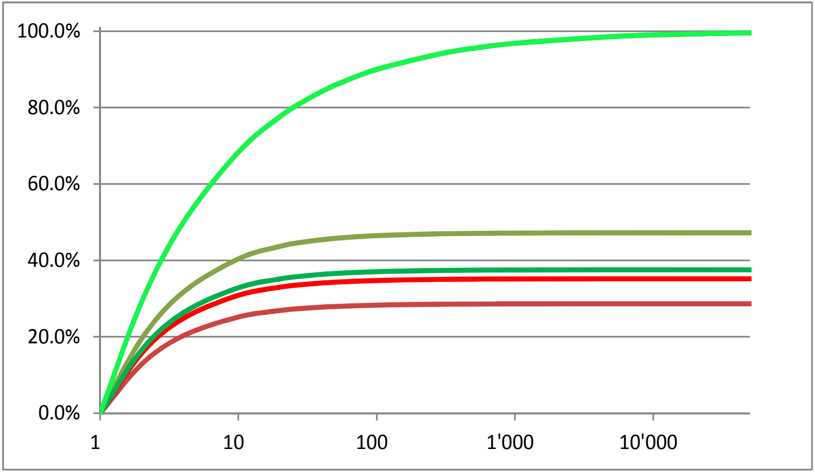
- The normalized TVAR of S_n, $TVaR_n/n$, decreases as n increases
- The TVaR decreases as α increases
- The rate of convergence of $TVaR_n/n$ increases with n
- The heavier the tail, the slower the convergence
- In the case of very heavy tail and strong dependence ($\alpha = 1.1$ and $\theta = 0.91$), we do not see any satisfactory convergence, even with 10 million simulations, and for any n
- When $\alpha = 2, 3$, the convergence is good from 1 million, 100'000 simulations onwards, respectively.
- Via the analytical expression
  - Beside the gain in precision, the analytical formula can be numerically evaluated 40 times faster, resp. 580 times faster (for $\alpha = 2$ and $n = 10$, resp. $n = 100$ for 1 million simulation) than the estimation given by Monte Carlo simulations
  - Hence, bypassing Monte Carlo simulation is a considerable gain in time and precision for heavy tail

Convergence of the diversification benefit of S_n (associated with TVaR at 99.5%) for $\alpha = 1.1, 2, 3$ (from left to right), for $n = 2, 10, 100$ (from up to down). Dark lines for the analytical values, light ones for average values obtained from the MC simulations. Same y-scale for all the graphs, for fair comparison.



Similar pattern as for the TVaR.

Plot of the diversification benefit D_n (y-axis) as a function of n (x-axis), for $TVaR_\kappa$ with $\kappa = 99.5\%$. The curves from bottom to up correspond, respectively, to Pareto ($\alpha = 2$)-Clayton ($\theta = 1/2$), Gauss-Gauss (Clayton; $r = 0.42$), Gauss-Gauss (Gumbel; $r = 0.39$), Weibulll ($\tau = 1/2$)-Gumbel ($\theta = 2$), independent Pareto ($\alpha = 2$)



II- Asymptotic (high threshold) risk concentration

Study with B. Das (SUTD)

1 - Various notions of Regular Variation

▷ Notation - definitions

Dimension 1

- *Regular variation* (Karamata)

  A random variable X is $\mathcal{RV}_{-\alpha}$ $\Leftrightarrow \lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-\alpha}\ (x>0)$

  $$\Leftrightarrow \frac{\mathbb{P}[t^{-1}X \in \cdot]}{\mathbb{P}[X>t]} \underset{t\to\infty}{\overset{v}{\longrightarrow}} \mu_\alpha(.)$$

  with $\mu_\alpha(dx) = \alpha x^{-\alpha-1}dx$.

- *Second order regular variation* (see e.g. Resnick (2000))
  $X \in 2\mathcal{RV}_{-\alpha,\rho}(b,A)$ if $\exists\, b(\cdot) \in \mathcal{RV}_{1/\alpha}$, $A(t) \underset{t\to\infty}{\longrightarrow} 0$ ultimately of the same sign and $|A(t)| \in \mathcal{RV}_\rho$ with $\rho \le 0$ such that

  $$\frac{t\overline{F}(b(t)x) - x^{-\alpha}}{A(b(t))} \underset{t\to\infty}{\longrightarrow} c x^{-\alpha}\frac{x^\rho - 1}{\rho}$$

Another equivalent definition of 2RV (De Haan & Ferreira (2006)):
$\overline{F}$ is $2\mathcal{RV}_{-\alpha,\rho}(A)$ if there exists an ultimately positive or negative function A with $A(t) \to 0$ as $t \to \infty$ such that

$$\frac{\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\alpha}}{A(t)} \underset{t\to\infty}{\longrightarrow} c x^{-\alpha} \frac{x^{\rho} - 1}{\rho}.$$

The function A coincides in both definitions, as well as α and ρ.

