The explicit formula and a motivic splitting

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Abstract We apply the Guinand-Weil-Mestre explicit formula to resolve two questions about how a certain hypergeometric motive splits into two irreducible motives.

1 Introduction

The classical explicit formula of Guinand and Weil was generalized to a broader context by Mestre in [1]. This formula applies to any L-function satisfying standard analytic properties, and gives a family of formulas for its conductor N. Mestre used it to get lower bounds on conductors of abelian varieties. This extended abstract gives an example of how it can be used in more exotic motivic contexts.

The example we pursue here has the form $M = M_8 \oplus M_6$, the factor motives being indexed by their degree. We assume that the associated L-functions really do have the required analytic properties, and work numerically to a precision that is adequate for being very confident in the assertions. Presently, we can compute directly with M, but not with the individual factors. We know that its conductor is $\mathrm{cond}(M) = 2^{15}$ and its local L-factor at 2 is just 1. These numerics imply that one of M_6 and M_8 is tame at 2, and the other is minimally wild. Also we know the order of central vanishing is $\mathrm{rank}(M) = 2$. This raises two questions:

Q1: $(\mathrm{rank}(M_6), \mathrm{rank}(M_8))$ can only be $(2,0)$, $(1,1)$, or $(0,2)$. Which is correct?
Q2: $(\mathrm{cond}(M_6), \mathrm{cond}(M_8))$ can only be $(2^6, 2^9)$ or $(2^7, 2^8)$. Which is correct?

The answers are given in the table at the end of this extended abstract. We provide enough computational details so that the reader can both reproduce our answers and attempt analogous calculations for other split motives.

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2 The motive $M = M_6 \oplus M_8$

One of the points of the talk was to illustrate how the *Magma* hypergeometric motives package by Mark Watkins lets one compute with hypergeometric motives of large degree. We use *Magma* language here as well [2], and the reader can repeat most computations using the free online *Magma* calculator.

To obtain the motive M and its L-function L, type

```
M:=HypergeometricData(
        [1/2: i in [1..16]], [0: i in [1..16]]);
L:=LSeries(M,1:Precision:=10,BadPrimes:=[<2,15,1>]);
```

Here *Magma* correctly understands that M has good reduction outside of 2. The optional argument ensures that it has the correct data at 2 as well, that being conductor 2^{15} and local L-factor 1. Other possibilities failing badly, correctness of the choice `<2,15,1>` is confirmed by `CheckFunctionalEquation(L)` returning 0.0000000000. The command `HodgeStructure(L:PHV)` says that M has weight $w = 15$ with Hodge vector

$$(h^{0,15}, h^{1,14}, \ldots, h^{14,1}, h^{15,0}) = (1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1).$$

In particular M can only appear in the cohomology of varieties of dimension ≥ 15.

In general, if d is even and the α_i's and the β_j's are obtained from one another by adding $1/2$ modulo $\mathbb{Z}$, then $H(\alpha,\beta|1)$ decomposes as a sum of two motives of specified degrees. In our case, we know *a priori* that $M = M_8 \oplus M_6$. `Factorization(EulerFactor(L,3))` then yields $f_3(x)$ in two seconds:

$$\begin{aligned}
&(1 - 268\cdot 3x + 204193\cdot 3^4x^2 - 1001800\cdot 3^9x^3 + 204193\cdot 3^{19}x^4 \\
&\qquad -268\cdot 3^{31}x^5 + 3^{45}x^6) \\
&(1 + 2992\cdot x + 39116\cdot 3^4x^2 - 7596496\cdot 3^6x^3 - 203836426\cdot 3^{12}x^4 \\
&\qquad -7596496\cdot 3^{21}x^5 + 39116\cdot 3^{34}x^6 + 2992\cdot 3^{45}x^7 + 3^{60}x^8).
\end{aligned}$$

Thus, M_6 and M_8 are both irreducible. Moreover Newton-over-Hodge forces the Hodge vector of M to decompose nicely into $h_6 + h_8$ with

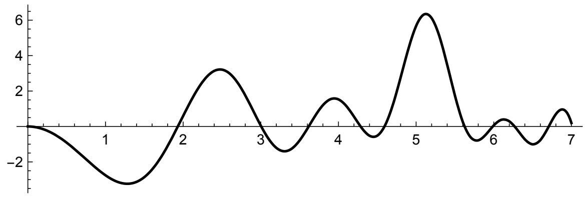
$$\begin{aligned}
h_6 :&= (0,1,0,1,0,1,0,0,0,0,1,0,1,0,1,0),\\
h_8 :&= (1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1).
\end{aligned}$$

Likewise, but in 30 seconds, 8 minutes, and 2.5 hours now,

$$\begin{aligned}
f_5(x) &= (1 + 1614\cdot 5^3x + \cdots + 5^{45}x^6)(1 - 41208x + \cdots + 5^{60}x^8),\\
f_7(x) &= (1 + 248232\cdot 7x + \cdots + 7^{45}x^6)(1 + 667104x + \cdots + 7^{60}x^8),\\
f_{11}(x) &= (1 - 883812\cdot 11x + \cdots + 11^{45}x^6)(1 + 34438544x + \cdots + 11^{60}x^8).
\end{aligned}$$

Any two of the $f_p(x)$ are completely different Galois-theoretically, implying that the two factor M_k each have motivic Galois group as large as possible, namely GSp_k.

