Dividend problems for insurance risk processes

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Economic point of view

- The word *dividend* comes from the Latin word *dividendum* meaning *the thing which is to be divided* and has got sense of *portion of interest on a loan, stock, etc.*
- Dividends are usually defined as the distribution of earnings in real assets among the shareholders of the firm (in proportion to their ownership).
- Dividends are paid from the firm's after-tax income. For the recipient, dividends are considered regular income and are therefore fully taxable.
- There are two sides of dividends policies in the modern corporate firms. The first are managers of the firm (insiders), the second are shareholders (outsiders). The interest of management and shareholders may not coincide. This has important consequences for dividend policy. There is a suggestion that former typically prefer a low payout in order to pursue growth maximizing strategies or consume additional benefits, while letters generally wish for a high payout since this will force the management to incur the inspection of the capital markets for each new project undertaken.
- We focus in this talk on the maximizing the cumulant dividend payments (we look at it only from the point of view of beneficiaries).

Cramér-Lundberg model

Usually the surplus $X = \{X_t, t \geq 0\}$ of an insurance company is described by Cramér-Lundberg process:

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where

C_k - i.i.d. claims with d.f. F

N_t - independent Poisson process with intensity λ

p - premium rate

Lévy process

X_t - spectrally negative Lévy process which is not subordinator

will model reserves of insurance company (in the absence of dividend payments), that is:

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De Finetti's problem

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$$\pi = (\tau^\pi, D_t^\pi) \text{ - "dividend-liquidation" policy}$$

is a pair of a **non-decreasing** left-continuous $\mathbf{F}$-adapted process D_t^π and an $\mathbf{F}$-stopping time τ^π.

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We observe process U^π up to $\tau^\pi \wedge \sigma^\pi$, where

$$\sigma^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$$

Impulse control

We assume that for $t < \sigma^\pi$

$$\Delta D_t^\pi := D_{t+}^\pi - D_t^\pi < U_t^\pi$$

When dividends are being paid it could be added fixed transaction cost $K > 0$ that are not transferred to the beneficiaries (case $K = 0$ means no transactions costs). In this case we assume additionally that:

$$\Delta D_t^\pi \geq K$$

In this case π is given by an increasing sequence $0 \leq T_1 \leq T_2 \leq \ldots$ of $\mathbf{F}$-stopping times representing the times at which a dividend payment is made and a sequence of positive $\mathcal{F}_{T_i}$-measurable random variables $J_i \geq K$ representing the sizes of the dividend payments. Then,

$$D_t^\pi = \sum_{k=1}^{N_t^\pi} J_k$$

where $N_t^\pi = \#\{k : T_k \leq t\}$

Value function

- cumulative discounted dividends received until the moment of ruin:

$$\mathcal{D}_\pi(x) = \mathbb{E}_x\left[\int_0^{\sigma^\pi \wedge \tau^\pi} \mathrm{e}^{-qt}(dD_t^\pi - K)_+\right]$$

where

$$\int_0^t (dD_s^\pi - K)_+ = \begin{cases} D_t^\pi & \text{if } K = 0, \\ \sum_{s \le t} \mathbf{1}_{\{\Delta D_t^\pi > K\}}(\Delta D_t^\pi - K) & \text{if } K > 0 \end{cases}$$

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- Gerber-Shiu penalty/reward function:

$$\mathcal{W}_w^\pi(x) = \mathbb{E}_x \left[\mathrm{e}^{-q(\sigma^\pi \wedge \tau^\pi)} w \left(U_{\sigma^\pi \wedge \tau^\pi}^\pi \right) \right]$$

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Value function:

$$v_\pi(x) = \mathcal{D}_\pi(x) + \mathcal{W}_w^\pi(x)$$

Goal

We want to find policy

$$\pi^* = (\tau^{\pi^*}, D^{\pi^*}) = (\tau^*, D^*)$$

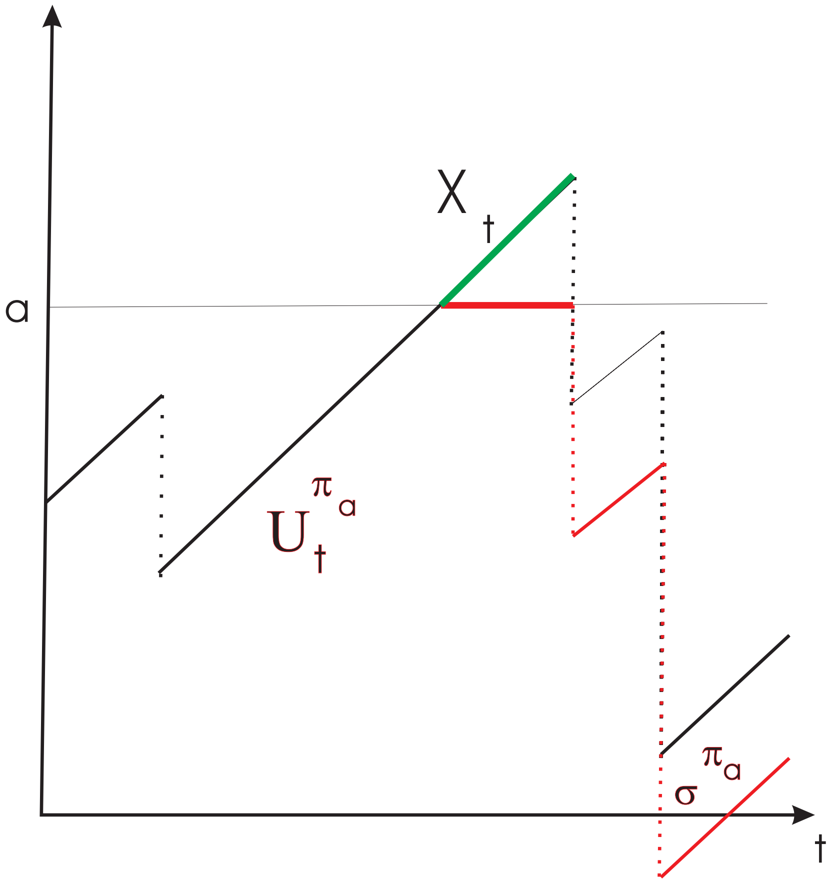
that maximizes

$$v_\pi(x) = \mathbb{E}_x\left[\int_0^{\sigma^\pi \wedge \tau^\pi} \mathrm{e}^{-qt}(dD_t^\pi - K)_+\right] + \mathbb{E}_x\left[\mathrm{e}^{-q(\sigma^\pi \wedge \tau^\pi)} w\left(U^\pi_{\sigma^\pi \wedge \tau^\pi}\right)\right]$$

