Affine Volterra processes and models for rough volatility

Martin Larsson

Department of Mathematics, ETH Zurich

(joint work with Eduardo Abi Jaber and Sergio Pulido)

Conference on Mathematical Modelling of Risk and Contiguous Topics,
MATRIX programme on the Mathematics of Risk

November 27 – December 1, 2017, University of Melbourne

D MATH

ETH zürich

Rough volatility models

- Empirical studies indicate **volatility is rougher than BM**: Gatheral, Jaisson & Rosenbaum ('14); Bennedsen, Lunde, Pakkanen ('16), ...
- Subsequent development of **stochastic models with this feature**: Gatheral, Jaisson & Rosenbaum ('14); Guennoun, Jacquier & Roome ('14); Bayer, Friz & Gatheral (15); Bennedsen, Lunde, Pakkanen ('16); **El Euch & Rosenbaum ('16,'17)**, ...
- Further literature: **sites.google.com/site/roughvol/home/risks-1**
- These models are able to
  - match roughness of time series data
  - fit implied volatility skew remarkably well
  - admit in some cases microstructural justification
- This rests on **fractional Brownian motion** in the tradition of Kolmogorov ('40), Lévy ('53), Mandelbrot & van Ness ('68), ...

Heston model

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \left(\kappa(\theta - X_s) ds + \sigma \sqrt{X_s} dW_s \right)$$

Heston model

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \left(\kappa(\theta - X_s)ds + \sigma \sqrt{X_s} dW_s \right)$$

Theorem (Heston, '93). Let $S_0 = 1$. Fix $u \in \mathrm{i}\mathbb{R}$. Assume ψ solves the **Riccati equation**

$$\psi' = \frac{1}{2}(u^2 - u) - (u\rho\sigma - \kappa)\psi + \frac{\sigma^2}{2}\psi^2, \qquad \psi(0) = 0,$$

and define $\phi(T) = \int_0^T \kappa\theta\psi(t)dt$. Then

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T) + \psi(T) X_0}$$

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t}\,d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$.

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t}\,d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$.

- Similar to Riemann–Liouville fractional Brownian. Lévy ('53).
- Hölder continuous paths of any order less than $H = \alpha - \frac{1}{2}$
- Microstructural foundation as scaling limit of Hawkes processes

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$
$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$.

- Similar to Riemann–Liouville fractional Brownian. Lévy ('53).
- Hölder continuous paths of any order less than $H = \alpha - \frac{1}{2}$
- Microstructural foundation as scaling limit of Hawkes processes

But:

- Existence and uniqueness is non-trivial.
- Not a semimartingale, not Markovian ...
- ... not clear how to usefully describe its law.

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\kappa(\theta - X_s) ds + \sigma \sqrt{X_s} dW_s \Big)$$

with $\alpha \in (\frac{1}{2}, 1)$.

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$. **Notation:** $D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds$

Rough Heston model of El Euch & Rosenbaum ('16)

$$\frac{dS_t}{S_t} = \sqrt{X_t}d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$. **Notation:** $D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}h(s)ds$

Theorem (El Euch & Rosenbaum, '16). Let $S_0 = 1$. Fix $u \in \mathrm{i}\mathbb{R}$. Assume ψ solves the **fractional Riccati equation**

$$D^\alpha\psi = \frac{1}{2}(u^2 - u) + (u\rho\sigma - \kappa)\psi + \frac{\sigma^2}{2}\psi^2, \qquad \psi(0) = 0,$$

and define $\phi(T) = \int_0^T \kappa\theta\chi(t)dt$ and $\chi(T) = \int_0^T D^\alpha\psi(t)dt$. Then

$$\mathbb{E}[e^{u\log S_T}] = e^{\phi(T)+\chi(T)X_0}$$

Why?

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

(i) Define $M_t = e^{\phi(T-t)+\psi(T-t)X_t}$ with ϕ, ψ from Riccati.

(ii) Itô and Riccati imply M is local martingale. $M_T = e^{uX_T}$.