L has a functional equation with respect to $s \leftrightarrow 16-s$. `Sign(L)` immediately returns 1, so the analytic rank r of L is even. `Evaluate(L,8)` takes four seconds and returns 0.000000000, so $r \geq 2$. `Evaluate(L,8:Derivative:=2)` takes fourteen seconds and returns 7.851654518, so $r = 2$. The Hardy Z-function $Z(t)$ is a vertically rescaled version of $L(M, 8+ti)$. On $[0,7]$ it graphs out to



The double zero at $t = 0$ is visible. The next three zeros are $\gamma_1 \approx 1.93195000805$, $\gamma_2 \approx 3.00559765$, and $\gamma_3 \approx 3.61679$. Note that this calculation does not give any hints as to the desired factorization $Z(t) = Z_6(t)Z_8(t)$. In other words, we do not know which motive a given γ_i belongs to.

3 The explicit formula

Let M be a motive of odd weight w with L-function assumed to satisfy the Riemann hypothesis. Then its Hodge vector h, conductor N, analytic rank r, Frobenius traces $c_{p^e} = \mathrm{Tr}(\mathrm{Fr}_p^e|M)$, and zeros $1/2 + \gamma_k i$ in the upper half plane are related by $\log N =$

$$2\pi r\widehat{F}(0) + 4\pi \sum_k \widehat{F}(\gamma_k) + 4\sum_{j>0} h^j \int_0^\infty \widehat{F}(t)E_j(t)dt + 2\sum_{p^e} c_{p^e} \frac{\log p}{p^{(ew+e)/2}} F(e\log p).$$

Here F is an allowed test function, $E_j(t) = \log 2\pi - \Psi((1+j)/2 + it)$ with $\Psi(s) = \mathrm{Re}(\Gamma'(s)/\Gamma(s))$, and $h^{p-q} = h^{p,q}$.

The standard Odlyzko test function and its Fourier transform are

$$F_{\mathrm{Od}}(x) = \chi_{[-1,1]}\left((1-|x|)\cos(\pi x) + \frac{\sin|\pi x|}{\pi}\right), \qquad \widehat{F}_{\mathrm{Od}}(t) = \frac{4\pi\cos^2(t/2)}{(\pi^2-t^2)^2}.$$

Also allowed are the scaled functions $F_z(x) = F_{\mathrm{Od}}(x/\log z)$ and their Fourier transforms $\widehat{F}_z(t) = (\log z)\widehat{F}_{\mathrm{Od}}(t\log z)$.

4 Applying the explicit formula to M_6 and M_8

Computing c_{p^e} for our motive M is easily done by *Magma*. However, to get the decomposition $c_{p^e} = c^6_{p^e} + c^8_{p^e}$, even for just $e = 1$, we need to factor $f_p(x)$, which we can do only for $p \leq 11$. From the factorizations above, one has $c^6_3 = 268 \cdot 3$,

$c_9^6 = (268 \cdot 3)^2 - 2(204193 \cdot 3^4)$, etc. The explicit formula using $(F_{13}, \hat{F}_{13})$, with all terms divided by $\log 2$ for greater clarity, answers Questions 1 and 2:

	term_6	(Tends to 6 or 7) total_6	term_8	(Tends to 8 or 9) total_8	Comments
h	3.11142	3.11142	4.85928	4.85928	Hodge contribution
3	0.17011	3.28154	−0.63306	4.22622	Contributions
5	−0.35472	2.92682	0.07245	4.29897	from the successively
7	−0.07386	2.85296	−0.02836	4.27031	harder factorizations
9	−0.02269	2.83027	0.00183	4.27214	of Frobenius
11	0.00028	2.83055	−0.00101	4.27114	polynomials $f_p(x)$
r	2.99946	5.83002	2.99946	7.27060	Forced! **A1** : $(1,1)$
γ_1		5.83002	1.68061	8.95121	Forced! **A2** : $(2^6, 2^9)$
γ_2	0.13610	5.96612		8.95121	Forced!
⋮	⋮	⋮	⋮	⋮	
Total		6.00000		9.00000	

Terms are positive after the double line, and so these terms must be associated with either M_6 or M_8 so as to keep $(\text{total}_6, \text{total}_8)$ coordinatewise less than either $(6,9)$ or $(7,8)$. This forces the indicated answers. Thus, both motives have analytic rank 1. The prime 2 is tamely ramified in M_6 and minimally wildly ramified in M_8

Remarkably, the talk just described relates directly to two collaborative projects begun at the MATRIX Institute. The decomposition studied here is the $d = 16$ case of the sequence of decompositions mentioned in §4.ii in the abstract with Rodriguez Villegas. The Hodge vectors h_6 and h_8 also arise for the L-functions denoted L_{16} and L_{18} in the abstract with Broadhurst; conductors there are $1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$ and $7560 = 2^3 \cdot 3^3 \cdot 5 \cdot 7$ respectively.

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References

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2. W. Bosma, J. J. Cannon, C. Fieker, A. Steel (eds.) *Handbook of Magma functions, Edition 2.19* (2013), 5291 pages.