Then

$$v_*(x) = \sup_\pi v_\pi(x) = v_{\pi^*}(x)$$

Barrier strategy π_a for $K = 0$



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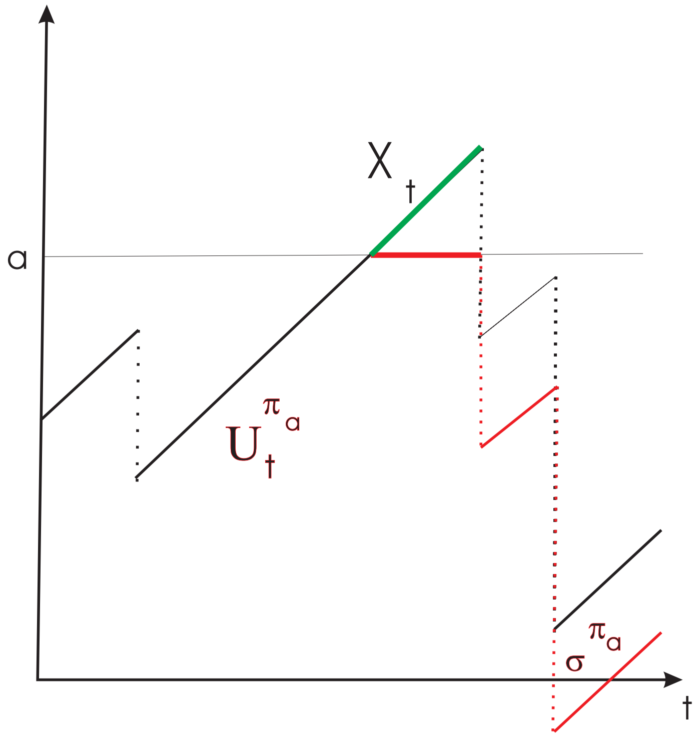


'If the barrier is too high, then we will wait too long for the risk process to hit the barrier and if we put the barrier too low then we derive the ruin too quickly.' We can then expect the existence of the 'optimal barrier'.

Local time at maximum

For the barier strategy at a and for $K = 0$:

$$D_t^{\pi_a} = a \vee \overline{X}_t - a, \qquad \text{where } \overline{X}_t = \sup_{s \le t} X_s$$

and

$$\{U_t^{\pi_a}, t \le \sigma^{\pi_a}; U_0^{\pi_a} = x\} \stackrel{D}{=} \{a - Y_t, t \le \sigma_a; Y_0 = a - x\}$$

