Default Contagion and Systemic Risk in the Presence of Credit Default Swaps

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Plan of Talk

Introduction

Background

- CDS and other credit derivatives are blamed as a major cause of the financial crisis in 2008.

  > $\cdots$, *in trying to understand the credit crisis, many observers have identified credit default swaps to be a prominent villain (Stulz, 2010).*

- The network linkage in financial markets through CDS transactions is said to have amplified the crises.
- Research motivation: to investigate how the cross-tradings of CDSs affect financial stability.

Literature review

- Theoretical papers:
  - Eisenberg and Noe (2001), Suzuki (2002); debt cross-holdings, no default cost.
  - Rogers and Veraart (2013); with default costs
  - Fischer (2014): no default cost, with seniority structure of debts.
- Difficulty to introduce CDSs: non-monotonicity of payoffs.
  ⇒Existence of clearing payment vector seems hard to show.
  - No default cost: Suzuki (2002), Fischer (2014), El Bitar et al. (2016),
  - With default cost: Rogers and Veraart (2013).

Illustration of balance sheet

- Balance sheet of bank i at maturity:

Asset	Liability
$\sum_{j,k} m_{ij}^k \lambda_j^k(\bar{p}^k - p_k^k)$	CDS $\sum_k \lambda_i^k(\bar{p}^k - p_k^k)$
$\sum_j m_{ij}^j p_j^j$	Debt $\bar{p}^i$
Business asset e_i (or $(1-c_i)e_i$ at default)	Equity p_i^0

- The payoff of straight debt p_j^j is non-decreasing in the asset values.
- The payoff of CDS $\lambda_i^k(\bar{p}^k - p_k^k)$ is non-increasing in the asset values.
- When bank i defaults, the business asset e_i is reduced to $(1-c_i)e_i$ due to the default costs.
- In Eisenberg and Noe (2001) and other related papers, the default of bank i is determined by $p_i^0 = 0$.

Clearing payment

- The payment vector consistent with the cross-trading structure is expressed as a fixed point:
$$\mathbf{p} = \mathbf{f}(\mathbf{p}).$$
- If there is no default cost, Banach's fixed-point theorem can be applied due to the continuity and contraction of $\mathbf{f}$.
- If there is only debt cross-trading with default costs, Tarski's fixed-point theorem can be applied due to the monotonicity of $\mathbf{f}$.
- How can we prove the existence of a clearing payment vector $\mathbf{p}$ in our case?

Summary of our paper (1)

Our model:

- Introduce CDS cross-holdings with default costs into Fischer (2014).
- Propose the fictitious default algorithm with financial covenants, reflecting *technical defaults* observed in actual markets (Kusnetsov and Veraart, 2016).
    - Debt service default: the borrower cannot make a scheduled payment.
    - Technical default: another condition such as safety covenant is violated.

	Mean	5%	95%	N
Market assets/Face debt	0.660	0.303	1.221	148

(Davydenk, 2012)

Summary of our paper (2)

Major results:

- Prove the existence of a clearing payment vector under the assumption of our algorithm.
- Show with numerical examples that CDS cross-tradings can have a negative impact on financial stability.
  - Cross-trading of debts; complete graph leads to the most stable market (Allen and Gale, 2000, etc.).
  - Cross-trading of CDSs; complete graph leads to a default contagion.

Model Setup

Notations

The following notations are used in this talk:

lower	x, y etc.	scalars
lower and bold	$\mathbf{x}$, $\mathbf{y}$, etc.	vectors
upper and bold	$\mathbf{X}$, $\mathbf{Y}$, etc.	matrices

$$\mathbf{0} = (0, \ldots, 0)^\top, \quad \mathbf{1} = (1, \ldots, 1)^\top, \quad \mathbf{I} = \begin{pmatrix} 1 & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & 1 \end{pmatrix},$$

$$\mathbf{x} \wedge \mathbf{y} = \begin{pmatrix} \min\{x_1, y_1\} \\ \vdots \\ \min\{x_n, y_n\} \end{pmatrix}, \quad \mathbf{x} \vee \mathbf{y} = \begin{pmatrix} \max\{x_1, y_1\} \\ \vdots \\ \max\{x_n, y_n\} \end{pmatrix}, \quad (\mathbf{x})^+ = \mathbf{x} \vee \mathbf{0},$$