(iii) If true martingale, then $\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{\phi(T-t)+\psi(T-t)X_t}$.

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

(i) Define $M_t = e^{\phi(T-t)+\psi(T-t)X_t}$ with ϕ, ψ from Riccati.

(ii) Itô and Riccati imply M is local martingale. $M_T = e^{uX_T}$.

(iii) If true martingale, then $\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{\phi(T-t)+\psi(T-t)X_t}$.

Won't work for the rough CIR.

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

(i) Define $M_t = e^{\phi(T-t)+\psi(T-t)X_t}$ with ϕ, ψ from Riccati.

(ii) Itô and Riccati imply M is local martingale. $M_T = e^{uX_T}$.

(iii) If true martingale, then $\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{\phi(T-t)+\psi(T-t)X_t}$.

Won't work for the rough CIR. Instead, use representation

$$\log \mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s)^2 a(\mathbb{E}[X_s \mid \mathcal{F}_t])ds.$$

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

(i) Define $M_t = e^{\phi(T-t)+\psi(T-t)X_t}$ with ϕ, ψ from Riccati.

(ii) Itô and Riccati imply M is local martingale. $M_T = e^{uX_T}$.

(iii) If true martingale, then $\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{\phi(T-t)+\psi(T-t)X_t}$.

Won't work for the rough CIR. Instead, use representation

$$\log \mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s)^2 a(\mathbb{E}[X_s \mid \mathcal{F}_t])ds.$$

Here there is hope.

Affine Volterra processes

A continuous E-valued solution X of the stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

is called an **affine Volterra process** (of convolution type). Data:

- State space $E \subseteq \mathbb{R}^d$ and initial condition $X_0 \in E$.
- Affine diffusion and drift coefficients

$$a(x) = A^0 + A^1 x_1 + \cdots + A^d x_d$$
$$b(x) = b^0 + b^1 x_1 + \cdots + b^d x_d$$

  with $A^i \in \mathbb{S}^d$, $b^i \in \mathbb{R}^d$, and $a(x) = \sigma(x)\sigma(x)^\top$ for all $x \in E$.
- Matrix-valued kernel $K \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$.

Affine Volterra processes

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

- **Example:** $K(t) \equiv \mathrm{id}$ gives standard affine diffusions.
- **Example:** The rough CIR process of Rosenbaum & El Euch uses

$$K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$$

- **Example:** The full rough Heston model uses $d = 2$ and

$$K(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \end{pmatrix}$$

Conditional characteristic function

Theorem (∗). Fix a row vector $u \in (\mathbb{C}^d)^*$. Assume the function $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$ solves the **Riccati–Volterra equation**

$$\psi = uK + \left(\psi B + \frac{1}{2}A(\psi)\right) * K$$

where $A(\psi) = (\psi A^1 \psi^\top, \ldots, \psi A^d \psi^\top)$. Fix $T < \infty$ and define

$$Y_t = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s) a(\mathbb{E}[X_s \mid \mathcal{F}_t]) \psi(T-s)^\top ds.$$

Then $\{\exp(Y_t),\, 0 \le t \le T\}$ is a local martingale and, if it is a true martingale, one has

$$\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{Y_t}, \quad t \le T.$$

Conditional expectations

- Take expectations in

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

to obtain

$$\mathbb{E}[X] = X_0 + (K * 1)\, b^0 + (KB) * \mathbb{E}[X]$$

- Get $\mathbb{E}[X]$ by variation of constants formula using resolvent of $-KB$.
- Conditional expectations are similar.

Concrete specifications

Given a specification of E, $K(t)$, $a(x)$, $b(x)$, three things need proof:

- Existence of X (hence uniqueness of ψ)
- Existence of ψ (hence uniqueness of X)
- Martingale condition

Concrete specifications

Given a specification of E, $K(t)$, $a(x)$, $b(x)$, three things need proof:

- Existence of X (hence uniqueness of ψ)
- Existence of ψ (hence uniqueness of X)
- Martingale condition

We do this for three classes of specifications:

- Volterra Ornstein–Uhlenbeck: $E = \mathbb{R}^d$
- Volterra square-root: $E = \mathbb{R}_+^d$
- Volterra Heston: $E = \mathbb{R} \times \mathbb{R}_+$

Volterra Ornstein–Uhlenbeck process, $E = \mathbb{R}^d$

- With $E = \mathbb{R}^d$ and $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma dW_s$$

Volterra Ornstein–Uhlenbeck process, $E = \mathbb{R}^d$

- With $E = \mathbb{R}^d$ and $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma dW_s$$

- This is a **Gaussian process**.