where

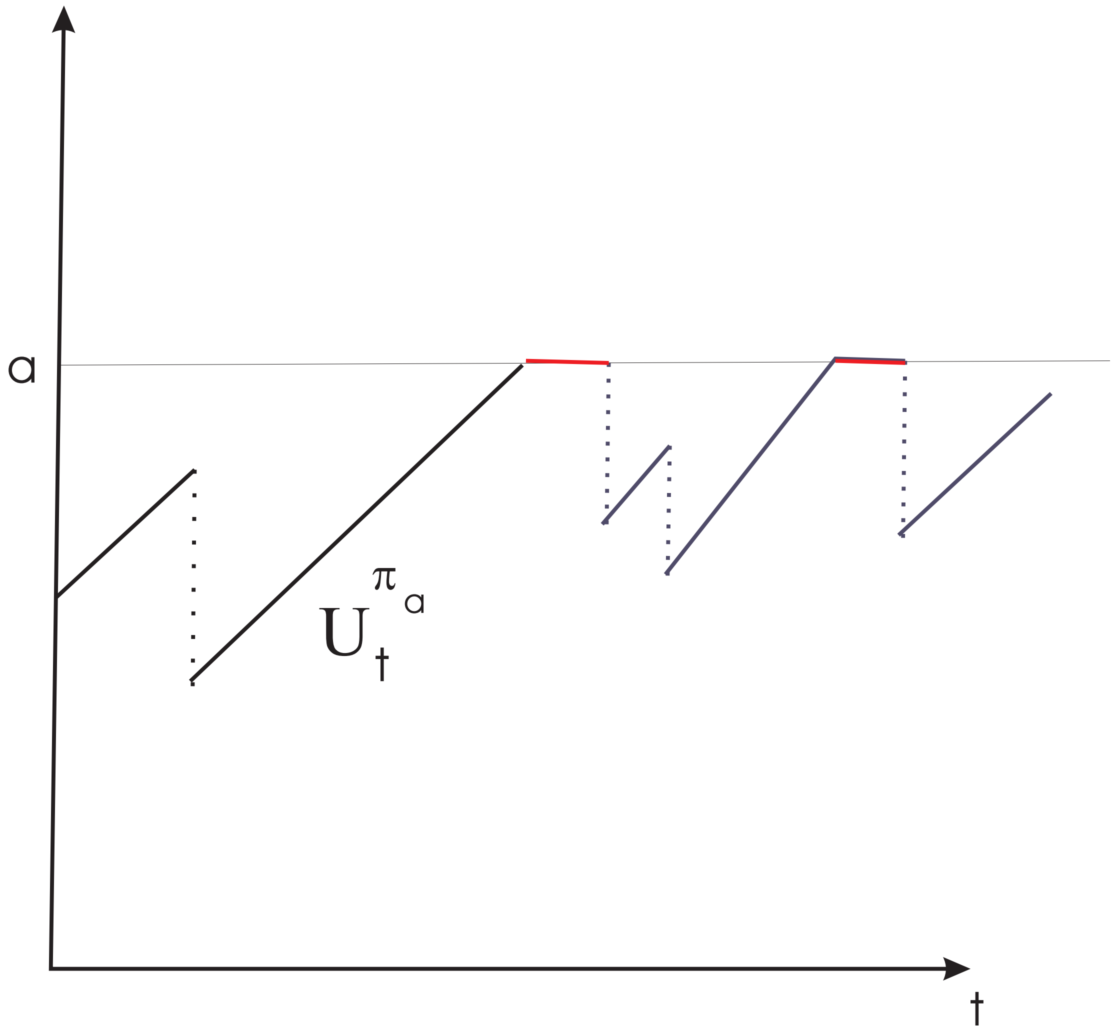
$$Y_t = (a \vee \overline{X}_t) - X_t$$

is reflected process at maximum and

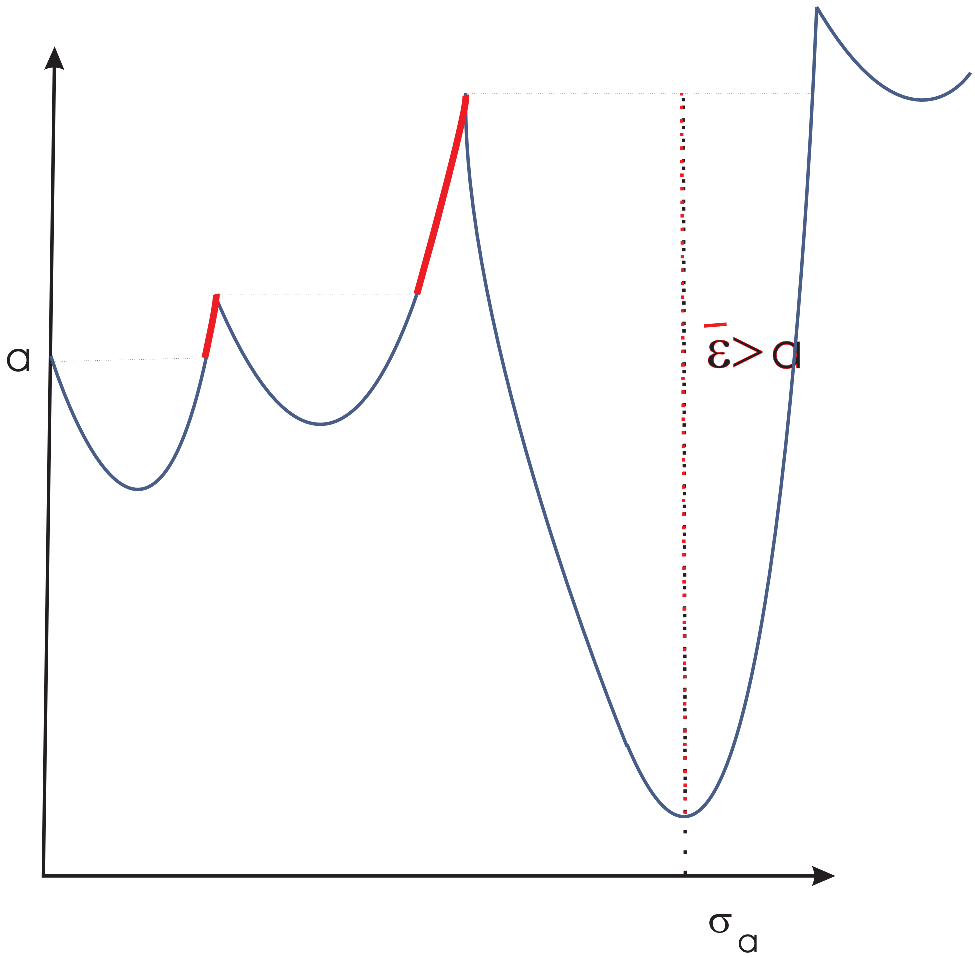
$$\sigma_a = \inf\{t > 0 : Y_t > a\}$$

is the first exit time of Y from interval $[0, a]$

Controlled process once again



Discounted local time



Bivariate subordinator

$$\mathcal{D}^{\pi_a}(x) = \int_0^{\sigma_a} e^{-qt} dD_t^{\pi_a} = \int_0^{\infty} e^{-qD_t^{\pi_a,-1}} \mathbf{1}_{(\sup_{s\leq t} \bar{\epsilon}_s \leq a)} dt$$
$$= \int_0^{\infty} e^{-q\xi_t} \mathbf{1}_{(\sup_{s\leq t} \Delta\eta_s \leq a)} dt$$

where $\Delta\eta_t = \eta_t - \eta_{t-}$ and $(\xi, \eta) = \{(\xi_t, \eta_t) : t \geq 0\}$ is a bivariate subordinator

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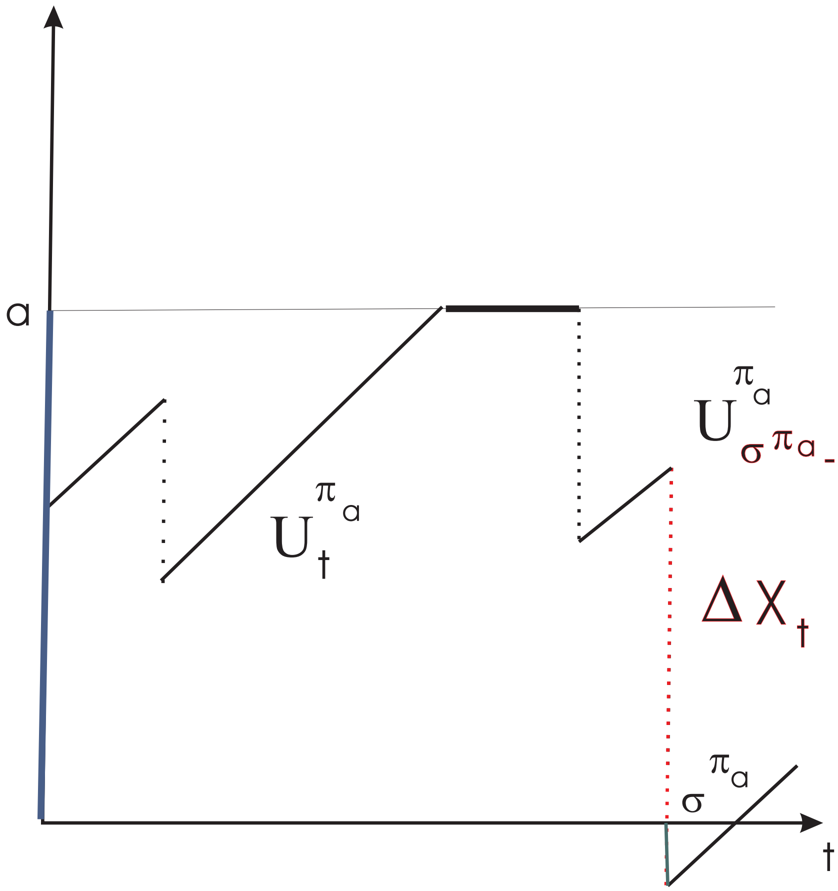
Theorem 1. (Kyprianou and Palmowski (2007)) For $n = 1, 2, 3, \ldots$ we have:

$$\mathbb{E}\left[\left(\int_0^{\infty} e^{-q\xi_t} \mathbf{1}_{(\sup_{s\leq t} \Delta\eta_s \leq a)} dt\right)^n\right] = n! \prod_{k=1}^{n} \frac{1}{\Lambda(qk) + \nu_{\Lambda(qk)}(a, \infty)}$$

where $\Lambda(q)$ is a Laplace exponent of ξ and $\nu_{\Lambda(q)}$ is a Lévy measure of η under the change of measure:

$$\frac{d\mathbb{P}_t^{\Lambda(q)}}{d\mathbb{P}_t} = e^{\Lambda(q)t - q\xi_t}$$

Barrier strategy π_a for $K = 0$



$$\mathcal{W}_w^{\pi_a}(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\sigma^{\pi_a}} w(U_{\sigma^{\pi_a}}^{\pi_a})\right]$$

