Some supercongruences for truncated hypergeometric series

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Abstract We prove various supercongruences involving truncated hypergeometric sums. These include a strengthened version of a conjecture of van Hamme. Our method is to employ various hypergeometric transformation and evaluation formulae to convert the truncated sums to quotients of Γ-values. We then convert these to quotients of Γ_p-values and use use Taylor's Theorem to make p-adic approximations. In the cases under consideration higher order coefficients often vanish leading to the supercongruences.

In [3], van Hamme conjectured that

$$
{}_7F_6\left[\begin{matrix} \frac{1}{3} & \frac{7}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{6} & 1 & 1 & 1 & 1 & 1 \end{matrix}\right]_{p-1} := \sum_{k=0}^{p-1}(6k+1)\frac{((1/3)_k)^6}{(k!)^6}
$$
$$
\equiv -p\Gamma_p(1/3)^9 \bmod p^4 \text{ for } p \equiv 1 \bmod 6.
$$

We have, in [2], proved

Theorem 1 Let $p > 11$. Then

$$
{}_7F_6\left[\begin{matrix} \frac{1}{3} & \frac{7}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{6} & 1 & 1 & 1 & 1 & 1 \end{matrix}\right]_{p-1} \equiv -p\Gamma_p(1/3)^9 \bmod p^6 \text{ for } p \equiv 1 \bmod 6
$$

and

$$
{}_7F_6\left[\begin{matrix} \frac{1}{3} & \frac{7}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{6} & 1 & 1 & 1 & 1 & 1 \end{matrix}\right]_{p-1} \equiv -\frac{10}{27}p^4\Gamma_p(1/3)^9 \bmod p^6 \text{ for } p \equiv 5 \bmod 6.
$$

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Note these are congruences cover (almost) all primes and are stronger than the van Hamme Conjecture. We proved a number of other supercongruences including the ${}_3F_2$ ones below:

Theorem 2

$$ {}_3F_2\left[\begin{matrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ & 1 & 1\end{matrix}\right]_{p-1} \equiv \Gamma_p(1/3)^6 \bmod p^3 \text{ for } p\equiv 1 \bmod 6$$

and

$$ {}_3F_2\left[\begin{matrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ & 1 & 1\end{matrix}\right]_{p-1} \equiv -\frac{p^2}{3}\Gamma_p(1/3)^6 \bmod p^3 \text{ for } p\equiv 5 \bmod 6.$$

For $p\equiv 1 \bmod 6$ the right side of the above congruence corresponds to Dwork's unit root for ordinary primes of a certain modular form that is part of the corresponding hypergeometric motive.

We outline the strategies for $p\equiv 1 \bmod 6$. Various minor differences (and one on medium-sized technical issue in Theorem 1) arise for $p\equiv 5 \bmod 6$. The idea in both theorems is to perturb the entries so that the series naturally truncate at $\frac{p-1}{3}$ (for $p\equiv 1 \bmod 6$).

Let ζ_3 be a primitive cube root of unity. For instance in Theorem 2 we study ${}_3F_2\left[\begin{matrix}\frac{1-p}{3} & \frac{1-\zeta_3 p}{3} & \frac{1-\zeta_3^2 p}{3}\\ & 1 & 1\end{matrix}\right]$ for $p\equiv 1 \bmod 6$. The corresponding infinite series truncates at $\frac{p-1}{3}$. The Galois symmetry and a simple congruence argument imply

$$ {}_3F_2\left[\begin{matrix}\frac{1-p}{3} & \frac{1-\zeta p}{3} & \frac{1-\zeta^2 p}{3}\\ & 1 & 1\end{matrix}\right] \equiv {}_3F_2\left[\begin{matrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ & 1 & 1\end{matrix}\right]_{p-1} \quad \bmod p^3.$$

At this point we can use the Pfaff-Saalschütz formula (see, for instance, Theorem 2.2.6 of [1])

$$ {}_3F_2\left[\begin{matrix}-n & a & b\\ & c & 1+a+b-c-n\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

with $n=\frac{p-1}{3}$ to write ${}_3F_2\left[\begin{matrix}\frac{1-p}{3} & \frac{1-\zeta p}{3} & \frac{1-\zeta^2 p}{3}\\ & 1 & 1\end{matrix}\right]$ as a quotient of Γ-values. One can rewrite this as a quotient of Γ_p-values and then use a Taylor approximation to get the desired result.

For Theorem 1 a similar argument with primitive 5th roots of unity and Dougall's formula below, which holds when $1+2a=b+c+d+e+f$,

$$
{}_7F_6\left[\begin{matrix} a & 1+\frac{a}{2} & b & c & d & e & f \\ & \frac{a}{2} & 1+a-b & 1+a-c & 1+a-d & 1+a-e & 1+a-f \end{matrix}\right]
$$

$$
= \frac{(a+1)_{-f}(a-b-c+1)_{-f}(a-b-d+1)_{-f}(a-c-d+1)_{-f}}{(a-b+1)_{-f}(a-c+1)_{-f}(a-d+1)_{-f}(a-b-c-d+1)_{-f}}
$$

gives the van Hamme congruences mod p^5.

To obtain the congruence mod p^6 involves an extra argument. It is not difficult to show the terminating

$$
{}_7F_6\left[\begin{matrix} \frac{1}{3} & \frac{7}{6} & \frac{1}{3}-\zeta_5 x & \frac{1}{3}-\zeta_5^2 x & \frac{1}{3}-\zeta_5^3 x & \frac{1}{3}-\zeta_5^4 x & \frac{1}{3}-x \\ & \frac{1}{6} & 1+\zeta_5 x & 1+\zeta_5^2 x & 1+\zeta_5^3 x & 1+\zeta_5^4 x & 1+x \end{matrix}\right]_{\frac{p-1}{3}} \in \mathbb{Z}_p[[x^5]].
$$

Call this power series $G(x)$. Using a result of Bailey relating ${}_9F_8$ expressions one can in fact prove the above series is in $p\mathbb{Z}_p[[x^5]]$. A somewhat subtle argument is required when $p \equiv 5 \bmod 6$ to obtain the divisibility of $G(x)$ by p.

Since $p \mid G(x)$, $G(0) \equiv G(p/3) \bmod p^6$. It is easy to show that $G(0)$ is congruent to the left side of Theorem 1 mod p^6. The argument using Dougall's formula gives $G(p/3)$ is congruent to the right side of Theorem 1 mod p^6.

References

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