Volterra Ornstein–Uhlenbeck process, $E = \mathbb{R}^d$

- With $E = \mathbb{R}^d$ and $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma dW_s$$

- This is a **Gaussian process**.
- The Riccati–Volterra equation has an **explicit solution**:

$$\psi = uE_B.$$

where $E_B = K - R_B * K$ with R_B the resolvent of $-KB$.

Volterra Ornstein–Uhlenbeck process, $E = \mathbb{R}^d$

- With $E = \mathbb{R}^d$ and $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma dW_s$$

- This is a **Gaussian process**.
- The Riccati–Volterra equation has an **explicit solution**:

$$\psi = uE_B.$$

where $E_B = K - R_B * K$ with R_B the resolvent of $-KB$.

- The quadratic variation of the process Y is deterministic,

$$\langle Y \rangle_t = \int_0^t \psi(T-s)\sigma\sigma^\top \psi(T-s)^\top ds.$$

Thus the martingale condition holds.

Volterra square-root process, $E = \mathbb{R}_+^d$

Volterra square-root process, $E = \mathbb{R}_+^d$

- **Dynamics:** Diagonal kernel $K = \mathrm{diag}(K_1, \dots, K_d)$,

$$X_{i,t} = X_{i,0} + \int_0^t K_i(t-s)\Big(b_i(X_s)ds + \sigma_i \sqrt{X_{i,s}} dW_{i,s}\Big),$$

where $b(x) = b^0 + Bx$ satisfies **inward-pointing drift** condition

$b^0 \in \mathbb{R}_+^d$ and $B_{ij} \geq 0$ for $i \neq j$.

Volterra square-root process, $E = \mathbb{R}^d_+$

- **Dynamics:** Diagonal kernel $K = \mathrm{diag}(K_1, \dots, K_d)$,

$$X_{i,t} = X_{i,0} + \int_0^t K_i(t-s)\Big(b_i(X_s)ds + \sigma_i \sqrt{X_{i,s}} dW_{i,s}\Big),$$

  where $b(x) = b^0 + Bx$ satisfies **inward-pointing drift** condition

$$b^0 \in \mathbb{R}^d_+ \text{ and } B_{ij} \geq 0 \text{ for } i \neq j.$$

- **Riccati–Volterra equation:**

$$\psi_i(t) = u_i K_i(t) + \int_0^t K_i(t-s)\left(\psi(s)b^i + \frac{\sigma_i^2}{2}\psi_i(s)^2\right) ds$$

Volterra square-root process, $E = \mathbb{R}_+^d$

- **Dynamics:** Diagonal kernel $K = \mathrm{diag}(K_1, \dots, K_d)$,

$$X_{i,t} = X_{i,0} + \int_0^t K_i(t-s)\Big(b_i(X_s)ds + \sigma_i \sqrt{X_{i,s}} dW_{i,s}\Big),$$

where $b(x) = b^0 + Bx$ satisfies **inward-pointing drift** condition

$$b^0 \in \mathbb{R}_+^d \text{ and } B_{ij} \geq 0 \text{ for } i \neq j.$$

- **Riccati–Volterra equation:**

$$\psi_i(t) = u_i K_i(t) + \int_0^t K_i(t-s) \left(\psi(s) b^i + \frac{\sigma_i^2}{2} \psi_i(s)^2 \right) ds$$

- **Assumption:** Each K_i is **completely monotone** and

$$t \mapsto t^{\gamma_i} K_i(t) \text{ is locally Lipschitz on } [0, \infty)$$

for some $\gamma_i < 1/2$.