Barrier strategy π_a for $K = 0$

$$\begin{aligned}\mathcal{W}_w^{\pi_a}(x) &= \mathbb{E}_x\left[\mathrm{e}^{-q\sigma^{\pi_a}} w(U^{\pi_a}_{\sigma^{\pi_a}})\right] \\ &= \mathbb{E}_x\left[\sum_{t\geq 0} \mathrm{e}^{-qt} w(U^{\pi_a}_{t-} + \Delta X_t)\mathbf{1}_{\{t<\sigma^{\pi_a},\ \Delta X_t < U^{\pi_a}_{t-}\}}\right] \\ &= \int_0^a \int_y^\infty w(y-z)\Pi_X(dz) R^{(q)}(x,dy) = \int_0^a K_w(y) R^{(q)}(x,dy)\end{aligned}$$

where $R^{(q)}(x,dy)$ is the resolvent of reflected at maximum process Y killed when exiting from the interval $[0,a]$:

$$\begin{aligned}R^{(q)}(x,dy) &= \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(Y_t \in dy, t < \sigma_a)dt \\ &= \left[\frac{W^{(q)}(x)}{W^{(q)\prime}(a)} W^{(q)\prime}(a-y) - W^{(q)}(x-y)\right] dy \\ &\quad + \frac{W^{(q)}(x)}{W^{(q)\prime}(a)} W^{(q)}(0)\delta_a(dy)\end{aligned}$$

Scale functions

Laplace exponent: $\psi(\theta)$:

$$\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\psi(\theta)}$$

$\Phi(q)$ - greatest root of equation $\psi(\theta) = q$

First scaling function: $W^{(q)} : [0, \infty) \to [0, \infty)$:

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(y) dy = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q)$$

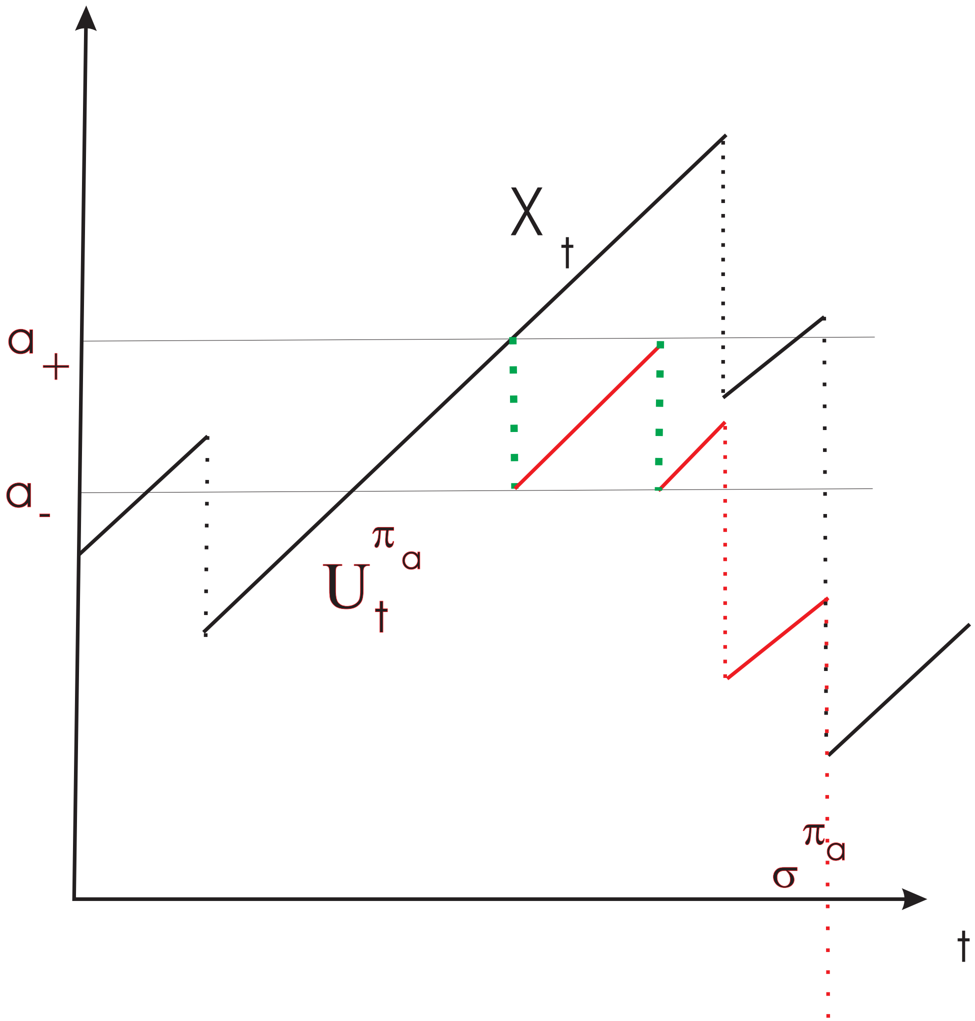
$W^{(q)}$ is differentiable (not necessary continuously) and

$$W(x) = W^{(0)}(x)$$

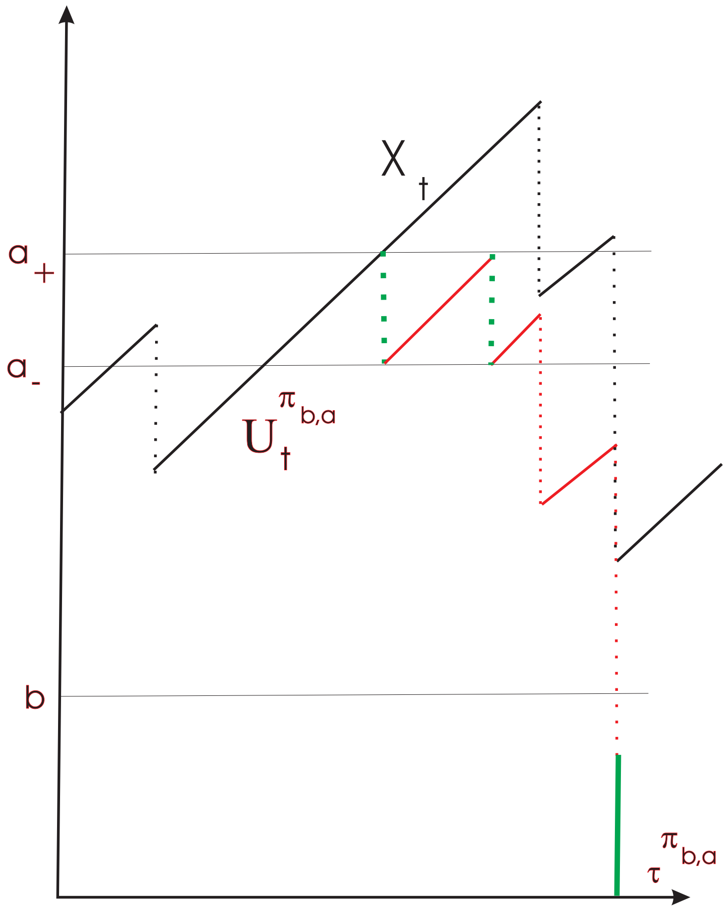
Second scaling function:

