Supercongruences Occurred to Rigid Hypergeometric Type Calabi–Yau Threefolds

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Abstract In this project, we establish the supercongruences for the 14 families of rigid hypergeometric Calabi–Yau threefolds conjectured by Roriguez-Villegas in 2003.

1 Main Result

The talk outlines the proof of the supercongruences for the 14 families of rigid hypergeometric Calabi–Yau threefolds conjectured by Roriguez-Villegas [9].

Theorem 1. *Let* $d_1, d_2 \in \{1/2, 1/3, 1/4, 1/6\}$ *or*

$$(d_1, d_2) = (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12).$$

Then for each prime $p > 5$*, we have*

$${}_4F_3 \begin{bmatrix} d_1 & 1-d_1 & d_2 & 1-d_2 \\ & 1 & 1 & 1 \end{bmatrix} ; 1 \Bigg]_{p-1} \equiv a_p(f_{d_1,d_2}) \mod p^3,$$

where the hypergeometric series on the left-hand side is truncated after $p-1$ *terms and* $a_p(f_{d_1,d_2})$ *is the pth coefficient of an explicit Hecke eigenform* f_{d_1,d_2} *of weight* 4

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associated to the corresponding rigid Calabi–Yau manifold via the modularity theorem.

2 Motivation

The term supercongruence refers to a congruence which is stronger than what the formal group law implies. In [3] Beukers proved

$$A\left(\frac{p-1}{2}\right) = {}_4F_3\left[\begin{matrix}\frac{1-p}{2} & \frac{1-p}{2} & \frac{p+1}{2} & \frac{p+1}{2} \\ & 1 & 1 & 1\end{matrix}; 1\right] \equiv a_p(f) \mod p,$$

where $A(n)$ are the Apéry numbers

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = {}_4F_3\left[\begin{matrix}-n & -n & n+1 & n+1 \\ & 1 & 1 & 1\end{matrix}; 1\right]$$

and $a_p(f)$ is the pth coefficient of the Hecke eigenform $\eta^4(2\tau)\eta^4(4\tau)$. In [3] Beukers also conjectured the supercongruence

$${}_4F_3\left[\begin{matrix}\frac{1-p}{2} & \frac{1-p}{2} & \frac{p+1}{2} & \frac{p+1}{2} \\ & 1 & 1 & 1\end{matrix}; 1\right] \equiv a_p(f) \mod p^2.$$

This was proved by Ahlgren and Ono [1]. Their key idea is using Green's hypergeometric function over finite fields to perform point counting on the Calabi–Yau threefold

$$\left\{x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0\right\},$$

which is modular. Later, Kilbourn [7] gives an extension of the supercongruence

$$a_p(f) \equiv \sum_{j=0}^{p-1} \binom{2j}{j}^4 2^{-8j} = {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \mod p^3, \tag{1}$$

which was conjectured by Van Hamme. Kilbourn's proof is mainly relying on p-adic tools. Using the techniques similar to the ones given by Ahlgren, Ono and Kilbourn, McCarthy [8] obtained the supercongruence

$${}_4F_3\left[\begin{matrix}\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \equiv a_p\left(f_{1/5,2/5}\right) \mod p^3, \tag{2}$$

where $f_{1/5,2/5}$ is an explicit Hecke eigenform conjectured by Rodriguez-Villegas. This supercongruence corresponds to the mirror quintic threefold in $\mathbb{P}^4$, whose modularity was first established by Schoen [10]. The supercongruences given by Kilbourn (1) and McCarthy (2) are particular instances of Rodriguez-Villegas's conjectures.

In this joint project, our main motivation is to study the arithmetic aspect of rigid hypergeometric type Calabi–Yau manifolds. The first step is verifying the supercongruences conjectured by Rodriguez-Villegas coming from the well-known 14 hypergeometric families of Calabi–Yau threefolds whose Picard–Fuchs equations are degree 4 hypergeometric differential equations with solution near 0 of the form

$$ {}_4F_3\left[\begin{matrix} d_1 & 1-d_1 & d_2 & 1-d_2 \\ & 1 & 1 & 1 \end{matrix} ; z\right], $$

where d_1, d_2 are as in Theorem 1. When $z = 1$, it corresponds to the singularity of the hypergeometric differential equation, which is equivalent to getting a rigid Calabi–Yau threefold in the fibre. Due to Gouvêa and Yui [6], a rigid Calabi–Yau threefold defined over $\mathbb{Q}$ is modular. This means, the L-function associated with the third étale cohomology group of a rigid Calabi–Yau threefold V in the 14 families is equal to the L-function of an explicit Hecke eigenform of weight 4 conjectured by Rodriguez-Villegas.

Very recently, Fuselier and McCarthy [**?**] establish the case $(d_1, d_2) = (1/2, 1/4)$. In this joint project, we provide a more general method to verify the remaining 11 cases of supercongruences conjectured by Rodriguez-Villegas.

3 Key Ideas and Example

The strategy of our proof is to use hypergeometric motives over $\mathbb{Q}$ to describe the arithmetic background. There are different versions of hypergeometric motives such as given by Katz, Greene and McCarthy. However, for our purposes, the most connivent one is the general version given by Beukers, Cohen and Mellit in [4]. They modify Katz's finite hypergeometric function $H(\alpha, \beta; \lambda)$ so that their version works for all the primes p. They also give a recipe to realize toric models as hypergeometric motives arising from certain type hypergeometric data $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, where $\alpha_i, \beta_j \in \mathbb{Q}$. For such case, one can express the number of rational points over finite fields on the given model in terms of $H(\alpha, \beta; \lambda)$.

For example, the multi-sets $\alpha = (1/3, 2/3, 1/3, 2/3)$ and $\beta = (1, 1, 1, 1)$ give the hypergeometric type Calabi–Yau threefold with $d_1 = d_2 = 1/3$. The toric model in this case corresponds to the resolution of singularities on the affine variety given by projective equations

$$ W : \; x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 0, \; (x_1 y_1)^3 = 3^6 x_2 x_3 x_4 y_2 y_3 y_4, \; x_i, y_j \neq 0. $$

The resulting manifold $\overline{W}$ is the rigid Calabi–Yau threefold labelled as $V_{3,3}$ by Batyrev and van Straten in [2].

In the talk, principal ideas of our proof are illustrated in the case $d_1 = d_2 = 1/3$.

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