Dimension $d > 1$

- *Multivariate regular variation* (see e.g. Resnick)

  A random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ in a cone $[0, \infty)^d$ is *multivariate regularly varying* ($\boldsymbol{X} \in \mathcal{MRV}_{-\alpha}(b)$) with limit measure ν if $\exists\, b(t) \uparrow \infty$ and a Radon measure $\nu \neq 0$ s.t. on $\mathbb{E} = [0, \infty]^d \setminus \{0\}$

  $$t\mathbb{P}\left[\frac{\boldsymbol{X}}{b(t)} \in \ \cdot\ \right] \stackrel{v}{\to} \nu(.)$$

  We can check that $b(\cdot) \in \mathcal{RV}_{1/\alpha}$, and that ν is homogeneous: $\nu(cA) = c^{-\alpha}\nu(A)$, $c > 0$, for $\alpha \geq 0$, relatively compact A, bounded away from 0, $A \subset \mathbb{E}$.

Another way: via polar coordinates (see e.g. R. Davis)
There exists a random vector $\mathbf{\Theta} \in S^{d-1}$ (unit sphere in $\mathbb{R}^d$) s.t. we have the following vague convergence on S^{d-1} as $t \to \infty$:

$$\frac{\mathbb{P}\left(|\boldsymbol{X}| > tx,\ \boldsymbol{X}/|\boldsymbol{X}| \in \bullet\right)}{\mathbb{P}\left(|\boldsymbol{X}| > t\right)} \underset{v}{\longrightarrow} x^{-\alpha}\, \mathbb{P}[\mathbf{\Theta} \in \bullet].$$

$\mathbb{P}[\mathbf{\Theta} \in \bullet]$ called the spectral measure ; $\alpha =$ index of $\boldsymbol{X}$

- *Second order multivariate regular variation* (see e.g. Resnick)

  $\boldsymbol{X}$ is *second order regularly varying* with parameters $\alpha \geq 0$ and $\rho \leq 0$, $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}(b, A, \nu, H)$, if $\boldsymbol{X} \in \mathcal{MRV}_{-\alpha}(b)$ and $\exists\, A(t) \underset{t\to\infty}{\to} 0$, ultimately of constant sign with $|A| \in \mathcal{RV}_\rho, \rho \leq 0$, such that

$$\frac{t\mathbb{P}\left(\frac{\boldsymbol{X}}{b(t)} \in [\mathbf{0}, \boldsymbol{x}]^c\right) - \nu([\mathbf{0}, \boldsymbol{x}]^c)}{A(b(t))} \to H(\boldsymbol{x})$$

  locally uniformly in $\boldsymbol{x} \in (0, \infty]^d \setminus \{\boldsymbol{\infty}\}$, and with $0 \neq H(\boldsymbol{x}) < \infty$ (no need to be a distribution, but can be related to a sign measure)

▷ Relating Univariate & Multivariate Second Order Regular Variation

- Recall that $\boldsymbol{X} \in \mathcal{MRV}_{-\alpha}(b) \quad \Rightarrow \quad S_d := \sum_{i=1}^{d} X_i \in \mathcal{RV}_{-\alpha}$
- Extension to the case where $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}$:

  Proposition.

  $$\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}(b, A, \nu, H) \ \Rightarrow \ S_d \in 2\mathcal{RV}_{-\alpha,\rho}(b_d, A_d, H_d)$$

  with $b_d(t) := (\nu(\Gamma_d))^{1/\alpha} b(t)$, $A_d(t) := A((\nu(\Gamma_d))^{-1/\alpha}\, t)$, $\forall t > 0$,

  $H_d(x) = \chi(x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d) = c_d\, x^{-\alpha}\dfrac{x^\rho - 1}{\rho}$ with

  $c_d = \dfrac{\rho 2^\alpha}{2^\rho - 1}\chi(2(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)$, $\chi([\mathbf{0}, \boldsymbol{x}]^c) = H(\boldsymbol{x})$, and

  $\Gamma_d = \{\boldsymbol{z} \in \mathbb{R}^d : z_1 + z_2 + \ldots + z_d > 1\}$.