Banks' business assets and default costs

- One-shot economy with current and maturity times.
- There are totally n banks in the financial market.
- Each bank has its own business (external) asset.
- $\mathbf{e} = (e_1, \ldots, e_n)^\top \in \boldsymbol{R}_+^n$ denotes the vectors of banks' business asset at maturity before default procedure.
- If bank i defaults, then its business asset, e_i is reduced to $(1 - c_i)e_i$, where c_i is a constant. We write

$$\mathbf{C} = \mathrm{diag}(\{c_i\}_{i=1}^n).$$

- Define an $n \times n$ diagonal matrix of default indicators:

$$\boldsymbol{\Delta} = \mathrm{diag}(\{1_{\mathcal{D}}(i)\}_{i=1}^n), \quad 1_{\mathcal{D}}(i) = \begin{cases} 1 & \text{if } i \in \mathcal{D}, \\ 0 & \text{if } i \notin \mathcal{D}, \end{cases}$$

  where $\mathcal{D} = \{i \in \{1, \ldots, n\} | \text{ bank } i \text{ defaults}\}$.
- Banks' business asset values after default cost reduction are described as

$$(\mathbf{I} - \mathbf{C}\boldsymbol{\Delta})\mathbf{e}.$$

Financial securities in the market

- Equities.
- Straight debts:
  - Bank j issues a straight debt with face value $\bar{p}^j$.
  - Contractual repayment should be $d_j^j = \bar{p}^j$.
- Credit Default Swaps:
  - Bank j writes a CDS with reference bank k.
  - λ_j^k: the proportion to cover the loss associated with debt $\bar{p}^k$ by bank j.
    - If bank k defaults and the payoff of its straight debt is p_k^k, bank j needs to repay $\lambda_j^k(\bar{p}^k - p_k^k)$.
  - Contractual repayment should be $d_j^k = \lambda_j^k(\bar{p}^k - p_k^k)$.
    - d_j^k is a derivative product written on p_k^k.
    - d_j^k is non-increasing in p_k^k.

Contract repayments

- $\mathbf{d}^k(p_k^k) = (d_1^k(p_k^k), \ldots, d_n^k(p_k^k))^\top$: contract payment function of debts and CDSs with reference on bank k if the payoff of bank k's straight debt is given by p_k^k.

$$d_j^k(p_k^k) = \begin{cases} \bar{p}^k & \text{if } k = j, \\ \lambda_j^k \left(\bar{p}^k - p_k^k\right) & \text{if } k \neq j. \end{cases}$$

- The issuance structure of CDSs are described by the matrix

$$\mathbf{\Lambda} = \left(\lambda_j^k\right)_{j,k=1,\ldots,n},$$

where we set $\lambda_j^j = 1$.

Sub-senior structure of liabilities

- Define $\phi_j(k) \in \{1, \dots, n\}$ to be the order function of repayment of bank j with reference on k, where

$$\begin{aligned} \phi_j(k_1) = 1 &\Leftrightarrow \text{bank } j \text{ should repay } d_j^{k_1} \text{ first,} \\ \phi_j(k_2) = 2 &\Leftrightarrow \text{bank } j \text{ should repay } d_j^{k_2} \text{ second,} \\ &\vdots \\ \phi_j(k_n) = n &\Leftrightarrow \text{bank } j \text{ should repay } d_j^{k_n} \text{ last.} \end{aligned}$$

- The sum of bank j's repayment that is senior to d_j^k is given by

$$\bar{d}_j^k(\mathbf{p}) = \sum_{\phi_j(k') < \phi_j(k)} d_j^{k'}(p_{k'}^{k'}).$$

- Total amount of bank j's liabilities to repay is written as

$$\bar{d}_j^0(\mathbf{p}) = \sum_{k=1}^{n} d_j^k(p_k^k).$$

Payoffs

- Equities:
  - p_k^0 denotes the final payoff of bank k's equity.
- Straight debts:
  - p_k^k denotes the final payoff of bank k's debt with face value $\bar{p}_k$.
- CDSs:
  - p_j^k denotes the final payoff of d_j^k, the CDS written by j with reference bank k.
- Write $\mathbf{p}^k = \left(p_1^k, p_2^k, \ldots, p_n^k\right)^\top \in \boldsymbol{R}^n$ and define payment vector in the market:

$$\mathbf{p} = \left((\mathbf{p}^0)^\top, (\mathbf{p}^1)^\top, \ldots, (\mathbf{p}^n)^\top\right)^\top \in \boldsymbol{R}^{n(n+1)}$$

