Sequences, modular forms and cellular integrals

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Abstract It is well-known that the Apéry sequences which arise in the irrationality proofs for $\zeta(2)$ and $\zeta(3)$ satisfy many intriguing arithmetic properties and are related to the pth Fourier coefficients of modular forms. In [6], we prove that the connection to modular forms persists for sequences associated to Brown's cellular integrals and state a general conjecture concerning supercongruences.

1 Introduction and statement of results

Recently, Brown [4] introduced a program where period integrals on the moduli space $\mathscr{M}_{0,N}$ of curves of genus 0 with N marked points play a central role in understanding irrationality proofs of values of the Riemann zeta function. The main idea of [4] is to associate a rational function f_σ and a differential $(N-3)$-form ω_σ to a given permutation $\sigma = \sigma_N$ on $\{1,2,\ldots,N\}$. Consider the *cellular integral*

$$I_\sigma(n) := \int_{S_N} f_\sigma^n \, \omega_\sigma,$$

where

$$S_N = \{(t_1,\ldots,t_{N-3}) \in \mathbb{R}^{N-3} : 0 < t_1 < \ldots < t_{N-3} < 1\}.$$

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For "convergent" σ, $I_\sigma(n)$ converges and

$$I_\sigma(n) = a_{n,\sigma}^{(N-3)} \zeta_{N-3} + a_{n,\sigma}^{(N-4)} \zeta_{N-4} + \ldots + a_{n,\sigma}^{(0)} \zeta_0,$$

where ζ_i are fixed $\mathbb{Q}$-linear combinations of multiple zeta values of weight i and the coefficients $a_{n,\sigma}^{(i)}$ are rational numbers. Specifically, $\zeta_{N-3} = I_\sigma(0)$ and we say that

$$A_\sigma(n) = A_{\sigma_N}(n) := a_{n,\sigma}^{(N-3)}$$

is the *leading coefficient* of the cellular integral $I_\sigma(n)$.

This construction recovers Beukers' integrals [3] which appear in the irrationality proofs of $\zeta(2)$ and $\zeta(3)$ and the Apéry numbers

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

as leading coefficients. This framework also raises some natural questions. Is there an analogue of the results in [1] and [2] to higher weight modular forms? Do the leading coefficients $A_\sigma(n)$ satisfy supercongruences akin to those in [5]?

In [6], we prove that there is a higher weight version of Theorem 5 in [2] and Theorem 3 in [1] can be extended to all odd weights greater than or equal to 3. Based on numerical evidence, we also conjecture that for each $N \geq 5$ and convergent σ_N, the leading coefficients $A_{\sigma_N}(n)$ satisfy

$$A_{\sigma_N}(mp^r) \equiv A_{\sigma_N}(mp^{r-1}) \pmod{p^{3r}}$$

for all primes $p \geq 5$ and integers $m, r \geq 1$.

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