Remarks:

- $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}(b, A, \nu, H) \Rightarrow \chi(k\Gamma_d) \neq 0$ for some $k > 0$ $\Rightarrow c_d \neq 0$
  Reverse not true: we can construct examples where $c_d = 0$, yet $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}(b, A, \nu, H)$ holds.
- $\boldsymbol{X} \in \mathcal{MRV}_{-\alpha} \Rightarrow$ at least one of the margins is $\mathcal{RV}_{-\alpha}$
  But $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho} \not\Rightarrow$ one of the components is $2\mathcal{RV}$.
  Ex: if the components of $\boldsymbol{X}$ are iid Pareto(α)- type 1, then $\boldsymbol{X} \in 2\mathcal{MRV}_{\alpha}$, although none of the margins are $2\mathcal{RV}$.
- The reverse implication of the Proposition, namely *if any convex combination of $\boldsymbol{X}$ is $2\mathcal{RV}$, then $\boldsymbol{X} \in 2\mathcal{MRV}$*, requires more conditions on the random variables to hold.
  (See e.g. Basrak, Davis & Mikosch (2002) for the conditions that allows this to happen for $\mathcal{RV}$ random vectors (not necessarily $2\mathcal{RV}$ or $2\mathcal{MRV}$).

Example of different degeneracies under $2\mathcal{RV}$

Take $X_1, X_2, \ldots, X_n$ iid $2\mathcal{RV}_{-\alpha,\rho}$. Then $\sum_{i=1}^{n} X_i \in 2\mathcal{RV}_{-\alpha,\rho}$ (Mao & Hu, 2013). But can we say $\boldsymbol{X} \in 2\mathcal{MRV}$?

Not always true!

Ex: X_1, X_2 iid rv with distribution function F ($\in$ Hall-Welsh class of heavy-tailed distributions) s.t.

$$1 - F(x) = \frac{1}{2} x^{-\alpha}(1 + x^{\rho}), \quad x \geq 1, \quad \alpha > 0,\, \rho < 0.$$

For any $\alpha > 0$ and $\rho < 0$, $X_1 \in 2\mathcal{RV}_{-\alpha,\rho}(b, A)$ where $b(t) = t^{1/\alpha}$ and $A(t) = t^{\rho}$.

Take a set of the form $[0, (x_1, x_2)]^c$ for $x_1 > 0, x_2 > 0$ and observe that as $t \to \infty$,

$$t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{t^{1/\alpha}} \in [0, (x_1, x_2)]^c\right) \to \frac{1}{2}\left(\frac{1}{x_1^{\alpha}} + \frac{1}{x_2^{\alpha}}\right) =: \nu([0, (x_1, x_2)]^c).$$

At the second level

$$\frac{t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{t^{1/\alpha}} \in [0,(x_1,x_2)]^c\right) - \frac{1}{2}\left(\frac{1}{x_1^\alpha}+\frac{1}{x_2^\alpha}\right)}{t^{\rho/\alpha}} = H^*(x_1,x_2,t)$$

and

$$\lim_{t\to\infty} H^*(x_1,x_2,t) = \begin{cases} \frac{1}{2}(x_1^{-\alpha+\rho}+x_2^{-\alpha+\rho}) & \text{if } \rho+\alpha>0 \\ \frac{1}{2}(x_1^{-\alpha+\rho}+x_2^{-\alpha+\rho}) - \frac{1}{4}x_1^{-\alpha}x_2^{-\alpha} & \text{if } \rho+\alpha=0 \\ -\infty & \text{if } \rho+\alpha<0. \end{cases}$$

Hence $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}$ iff $\alpha+\rho\geq 0$. We can check that no other choice of $A(\cdot)$ (up to equivalent tail behavior) would provide a finite limit as $t\to\infty$.

Main steps of the proof:

- Using a result by Resnick, $\boldsymbol{X}$ being $2\mathcal{MRV}-\alpha, \rho(b, A, \nu, H)$,

$$\frac{t\mathbb{P}\left(\frac{\boldsymbol{X}}{b(t)} \in \Lambda_d\right) - \nu(\Lambda_d)}{A(b(t))} \to \chi(\Lambda_d)$$

for any relatively compact $\Lambda_d \subset \mathbb{E}$, and$\chi$ s.t. $\chi([\mathbf{0}, \boldsymbol{x}]^c) = H(\boldsymbol{x})$.