Financial assets cross-held

- Banks cross-hold debts, equities, CDSs issued by other banks.
- Denote by m_{ij}^k bank i's proportion of ownership of the CDS issued by bank j with reference on bank k.
  - Bank i has a right to receive $m_{ij}^k d_j^k(p_k^k)$, but the actual amount received is $m_{ij}^k p_j^k$.
- The ownership structure in the interbank market can be written by

$$\mathbf{M}^k = \left(m_{ij}^k\right)_{i,j=1,\ldots,n} \text{ for } k = 0, \ldots, n.$$

- Note that
  - $\mathbf{M}^0$ means equity ownership structure.
  - $\mathbf{M}^k$ includes debt ownership structure, $m_{ik}^k, i, k = 1, \ldots, n$.
- The total assets of all banks are written as

$$\mathbf{a}(\mathbf{p}; \boldsymbol{\Delta}) = (\mathbf{I} - \mathbf{C}\boldsymbol{\Delta})\mathbf{e} + \sum_{k=0}^{n} \mathbf{M}^k \mathbf{p}^k.$$

Clearing payment vector

- The payoff of d_j^k is written as

$$p_j^k = d_j^k(p_k^k) \wedge \Big(a_j(\mathbf{p};\Delta) - \bar{d}_j^k(\mathbf{p})\Big)^+ .$$

- The clearing payment vector is expressed as the fixed point

$$\mathbf{p} = \mathbf{f}(\mathbf{p}).$$

- Eisenberg and Noe (2001) and other papers determine the default banks by

$$\mathcal{D} = \{i \in \{1, \ldots, n\} | p_i^0 = 0\}. \qquad (*)$$

- In our model, the vector function **f** is neither continuous nor monotonic if we assume $(*)$.

Existence of Clearing System

Fictitious default algorithm with financial covenants

Assumption 1 (Fictitious default algorithm with financial covenants)

At maturity, the clearing default matrix is determined in the following way.

0. *Set* $\boldsymbol{\Delta}^{(0)} = \mathbf{O}$.
1. *For the first step:*
   (i) *Calculate* $\mathbf{p}^{(1)}$ *satisfying* $\mathbf{p}^{(1)} = \mathbf{f}(\mathbf{p}^{(1)}; \boldsymbol{\Delta}^{(0)})$.
   (ii) *Set* $\mathcal{D}^{(1)} = \{i \in \{1, \dots, n\} | p_i^0 = 0\}$, *and*
   (iii) *Update* $\boldsymbol{\Delta}^{(1)} = \mathrm{diag}(\{1_{\mathcal{D}^{(1)}}(i)\}_{i=1}^n)$.
2. *For the* ℓ*-th step:*
   (i) *Calculate* $\mathbf{p}^{(\ell)}$ *satisfying* $\mathbf{p}^{(\ell)} = \mathbf{f}(\mathbf{p}^{(\ell)}; \boldsymbol{\Delta}^{(\ell-1)})$.
   (ii) *Set* $\mathcal{D}^{(\ell)} = \mathcal{D}^{(\ell-1)} \cup \{i \in \{1, \dots, n\} | p_i^0 = 0\}$.
   (iii) *Update* $\boldsymbol{\Delta}^{(\ell)} = \mathrm{diag}(\{1_{\mathcal{D}^{(\ell)}}(i)\}_{i=1}^n)$.
3. *Stop when* $\boldsymbol{\Delta}^{(\ell)} = \boldsymbol{\Delta}^{(\ell-1)}$ *and set* $(\mathbf{p}, \Delta) = (\mathbf{p}^{(\ell)}, \Delta^{(\ell)})$.

Some remarks on our algorithm

- Our algorithm coincides with a generalised clearing vector in Kusnetsov and Veraart (2016).
- A natural assumption for default is
$$\mathcal{D} = \{i \in \{1, \ldots, n\} | p_i^0 = 0\}$$
with a clearing payment vector $\mathbf{p}$ as in Eisenberg and Noe (2001) and other related studies.
- Under Assumption 1, the above result does not necessarily hold and it can be that $i \in \mathcal{D}$ and $p_i^0 > 0$ at the same time.
    - Once a bank is taken as default in the sequential procedure, it should incur default costs and cannot be solvent for ever.
    $\Rightarrow$Technical default.
- $\mathcal{D}^{(\ell)}$ is non-decreasing in ℓ.
    - There always exists a clearing system if we have a vector $\mathbf{p}$ such that $\mathbf{p} = \mathbf{f}(\mathbf{p}; \boldsymbol{\Delta})$ for any $\boldsymbol{\Delta}$.

Contraction mapping

Assumption 2

$$\sum_{i=1}^{n} m_{ij}^{k} < 1$$

for $k = 0, \ldots, n$.