Volterra square-root process, $E = \mathbb{R}_+^d$

Theorem.

- The stochastic Volterra equation has a unique in law $\mathbb{R}_+^d$-valued continuous weak solution for any initial condition $X_0 \in \mathbb{R}_+^d$. The paths of X_i are Hölder continuous of any order less than $H_i = 1/2 - \gamma_i$, for each $i = 1, \ldots, d$.
- For any $u \in (\mathbb{C}^d)^*$ with $\operatorname{Re} u_i \leq 0$ for each $i = 1, \ldots, d$, the Riccati–Volterra equation has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$, which satisfies $\operatorname{Re} \psi_i \leq 0$.
- The martingale condition in Theorem (∗) holds, as does the affine transform formula.

Volterra Heston model, $E = \mathbb{R} \times \mathbb{R}_+$

Volterra Heston model, $E = \mathbb{R} \times \mathbb{R}_+$

- **Dynamics** with $\kappa\theta \geq 0$, $\sigma \geq 0$, $d\langle \widetilde{W}, W \rangle_t = \rho\, dt$:

$$\frac{dS_t}{S_t} = \sqrt{V_t}\, d\widetilde{W}_t$$

$$V_t = V_0 + \int_0^t K(t-s) \left(\kappa(\theta - V_s)ds + \sigma \sqrt{V_s}\, dW_s \right)$$

Volterra Heston model, $E = \mathbb{R} \times \mathbb{R}_+$

- **Dynamics** with $\kappa\theta \geq 0$, $\sigma \geq 0$, $d\langle \widetilde{W}, W \rangle_t = \rho \, dt$:

$$\frac{dS_t}{S_t} = \sqrt{V_t} \, d\widetilde{W}_t$$

$$V_t = V_0 + \int_0^t K(t-s) \left(\kappa(\theta - V_s) ds + \sigma \sqrt{V_s} \, dW_s \right)$$

- **Riccati–Volterra equation:**

$$\psi_1 = u_1$$

$$\psi_2 = u_2 K + K * \left(\frac{1}{2} \left(u_1^2 - u_1 \right) + (\rho\sigma u_1 - \kappa)\psi_2 + \frac{1}{2}\sigma^2 \psi_2^2 \right)$$

Volterra Heston model, $E = \mathbb{R} \times \mathbb{R}_+$

- **Dynamics** with $\kappa\theta \geq 0$, $\sigma \geq 0$, $d\langle \widetilde{W}, W \rangle_t = \rho \, dt$:

$$\frac{dS_t}{S_t} = \sqrt{V_t} \, d\widetilde{W}_t$$

$$V_t = V_0 + \int_0^t K(t-s) \left(\kappa(\theta - V_s) ds + \sigma \sqrt{V_s} \, dW_s \right)$$

- **Riccati–Volterra equation:**

$$\psi_1 = u_1$$

$$\psi_2 = u_2 K + K * \left(\frac{1}{2} \left(u_1^2 - u_1 \right) + (\rho\sigma u_1 - \kappa)\psi_2 + \frac{1}{2}\sigma^2 \psi_2^2 \right)$$

- **Assumption:** K is **completely monotone** and

$$t \mapsto t^\gamma K(t) \text{ is locally Lipschitz on } [0, \infty)$$

for some $\gamma < 1/2$.

Volterra Heston model, $E = \mathbb{R} \times \mathbb{R}_+$

Theorem.

- The stochastic Volterra equation has a unique in law $\mathbb{R} \times \mathbb{R}_+$-valued continuous weak solution $(\log S, V)$ for any initial condition $(\log S_0, V_0) \in \mathbb{R} \times \mathbb{R}_+$. The paths of V are Hölder continuous of any order less than $H = 1/2 - \gamma$.
- For any $u \in (\mathbb{C}^2)^*$ such that

$$\operatorname{Re} u_1 \in [0,1] \text{ and } \operatorname{Re} u_2 \leq 0,$$

  the Riccati–Volterra equation has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^*)^2)$, which satisfies $\operatorname{Re} \psi_2 \leq 0$.
- The martingale condition in Theorem (∗) holds, as does the affine transform formula.
- The process S is a martingale.