$$Z^{(q)}(y) = 1 + q \int_0^y W^{(q)}(z)\, dz$$

Barrier strategy π_a for $K > 0$

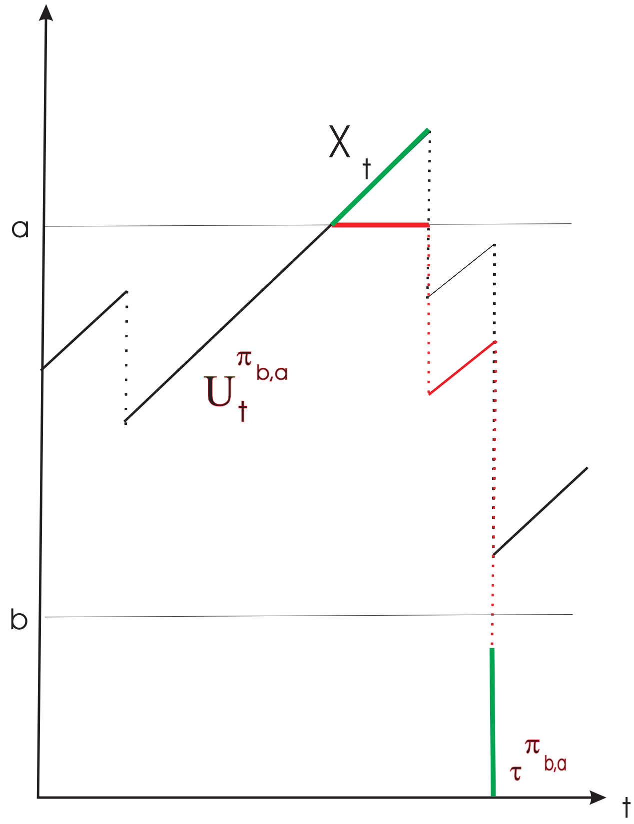


Barrier-liq. strategy $\pi_{b,a}$, $K > 0$



$\pi_{b,a}$ **for** $a = (a_-, a_+)$ **and** $b > 0$

Barrier-liq. strategy $\pi_{b,a}$, $K = 0$



$\pi_{b,a}$ for $a > b$ and $b > 0$ ('limiting case' when $a_+ = a_- = a$)

For simplicity we will use the same notation $\pi_{b,a}$ for both cases

Value function for $\pi_{b,a}$

Define:

$$\tilde{w}(x) = w(x-b), \qquad K_{\tilde{w}}(y) = \int_{-\infty}^{-y} \tilde{w}(y+z)\Pi_X(dz) < \infty$$

$$F_{\tilde{w}}(x) = -\int_0^x W^{(q)}(x-y)K_{\tilde{w}}(y)dy$$

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Theorem 2.

$$v_{\pi_{b,a}}(x) := v_{b,a}(x) = \begin{cases} w(x) & x < b \\ W^{(q)}(x-b)G_b(a) + F_{\tilde{w}}(x-b) & x \in [b, a_+] \end{cases}$$

where

$$G_b(a) := \begin{cases} \dfrac{[\Delta a - K - \Delta F_{\tilde{w}}(a-b)}{\Delta W^{(q)}(a-b)} & \text{if } K > 0 \\ \dfrac{1 - F'_{\tilde{w}}(a-b)}{W^{(q)\prime}(a-b)} & \text{if } K = 0 \end{cases}$$

and $\Delta a = a_+ - a_-$, $\quad \Delta g(a-b) = g(a_+ - b) - g(a_- - b)$.

Optimality of $\pi_{b,a}$

We choose optimal barriers $a^* = (a^*_-, a^*_+)$ and b^* (for $K = 0$: $a^* = a^*_- = a^*_+$). Let:

$$\mathcal{I}^*_w := \sup_{x>0} \mathcal{I}_w(x), \qquad \mathcal{I}_w(x) = \mathcal{A}w(x) - qw(x)$$

where

$$\mathcal{A}f(x) = \frac{\sigma^2}{2} f''(x) + pf'(x) + \int_{-\infty}^{0} \left[f(x+y) - f(x) - f'(x) y 1_{\{|y|<1\}} \right] \Pi_X(dy)$$

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Theorem 3. (APP (2013))
(i) If $\mathcal{I}^*_w \leq 0$, then it is optimal to stop immediately for all levels $x > 0$ of the reserves ($\tau_* \equiv 0$).

Assume that $\mathcal{I}^*_w > 0$. Then

(ii) π_{b^*,a^*} is the optimal strategy in the set of optimal strategies bounded by a^*_+;

(iii) if $(\mathcal{A}v_{b^*,a^*} - qv_{b^*,a^*})(x) \leq 0$ for $x > a^*_+$, then π_{b^*,a^*} is optimal strategy and $v_* = v_{b^*,a^*}$. In particular, if Lévy measure admits a convex density then barrier-liquidation strategy is optimal.

HJB equation and Verification thm.

Hamilton-Jacobi-Bellman's (HJB) system of equations:

$$\mathcal{A}f(x) - qf(x) \leq 0$$
$$f(x) \geq f(y) + x - y - K$$
$$f(x) = w(x), \qquad \text{for } x < 0$$
$$f(x) \geq w(x), \qquad \text{for } x \geq 0$$

Verification Theorem

Let π be an admissible dividend strategy such that v_π is absolutely continuous and ultimately dominated by some affine function. If HJB equation holds for v_π then $v_\pi(x) = v_*(x)$ for all $x \geq 0$.

Idea of the proof

Let g be a càdlàg function satisfying:
$g'(x) \geq 1$ for $x > 0$
$g(x) \leq w(x)$ for $x < 0$
and there exists linear function dominating g

For any closed interval $I \subset [0, \infty)$ we denote

$$T_I := \inf\{t \geq 0 : X_t \notin I\}$$

We will say that g is a local stochastic supersolution on I of the HJB equation if

$$\overline{M}_t^{g,T_I} := \left\{ e^{-q(t \wedge T_I)} g\left(X_{t \wedge T_I}\right), \ t \geq 0 \right\}$$

is a UI supermartingale.

Function g is a stochastic supersolution if g is a local stochastic supersolution for all closed intervals I.