Lemma 1

For any given $\mathbf{\Delta}$*, the equation system* $\mathbf{p} = \mathbf{f}(\mathbf{p}; \mathbf{\Delta})$ *has a unique solution under Assumption 2.*

Sketch of the Proof: For a fixed $\mathbf{\Delta}$, $\mathbf{f}$ is continuous. Further under Assumption 2, the mapping $\mathbf{f}$ is contractive in l^1-norm as shown by Fischer (2014). Therefore we can apply Banach's fixed point theorem. □

Main theorem

Theorem 2

There exists a clearing system under Assumptions 1 and 2.

Proof: The theorem easily follows from Lemma 1 and the monotonicity of $\boldsymbol{\Delta}^{(\ell)}$. □

Remark 1

- *The result on existence does not depend on* $\boldsymbol{\Lambda}$*, issuing structure of CDSs.*
- *Banks can issue leveraged CDSs on other banks in our setting. In other words, we do not need to impose the condition* $\lambda_j^k < 1$*.*

Numerical Analysis

Simulations

1. Suppose a multivariate Merton (1974) model.
$$e_i = e_{i0} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma\omega_i\right\},$$
where $\omega_i \sim N(0,1)$. $\mathrm{Corr}[\omega_i, \omega_j] = \rho$ for $i \neq j$.
2. Conduct Monte Carlo simulations to get
$$\tilde{\mathbf{e}}^{(h)} = (\tilde{e}_1^{(h)}, \ldots, \tilde{e}_n^{(h)})^\top \text{ for } h = 1, \ldots, \eta$$
where η is the number of simulations.
3. For each $\tilde{\mathbf{e}}^{(h)}$, obtain the clearing system $(\mathbf{p}, \mathbf{\Delta})$ with our algorithm.
4. Calculate the probability
$$\mathbb{P}\{\#\text{ of defaulted banks is } \xi\}$$
for $\xi = 0, 1, \ldots, n$.

Market structure

- We say that the market is of type-(ℓ_1, ℓ_2, ℓ_3) for $\ell_1, \ell_2, \ell_3 \leq n-1$ if
  - each debt is cross-held by ℓ_1 banks,
  - each bank writes ℓ_2 names of CDSs,
  - each CDS is cross-held by ℓ_3 banks.
- Concretely, we set

$$\lambda_j^k = 1_{\{j=k\}} + \frac{1}{n} \times 1_{\left\{j \in \bigcup_{h=1}^{\ell_2} \{\mathrm{mdl}_n^{k+h}\}\right\}},$$

$$\phi_j(k) = \mathrm{mdl}_n^{k+1-j},$$

$$m_{ij}^k = \frac{1}{n} \times 1_{\left\{j=k, i \in \bigcup_{h=1}^{\ell_1} \{\mathrm{mdl}_n^{k-h}\}\right\}} + \frac{1}{n} \times 1_{\left\{j \in \bigcup_{h=1}^{\ell_2} \{\mathrm{mdl}_n^{k+h}\}, i \in \left(\bigcup_{h=1}^{\ell_3} \{\mathrm{mdl}_n^{k-h}\}\right) \setminus \{k\}\right\}},$$

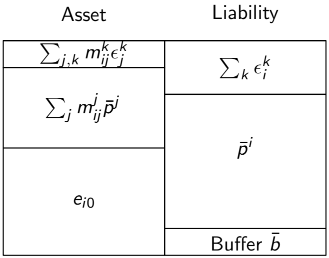
where

$$\mathrm{mdl}_n^h = \{i \in \{1, \ldots, n\} |\; i \equiv h \pmod{n}\}.$$

- Structure is symmetric.

Balance sheet

- Illustration of bank i's balance sheet at time 0:



  - ϵ_j^k represents the risk exposure (initial market value) of CDS d_j^k.
  - If we consider the market of type-(ℓ_1, ℓ_2, ℓ_3), we set

$$\epsilon_j^k = c_i \times \lambda_j^k \times \bar{p}^k \times \mathbb{P}\{\text{bank } k \text{ defaults}\}$$

    in the market of type-$(\ell_1, 0, 0)$.