- Define $b_d(t) = (\nu(\Gamma_d))^{1/\alpha} b(t)$. Then

$$t\,\mathbb{P}\left(\frac{S_d}{b_d(t)} > x\right) = t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{b(t)} \in x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d\right) \underset{t\to\infty}{\longrightarrow} \nu\left(x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d\right) = x^{-\alpha}$$

- Let $A_d(t) = A((\nu(\Gamma_d))^{-1/\alpha} t)$ for $t > 0$. Then,

$$\frac{t\,\mathbb{P}\left(\frac{S_d}{b_d(t)} > x\right) - x^{-\alpha}}{A_d(b_d(t))} = \frac{t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{b(t)} \in x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d\right) - \nu(x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)}{A(b(t))}$$

$$\underset{t\to\infty}{\rightarrow} \chi(x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)$$

- Setting $H_1(x) := \chi(x(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)$ and using De Haan & Fereira (Th. 2.3.9), we can check that for $x > 0$

$$H_1(x) = c\,x^{-\alpha}\frac{x^\rho - 1}{\rho}, \text{ where } c = \frac{\rho 2^\alpha H_1(2)}{2^\rho - 1}, \; H_1(2) = \chi(2(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)$$

- Conclude that $S_d \in 2\mathcal{RV}_{-\alpha,\rho}(b_d, A_d)$.

2 - Risk concentration for high threshold

Let ρ be a risk measure, e.g. $\rho = VaR_\alpha$.
Note that *asymptotic* results on the diversification index using TVAR may be deduced from those with VaR, using the property

$$\text{if } Z \text{ is } \mathcal{RV}_\gamma\text{, then} \quad \lim_{\kappa\to 1}\frac{TVaR_\kappa(Z)}{VaR_\kappa(Z)} = \frac{\gamma}{\gamma-1}.$$

For notational convenience: let, for $0<\beta<1$,

$Q_{1-\beta}(X) = VaR_\beta(X) := \overline{F}^{\leftarrow}(1-\beta) = \inf\{x\in\mathbb{R} : \mathbb{P}(X>x)\le 1-\beta\}$.

▷ **iid rv's:**

- **Lemma.** (Embrechts et al., 2009) Let $X_1,\ldots,X_n$ iid $\mathcal{RV}_{-\alpha}$ ($\alpha>0$), $S_n=\sum_{i=1}^n X_i$. Then
  $\lim_{\beta\to 1}\frac{VaR_\beta(S_n)}{VaR_\beta(X)} = \lim_{\gamma\to 0}\frac{Q_\gamma(S_n)}{Q_\gamma(X_1)} = n^{1/\alpha}$.
- 2nd order: e.g. Degen et al. (IME 2010); Mao et al. (IME 2012); Mao and Hu (Extremes 2013)

▷ **Dependent rv's**: with alternative approaches (not MRV); e.g. Albrecher et al. (specific copulas) (Scand. Actuar. J. 2010), Kortschak (tail independence) (Extremes 2012)

▷ Main result

Theorem. Assume $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,\rho}(b, A, \nu, H)$, with $\alpha > 0$ and $\rho \leq 0$, and a Resnick technical condition.

Then, for any $d \geq 2$, $S_d = \sum_{i=1}^{d} X_i \in 2\mathcal{RV}_{-\alpha,\rho}(b_d, A_d)$ (with b_d, A_d defined in the Proposition), and satisfies

$$\lim_{\beta \uparrow 1} \frac{VaR_\beta(S_d)}{\sum_{i=1}^{d} VaR_\beta(X_i)} = \lim_{\beta \uparrow 1} D_\beta(\boldsymbol{X}) = K_d \text{ with } K_d := \frac{1}{d}\left(\frac{\nu(\Gamma_d)}{\nu(\Gamma_1)}\right)^{\frac{1}{\alpha}}.$$

Moreover, if for all $d \geq 1$, $|\chi(2(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)| < \infty$, and

$$|\chi(2(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)| \neq |\chi(2(\nu(\Gamma_1))^{1/\alpha}\Gamma_1)|, \quad \forall d \geq 2,$$

$$\text{then } \lim_{\gamma \downarrow 0} \frac{D_{1-\gamma x}(\boldsymbol{X}) - K_d}{A\left(b\left(1/\gamma\right)\right)} = C \frac{K_d}{\alpha\rho}(x^{-\rho/\alpha} - 1),$$

for a constant $C \neq 0$ s.t., for $d \geq 2$,

$$C = \begin{cases} c_d - c_1 & \text{if } X_1 \sim 2RV \\ c_d\,(\neq 0) & \text{otherwise} \end{cases}$$

Remarks:

- Note that if $|\chi(2(\nu(\Gamma_d))^{1/\alpha}\Gamma_d)| = \infty$, the question about the limiting rate of convergence above remains open.
- Even if $c_d \neq 0$ and $c_1 \neq 0$, it is possible that $C = c_d - c_1 = 0$; we have not found an example of this case.