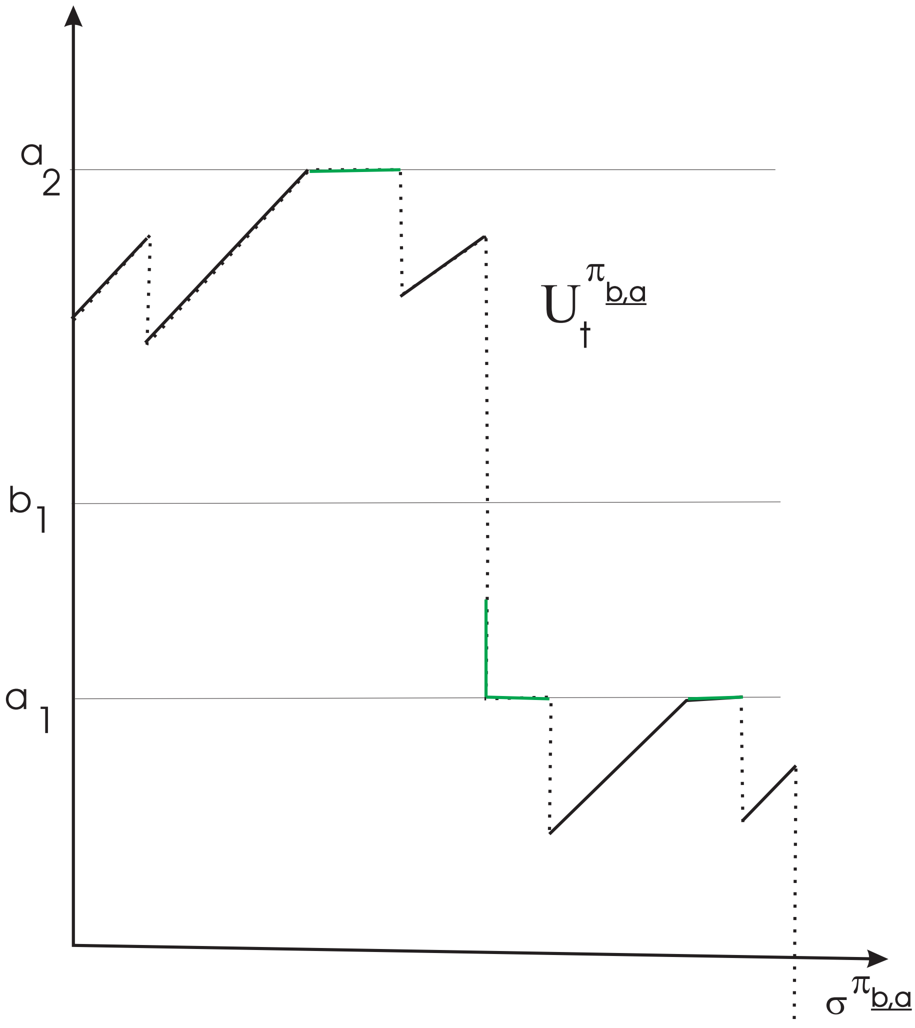
Dual representation

Theorem 4.
The value function v_* is the smallest stochastic supersolution of the HJB equation:

$$v_*(u) = \min_{g \in \mathcal{G}} g(u)$$

where $\mathcal{G}$ denotes the family of stochastic supersolutions.

Band strategy $\pi_{\underline{b},\underline{a}}$



$\pi_{\underline{b},\underline{a}}$, $K = 0$

Theorem 5. Let $K = 0$. For $i \geq 1$ and $v_{\pi_{\underline{b},\underline{a}}} := v_{\underline{b},\underline{a}}(x)$:

$$v_{\underline{b},\underline{a}}(x) = \begin{cases} W^{(q)}(x - b_{i-1})G_{w_{i-1}}(a_i, b_{i-1}) + F_{w_{i-1}}(x - b_{i-1}) & x \in [b_{i-1}, a_i) \\ \\ v_{\underline{b},\underline{a}}(a_i -) + x - a_i & x \in [a_i, b_i) \end{cases}$$

where

$$F_{w_{i-1}}(x - b_{i-1}) = w'_{i-1}(b_{i-1}-)F_1(x) + w_{i-1}(b_{i-1}-)F_0(x) + F_{w_{i-1,0}}(x)$$

and

$$w_{i-1,0}(x) = v_{\underline{b},\underline{a}}(x - b_{i-1}) - v_{\underline{b},\underline{a}}(b_{i-1}) - (x - b_{i-1})v_{\underline{b},\underline{a}}(b_{i-1})$$

Optimality of $\pi_{\underline{b}^*,\underline{a}^*}$

Optimal bands:

a_i^* is determined by smooth fit condition of singular control:

$$0 = \lim_{x \downarrow a_i^*} v''_{\underline{a}^*,\underline{b}^*}(x) = \lim_{x \uparrow a_i^*} v''_{\underline{a}^*,\underline{b}^*}(x),$$

$b_i^* > 0$ is determined by smooth fit condition:

$$1 = \lim_{x \uparrow b_i^*} v'_{\underline{a}^*,\underline{b}^*}(x) = \lim_{x \downarrow b_i^*} v'_{\underline{a}^*,\underline{b}^*}(x)$$

if X has unbounded variation and determined by the continuous fit condition:

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Theorem 6. Band strategy $\pi_{\underline{b}^*,\underline{a}^*}$ is always optimal.

Exponential claims $\mathrm{Exp}(\mu)$

Let

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where $C_k = \mathrm{Exp}(\mu)$.

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where $A_\pm = \frac{\mu + q^\pm(q)}{q^+(q) - q^-(q)}$ and $q^+(q) = \Phi(q)$ and $q^-(q)$ solve equation $\psi(\theta) = q$:

$$q^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}$$

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Let consider piecewise-linear Gerber-Shiu function:

$$w(x) = cx\mathbf{1}_{\{x<0\}} + (x - K)\mathbf{1}_{\{x>0\}}$$

Exponential claims $\mathrm{Exp}(\mu)$

Liquidation strategy is optimal when:

$$\mathcal{I}_w^* \leq 0$$

$$\Leftrightarrow$$

$$c \geq \max\{c_1, c_2\}$$

where

$$c_1 = (p\mu/\lambda + K(\mu + \frac{q\mu}{\lambda})), \qquad c_2 = 1 + K\mu + \frac{q}{\lambda}\exp\left\{(\frac{p\mu - \lambda}{q} + K\mu - 1)_+\right\}$$

If $c \in [c_1, c_2)$, then barrier-liquidation strategy π_{b^*,a^*} with $b^* > 0$ is optimal.

If $c < \min\{c_1, c_2\}$, then barrier strategy π_{a^*} is optimal (hence $b^* = 0$).

Premium dependent risk process

Risk process with state-dependent premium:

$$X_t = x + \int_0^t p(X_s)ds - \sum_{k=1}^{N_t} C_k$$

where p is a deterministic, increasing and Lipschitz function

We assume that

$$\int_0^\infty e^{-qt} p(x_t^x)dt \leq Ax + B$$

for some $A, B \geq 0$ and $x_t^x = x + \int_0^t p(x_s^x)ds$

De Finetti’s problem

Regulated risk process X_t^π satisfies:

$$U_t^\pi = x + \int_0^t p(U_s^\pi)ds - \sum_{k=1}^{N_t} C_k - D_t^\pi$$

where π is a admissible strategy such that D_t^π denotes cumulative amount of dividends transferred up to time t

Remark

For $p \neq$ const

$$U_t^\pi = X_t - D_t^\pi$$

DOES NOT HOLD TRUE !

This is true for the Cramér-Lundberg risk process X and for Lévy risk process X though.

Barrier strategy π_a

We define:

$$
\begin{aligned}
W_q(x) &= \lim_{y\to\infty} \mathbb{E}_x[e^{-q\tau_y^+}, \tau_y^+ < T]/\mathbb{E}_0[e^{-q\tau_y^+}, \tau_y^+ < T] \\
G_{q,w}(x) &= \mathbb{E}_x\left[e^{-qT} w(X_T)\mathbb{I}_{\{T<\infty\}}\right]
\end{aligned}
$$

where $\tau_a^+ = \inf\{t \geq 0 : X_t > a\}$, $T = \inf\{t \geq 0 : X_t < 0\}$.