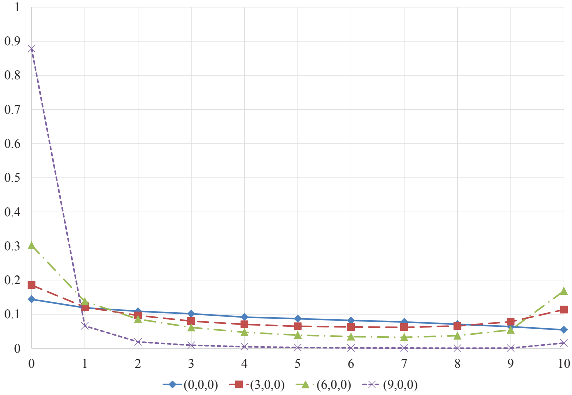
Parameter values

- Basecase parameters:

n	number of banks	10
$\bar{p}^i$	face value of debts	1
$\bar{b}$	buffer	0.2
c_i	default cost ratio	0.5
μ	growth rate of asset	0.05
σ	volatility of asset	0.5
ρ	correlation of assets	0.5
ℓ_1	connection via debt holding	6
ℓ_2	connection via CDS writing	6
ℓ_3	connection via CDS holding	6

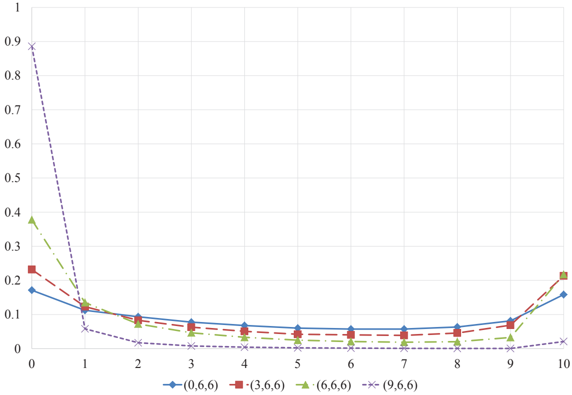
- Number of simulations: $\eta = 100,000$.

Default probabilities: effect of debt cross-holding (1)



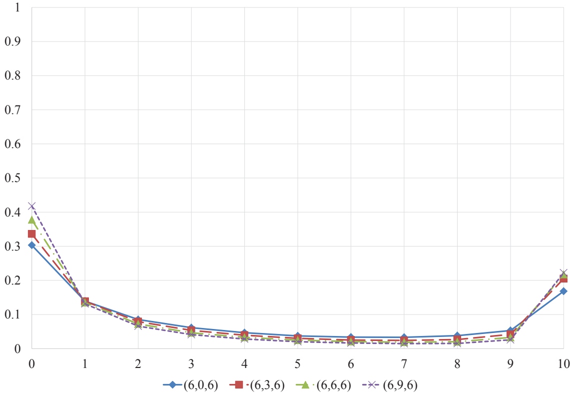
Market of type-$(\ell_1, 0, 0)$

Default probabilities: effect of debt cross-holding (2)



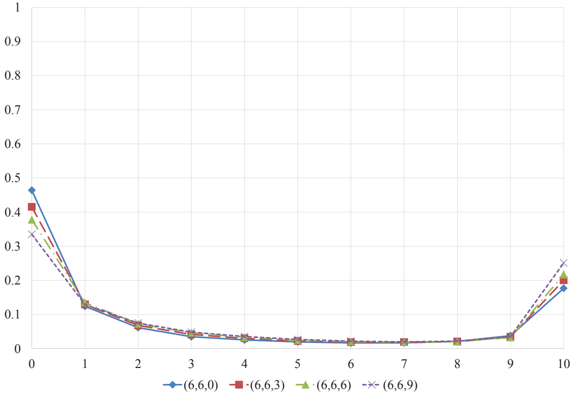
Market of type-$(\ell_1, 6, 6)$

Default probabilities: effect of CDS cross-writing



Market of type-$(6, \ell_2, 6)$

Default probabilities: effect of CDS cross-holding



Market of type-$(6, 6, \ell_3)$

Major observation

- Complete connections:
  - Straight debts; most stable financial market (Allen and Gale, 2000; Acemuglu et al., 2015).
  - CDS writing; extreme result (all survive or all default)
  - CDS holding; least stable (more banks default)
- Why?
  - Debt cross-trading works as profit-sharing since its payoff is non-decreasing in asset values.
  - CDS cross-trading works as loss-sharing since its payoff is non-increasing in asset values.
- This is the first study to show that the strongest connectedness (complete graph) leads to market vulnerability and systemic risk.

Conclusion

Conclusion

- Extended Fischer (2014) to the model with cross-tradings of CDS as well as banks' default costs and focus on CDS market.
- Proposed fictitious default algorithm with financial covenants.
- Proved existence theorem for clearing system.
- Showed with numerical examples that the cross-trading of CDS may increase the systemic risk of financial markets.

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Thank you for your attention