In order to prove the theorem, we need the following result (direct application of a lemma from Vervaat, 1971)

Lemma. *For any positive rv X assumed to be $2\mathcal{RV}_{-\alpha,\rho}(b, A)$, we have*

$$\lim_{\gamma\to 0} \frac{\frac{1}{b(1/\gamma)} Q_{\gamma x}(X) - x^{-1/\alpha}}{A(b(1/\gamma))} = \frac{c}{\alpha\rho} x^{-1/\alpha}(x^{-\rho/\alpha} - 1),$$

for a non-zero constant c.

Main steps of the proof:

- Apply our Proposition to write

$$\frac{t\,\mathbb{P}\left(S_n > b_n(t)x\right) - x^{-\alpha}}{A_n(b_n(t))} \underset{t\to\infty}{\longrightarrow} H_1(x) = k_1 x^{-\alpha}\frac{x^\rho - 1}{\rho}.$$

  Moreover

$$\frac{t\,\mathbb{P}\left(X > b(t)x\right) - x^{-\alpha}}{A(b(t))} \underset{t\to\infty}{\longrightarrow} H_2(x) = H(x,\infty,\ldots,\infty) = k_2 x^{-\alpha}\frac{x^\rho - 1}{\rho}$$

  where $k_2 = \frac{\rho 2^\alpha H_2(2)}{2^\rho - 1}$.

- Apply on each the following result by Vervaat:

  Suppose y is a continuous function and $\{z_t(x)\}_{t\geq 0}$ is a family of non-increasing functions. Also assume that the function g has a negative continuous derivative. Let $\delta(t)\to 0$ with $\delta(t)>0$ eventually and

$$\lim_{t\to\infty}\frac{z_t(x) - g(x)}{\delta(t)} = y(x)$$

  locally uniformly on $(0,\infty)$. Then, locally uniformly on $(g(0), g(\infty))$,

$$\lim_{t\to\infty}\frac{z_t^{\leftarrow}(x) - g^{\leftarrow}(x)}{\delta(t)} = -\left(g^{\leftarrow}\right)'(x)\; y(g^{\leftarrow}(x))$$

Examples:

1. Pareto-Lomax marginal distribution with survival Clayton copula ($2\mathcal{RV}$ margins)
   $\boldsymbol{X} = (X_1, X_2) \sim F$ with identical $(\alpha, 1)$-Pareto-Lomax marginal distributions with $\alpha > 1$ given by $\bar{F}_1(x) = \bar{F}_2(x) = 1 - (1+x)^{-\alpha}$ ($\forall x > 0$). Notice that $\overline{F_i} \in 2\mathcal{RV}$. Dependence structure of $\boldsymbol{X}$: survival Clayton copula defined on $[0,1]^2$, with $\theta > 0$, s.t.

$$\begin{aligned}\mathbb{P}[X_1 > x_1, X_2 > x_2] &= \left[(\overline{F}_1(x_1))^\theta + (\overline{F}_2(x_2))^\theta - 1\right]^{-1/\theta} \\ &= \left[(1+x_1)^{\alpha\theta} + (1+x_2)^{\alpha\theta} - 1\right]^{-1/\theta}.\end{aligned}$$

2. Pareto-Type 1 marginal distribution with survival Clayton copula (not $2\mathcal{RV}$ margins)
   $F_1(x) = F_2(x) = 1 - x^{-\alpha}$, $\forall x > 1$
3. Mixture model with Hidden Regular Variation

▷ Example: Pareto-type 1 margins with survival Clayton copula

- *Framework*
  Margins: $(\alpha, 1)$-Pareto distributed, with $\alpha > 1$, *i.e.* with cdf F defined by $\overline{F}(x) := 1 - F(x) = (1+x)^{-\alpha}$, $\forall x > 0$.