Function $G_{q,w}(x)$ is so-called Gerber-Shiu function when the penalty function depend only on deficit at ruin

Additionally, for $x \in (0, a)$:

$$
\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<T\}}\right] = \frac{W_q(x)}{W_q(a)}
$$

Value function for π_a

We will assume form now on that d.f. F of claim sizes C_k has density f.
Then the functions W_q, $G_{q,w}$ are differentiable.

Theorem 7.
For the barrier strategy π_a we have:

$$v_a(x) \;=\; v_{\pi^a}(x) = \begin{cases} \frac{W_q(x)}{W_q'(a)}\Big(1 - G_{q,w}'(a)\Big) + G_{q,w}(x) & x \leq a \\ x - a + v_a(a) & x > a \end{cases}$$

Furthermore, v_a is continuously differentiable for all $x > 0$.
Note that

$$v_a'(a) = 1$$

Verification theorem

Hamilton-Jacobi-Bellman's (HJB) system of equations:

$$\max\{\mathcal{A}f(x) - qf(x), 1 - f'(x)\} = 0 \text{ for } x \geq 0$$
$$f(x) = w(x) \text{ for } x < 0$$

where

$$\mathcal{A}f(x) = p(x)f'(x) + \int_0^\infty (f(x-y) - f(x))\, \lambda dF(y)$$

is an extended generator of X

(f' is understood above as Radon-Nikodym derivative with respect of Lebesgue measure)

Verification Theorem

Let π be an admissible dividend strategy such that v_π is absolutely continuous and ultimately dominated by some affine function. If HJB equation holds for v_π then $v_\pi(x) = v_*(x)$ for all $x \geq 0$.

Optimal barrier strategy

Let

$$H_q'(y) := \frac{W_q'(y)}{1 - G_{q,w}'(y)}.$$

Optimal barrier:

$$a^* = \sup \left\{ a \geq 0 : H_q'(a) \leq H_q'(y) \text{ for all } y \geq 0 \right\}$$

where $H_q'(0) = \lim_{y \downarrow 0} H_q'(y)$

Theorem 8.
Suppose that

$$H_q'(a) \geq H_q'(b) \qquad \text{for all } a^* \leq a \leq b$$

Then the barrier strategy at a^* is optimal, that is, $v(x) = v_{a^*}(x)$ for all $x \geq 0$.

Sufficient conditions

Theorem 9.
Suppose that either

(i) f is convex and p is concave
or
(ii) there is no penalty function $(w \equiv 0)$ and that f is decreasing and

$$p'(x) \leq q + \lambda, \qquad x \geq a^*$$

where p' is the density of the premium rate p.

Then the barrier strategy at a^* is optimal, that is, $v_*(x) = v_{a^*}(x)$ for all $x \geq 0$.

Exponential claim size

Assume that the claim size has an exponential distribution with parameter μ

Let $p(x) = c + \epsilon x$ for some $\epsilon < q$.

Functions W_q i $G_{q,w}$ solve:

$$\mathcal{A}W_q(x) = qW_q(x) \quad \text{for } x \geq 0$$

$$W_q(x) = 0 \quad \text{for } x < 0$$

$$\mathcal{A}G_{q,w} = qG_{q,w}(x) + \omega(x) \quad \text{for } x \geq 0$$
$$G_{q,w}(x) = w(x) \quad \text{for } x < 0$$

where

$$\omega(x) = \int_x^\infty w(x - z)\mu e^{-\mu z} dz$$

Example

Let:

$$s(x) = U\big(\tfrac{q}{\epsilon} + 1,\ \tfrac{\lambda+q}{\epsilon} + 1,\ \mu x + \tfrac{\mu c}{\epsilon}\big)\,(\epsilon x + c)^{\frac{\lambda+q}{\epsilon}}\ \exp(-\mu x)$$

and

$$Gg(x) = \tfrac{\Gamma(q/\epsilon+1)}{\Gamma((q+\lambda)/(1+\epsilon))}\tfrac{1}{\epsilon}\,\big(\tfrac{\mu}{\epsilon}\big)^{(\lambda+q)/\epsilon}\,(\epsilon x + c)^{(\lambda+q)/\epsilon}\exp(-\mu x - \tfrac{\mu c}{\epsilon})\times$$
$$\Big(-U(x)\int_0^x M(v) - M(x)\int_x^\infty U(v) + \tfrac{M(0)}{U(0)}\,U(x)\int_0^\infty U(v)\Big)\,g(v)\,dv$$

where $U(x)$ and $M(x)$ are Kummer's functions and

$$g(x) = -\frac{\lambda}{p(x)}\Big(\frac{\partial}{\partial x} + \mu\Big)\,\omega(x)$$

Then from Albreceher et al. (2013) we can conclude that:

$$G_{q,w}(x) = -(s(x) + Gg(x))$$

and

$$W_q(x) = M\big(\tfrac{q}{\epsilon} + 1,\ \tfrac{\lambda+q}{\epsilon} + 1,\ \mu x + \tfrac{\mu c}{\epsilon}\big)\,(\epsilon x + c)^{\frac{\lambda+q}{\epsilon}}\ \exp(-\mu x)$$