Why complete monotonicity?

- A **resolvent of the first kind** of K is a kernel L such that

$$K * L = L * K \equiv \mathrm{id}$$

In general L is a measure, for instance $L(dt) = \delta_0(dt)$ if $K \equiv \mathrm{id}$.

Why complete monotonicity?

- A **resolvent of the first kind** of K is a kernel L such that

$$K * L = L * K \equiv \mathrm{id}$$

In general L is a measure, for instance $L(dt) = \delta_0(dt)$ if $K \equiv \mathrm{id}$.

- **Example:** If $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $\alpha \in (\frac{1}{2}, 1)$, then

$$L(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$$

Why complete monotonicity?

- A **resolvent of the first kind** of K is a kernel L such that

$$K * L = L * K \equiv \mathrm{id}$$

In general L is a measure, for instance $L(dt) = \delta_0(dt)$ if $K \equiv \mathrm{id}$.

- **Example:** If $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $\alpha \in (\frac{1}{2}, 1)$, then

$$L(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$$

- **Example:** If K is **completely monotone**, then L exists and is the sum of a point mass in zero and a completely monotone function.

Why complete monotonicity?

Shift operator: $\Delta_h f(t) = f(t+h)$

Why complete monotonicity?

Shift operator: $\Delta_h f(t) = f(t+h)$

Lemma. Setting of Theorem (∗). Assume K has resolvent of the first kind L. Define

$$\pi_h = \Delta_h \psi * L - \Delta_h(\psi * L),$$

and assume $\pi_h \in BV_{\text{loc}}(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ for $h = T - t$. Then

$$Y_t = \phi(h) + (\Delta_h \psi * L)(0) X_t - \pi_h(t) X_0 + (d\pi_h * X)_t$$

with $h = T - t$ and $\phi(h) = \int_0^h \left(\psi(s) b^0 + \frac{1}{2}\psi(s) A^0 \psi(s)^\top\right) ds$.

Why complete monotonicity?

Shift operator: $\Delta_h f(t) = f(t+h)$

Lemma. Setting of Theorem $(*)$. Assume K has resolvent of the first kind L. Define

$$\pi_h = \Delta_h \psi * L - \Delta_h(\psi * L),$$

and assume $\pi_h \in BV_{\mathrm{loc}}(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ for $h = T - t$. Then

$$Y_t = \phi(h) + (\Delta_h \psi * L)(0) X_t - \pi_h(t) X_0 + (d\pi_h * X)_t$$

with $h = T - t$ and $\phi(h) = \int_0^h \left(\psi(s) b^0 + \frac{1}{2} \psi(s) A^0 \psi(s)^\top \right) ds$.

In particular: For $E = \mathbb{R}^d_+$, verify martingale condition by controlling the signs of the real parts of $(\Delta_h \psi * L)(0)$, $\pi_h(t)$, and $d\pi_h$.

Why complete monotonicity?

Moreover: Fourier–Laplace transform exponential-affine in $\{X_s \colon s \leq t\}$,

$$\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = \exp\Big(\phi(h) + (\Delta_h \psi * L)(0) X_t - \pi_h(t) X_0 + (d\pi_h * X)_t\Big)$$

Conclusion

- Affine Volterra processes admit affine transform formulas despite lack of semimartingale and Markov properties
- Tools from deterministic theory of Volterra equations, e.g. resolvents of first and second kind. Read the spectacular book *Volterra integral and functional equations*, 1990, by Gripenberg, Londen, Staffans.
- Unknown or in progress :
  - Regularity of ψ (currently L^2_{loc})
  - Pathwise uniqueness for X (currently uniqueness in law)
  - Numerical methods for ψ
  - Numerical methods for X
  - Boundary attainment for Volterra square-root processes
  - Non-convolution kernels $K(t,s)$
  - Stationary case $X_t = \int_{-\infty}^{t}(\cdots)$
  - Etc.