  Dependence structure of the $\boldsymbol{X}$: Clayton copula with parameter $\theta > 0$, defined on $[0,1]^2$ by

$$C_\theta(u,v) = \varphi_\theta^{-1}(\varphi_\theta(u) + \varphi_\theta(v)) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

  with its generator φ given by $\varphi_\theta(t) = t^{-\theta} - 1$, $t \in [0;1]$.

- We can check that $\boldsymbol{X} \in 2\mathcal{MRV}_{-\alpha,-1}(b, A, \nu, H)$ where $b(t) = t^{1/\alpha}$, $A(t) = -t^{-1}$, $\nu\left(([0,x_1] \times [0,x_2])^c\right) = x_1^{-\alpha} + x_2^{-\alpha} - (x_1+x_2)^{-\alpha}$, $H(x_1,x_2) := \alpha(x_1+x_2)^{-(\alpha+1)}$.

- We write $\chi(([0,x_1] \times [0,x_2])^c) = H(x_1,x_2)$, which can be considered as a signed measure with density given by

$$h(x_1,x_2) = \alpha(\alpha+1)(\alpha+2)(x_1+x_2)^{-(\alpha+3)}, \quad x_1 > 0, x_2 > 0.$$

  Then we can compute $\chi(k\Gamma_2)$ as

$$\chi(k\Gamma_2) = \iint_{x_1+x_2>k} h(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = \alpha(\alpha+2)k^{-(\alpha+1)}.$$

- Check Resnick assumption
- Applying our proposition gives

$$S_2 = X_1 + X_2 \in 2\mathcal{RV}_{-\alpha,-1}(b_2, A_2)$$

where $b_2(t) = (\nu(\Gamma_2))^{1/\alpha} b(t) = (\alpha+1)^{1/\alpha} t^{1/\alpha}$ and
$A_2(t) = A((\nu(\Gamma_2))^{-1/\alpha}\, t) = -(\alpha+1)^{1/\alpha} t^{-1}$. We also have

$$\lim_{t\to\infty} \frac{t\,\mathbb{P}[S_2/b_2(t) > x] - x^{-\alpha}}{A_2(b_2(t))} = c_2 x^{-\alpha}(1 - x^{-1}) =: H_2(x),$$

where $c_2 = 2^{\alpha+1}\chi(2(\nu(\Gamma_2))^{1/\alpha}\Gamma_2) = \dfrac{\alpha(\alpha+2)}{(\alpha+1)^{1+1/\alpha}}$.

- Applying our theorem, we obtain, for any $x > 0$,

$$\lim_{\gamma\downarrow 0} \frac{D_{1-\gamma x}(\boldsymbol{X}) - K_2}{A_2(b_2(1/\gamma))} = c_2 \frac{K_2}{-\alpha}(x^{1/\alpha} - 1),$$

where A_2, b_2, c_2 are as defined in the previous steps and
$K_2 = \frac{1}{2}\left(\frac{\nu(\Gamma_2)}{\nu(\Gamma_1)}\right)^{1/\alpha} = \frac{1}{2}(1+\alpha)^{1/\alpha}$.
Therefore we can rewrite (using the definitions of A_2, b_2, c_2, K_2), and noting that $A_2(b_2(1/\gamma)) = -\gamma^{1/\alpha}$,

$$\lim_{\gamma\downarrow 0} \frac{1}{\gamma^{1/\alpha}} \left[\frac{VaR_{1-\gamma x}(S_2)}{VaR_{1-\gamma x}(X_1)} - (1+\alpha)^{1/\alpha}\right] = \frac{\alpha+2}{\alpha+1}(x^{1/\alpha} - 1).$$

Conclusion

- We apply mixing techniques to obtain analytical results for diversification benefit of copula models for any threshold of the associated risk measure to bypass MC simulations approach.
- It is a precious tool for validating results of internal models, which are based on MC simulations.
- This study is a first step towards a general approach where we could choose the tail index and the copula parameter independently.
- We also consider a $2MRV$ structure to compute the tail probabilities for sums of dependent heavy-tailed random variables, and infer the local asymptotic behavior of risk concentration
- It exhibits the strength of the assumption of 2MRV, which encompasses a broad variety of dependence structures, for understanding diversification benefits in a portfolio of risk factors.
- A few questions still remain open; e.g. a characterization of 2MRV in terms of linear combination of its marginals akin to a Cramér-Wold Theorem, or finding the effects of the related concept of hidden regular variation on diversification.