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and that

$$H_q'(x) = \frac{W_q'(x)}{1 - G_{q,w}'(x)}$$

Now we can find the optimal barrier a^* solving the equation

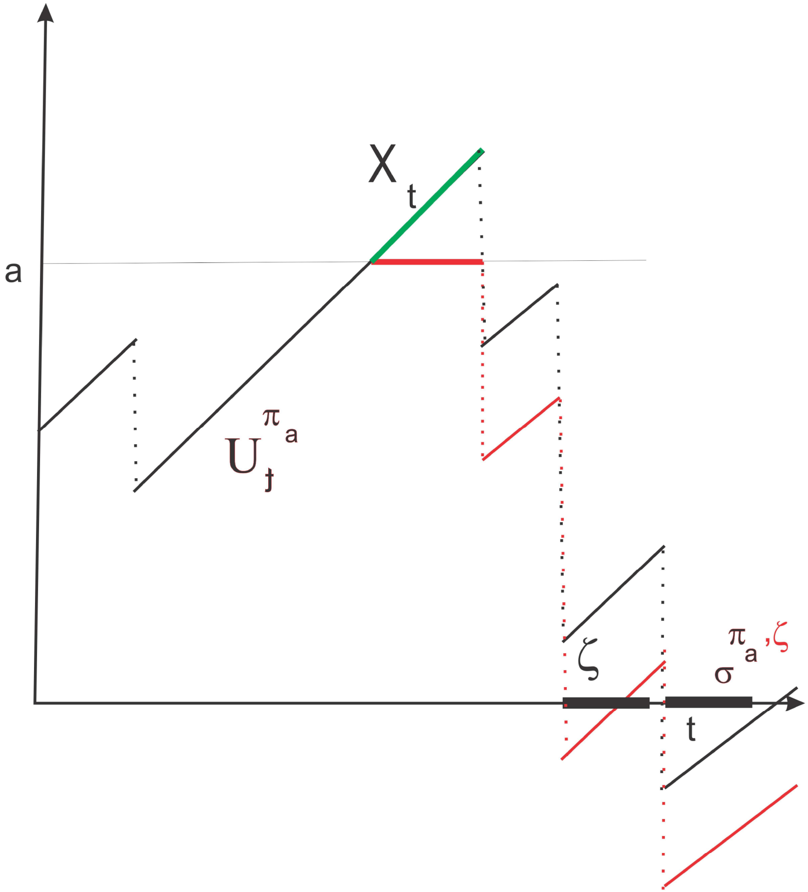
$$H_q''(a^*) = 0$$

DCF pricing

DCF - Discounted Cash Flow (Gajek and Kuciński (2011), only for Cramér-Lundberg process)

Pricing PZU This method produces price 8,03 bn EUR, the true market price in 2010 was: 7,97 bn EUR

Parisian barrier startegy π_a



Parisian scale function

Everything remains the same if one takes (Czarna and Palmowski (2013)):

$$V^{(q)}(x) = e^{\Phi(q)x}\mathbb{P}_x^{\Phi(q)}(\tau^\zeta = \infty),$$

instead of $W^{(q)}(x)$, where

$$\tau^\zeta = \inf\{t > 0 : t - \sup\{s \le t : X_s \ge 0\} \ge \zeta, X_t > 0\}$$

Theorem 10. For $x \ge 0$:

$$\mathbb{P}_x(\tau^\zeta < \infty) = 1 - \mathbb{E}X_1 \frac{\int_0^\infty W(x+z) z \mathbb{P}(X_\zeta \in \mathrm{d}z)}{\int_0^\infty z \mathbb{P}(X_\zeta \in \mathrm{d}z)}$$

Exponential claims $\mathrm{Exp}(\mu)$

If

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k$$

where $C_k = \mathrm{Exp}(\mu)$, then

$$\mathbb{P}_x(\tau^\zeta < \infty) = \mathrm{e}^{(\frac{\lambda}{p}-\mu)x}\left(1 - \frac{p\mu - \lambda}{e^{-\lambda\zeta}p\mu + e^{-\lambda\zeta}\frac{\mu}{\zeta}D}\right)$$

for

$$D = \int_0^{p\zeta}(p\zeta - t)\mathrm{e}^{-\mu t}\sqrt{\frac{\mu\lambda\zeta}{t}}I_1(2\sqrt{t\mu\lambda\zeta})\mathrm{d}t$$

C-L - numerical an.

We take the following parameters:
$w(x) = 0$ for $x \leq 0$, $\mu = 2, \lambda = 2, q = 0.1, p = 2.5$. Then

$$a^* = 3.78.$$

ζ	0.1	0.3	0.7	2
$a^{*,\zeta}$	3.54	3.09	2.40	0.84

Table 1: Optimal barrier for various Parisian delays.

x	2	5	10	50
$v_*(x)$	12.57	15.71	20.71	60.71
$v_{a^{*,\zeta}}(x)$	13.38	16.40	21.40	61.40

Table 2: Discounted cumulative dividends for classical and Parisian ruin for various initial capitals.

C-L - numerical an.

Let

$$\zeta = 0.3.$$

x_1	2.69	5.69	10.69	50.69
x	2	5	10	50
$v(x_1) = v_{a^*,\zeta}(x)$	13.38	16.40	21.40	61.40

Table 3: How much more initial capital is required to have the same amount of dividend payments for classical and Parisian ruin.

Brownian motion with drift

$$X_t = x + \sigma B_t + pt$$

For $x \geq 0$

$$\mathbb{P}_x(\tau^\zeta < \infty) = e^{-(2p\sigma^{-2})x} \frac{\Xi\left(\frac{p}{\sigma}\sqrt{\frac{\zeta}{2}}\right) - \frac{p}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}{\Xi\left(\frac{p}{\sigma}\sqrt{\frac{\zeta}{2}}\right) + \frac{p}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}$$

where

$$\Xi(x) = 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2}$$

and $\mathcal{N}(\cdot)$ is a distribution function of standard normal random variable.

Brownian motion with drift

Let

$$X_t = \sigma B_t + pt$$

Then:

$$W^{(q)}(x) = \frac{1}{\sigma^2 \delta}[\mathrm{e}^{(-\omega+\delta)x} - \mathrm{e}^{-(\omega+\delta)x}]$$

where

$$\delta = \sigma^{-2}\sqrt{p^2 + 2q\sigma^2}$$

and

$$\omega = p/\sigma^2$$

Brownian motion with drift

$$a^* = \log \left| \frac{\delta + \omega}{\delta - \omega} \right|^{1/\delta}$$

for $\delta{=}\sigma^{-2}\sqrt{p^2 + 2q\sigma^2}$ and $\omega = p/\sigma^2$ and

$$a^{*,\zeta} = \frac{\sigma^2}{p_q} \log \left[\frac{\Psi\left(\frac{p_q}{\sigma}\sqrt{\frac{\zeta}{2}}\right) - \frac{p_q}{\sigma}\sqrt{\frac{\zeta\pi}{2}}}{\Psi\left(\frac{p_q}{\sigma}\sqrt{\frac{\zeta}{2}}\right) + \frac{p_q}{\sigma}\sqrt{\frac{\zeta\pi}{2}}} \left(1 - \frac{2p_q}{p_q - p}\right) \right]$$

for

$$\Psi(x) = 2\sqrt{\pi} x \mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2}$$

and

$$p_q = \sqrt{p^2 + 2q\sigma^2}$$

Brownian mot. - numerical an.

We take the following parameters:
$w(x) = 0$ for $x \leq 0$, $\sigma = 2, p = 2.5$. Then

$$a^* = 5.28.$$

ζ	0.1	0.3	0.7	2
$a^{*,\zeta}$	4.48	3.89	3.12	1.17

Table 4: Optimal barrier for various Parisian delays.

x	2	5	10	50
$v_*(x)$	20.49	24.72	29.72	69.72
$v_{a^{*,\zeta}}(x)$	23.00	26.11	31.11	71.11

Table 5: Discounted cumulative dividends for classical and Parisian ruin for various initial capitals.

Brownian mot. - numerical an.

Let

$$\zeta = 0.3.$$

x_1	3.40	6.39	11.39	51.39
x	2	5	10	50
$v(x_1) = v_{a^*,\zeta}(x)$	23.00	26.11	31.11	71.11

Table 6: How much more initial capital is required to have the same amount of dividend payments for classical and Parisian ruin.

THANK YOU

- for the Invitation !
- for Your Attention !