

Hypergeometric supercongruences

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Abstract We discuss two related principles for hypergeometric supercongruences, one related to accelerated convergence and the other to the vanishing of Hodge numbers.

1 Introduction

At the conference, we added two related principles to the study of supercongruences involving the polynomials obtained by truncating hypergeometric series. By a *supercongruence* we mean a congruence which somewhat unexpectedly remains valid when the prime modulus p is replaced by p^r for some integer $r > 1$. We call r the *depth* of the supercongruence.

The first principle is that a supercongruence is the first instance of a sequence of similar supercongruences, reflecting accelerated convergence of certain Dwork quotients. The second is that splittings of underlying motives can be viewed as the conceptual source of supercongruences, with the depth of the congruence being governed by the vanishing of Hodge numbers.

We present these principles here in a limited context, so that they can be seen as clearly as possible. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a length d vector of rational numbers in $(0, 1)$ and let $\beta = 1^d = (1, \dots, 1)$. We assume that that multiplication by any integer coprime to the least common multiple m of the denominators of the α_i 's preserves the multiset $\{\alpha_1, \dots, \alpha_d\}$ modulo \mathbb{Z} .

The associated classical hypergeometric series and its p -power truncations, for p prime, are as follows.

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$$F(\alpha, 1^d | t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_d)_k}{k!^d} t^k, \quad F_s(\alpha, 1^d | t) := \sum_{k=0}^{p^s-1} \frac{(\alpha_1)_k \cdots (\alpha_d)_k}{k!^d} t^k.$$

Our starting point was the list *CY3* of fourteen $\alpha = (\alpha_1, \dots, \alpha_4)$ associated to certain families of Calabi–Yau threefolds discussed in [8]. Each has a corresponding normalized Hecke eigenform $f = \sum a_n q^n$ of weight four and trivial character. For each, it was conjectured in [8] that

$$F_1(\alpha, 1^d | 1) \equiv a_p \pmod{p^3}, \quad p \nmid ma_p. \quad (1)$$

Some of these cases have been settled. For example, the case $\alpha = (1/5, 2/5, 3/5, 4/5)$ was proved by McCarthy [6], the corresponding modular form having level 25 [9]. Just before submitting this note, Long, Tu, Yui, and Zudilin [4] announced two different proofs of (1) for all fourteen cases in *CY3*.

2 Convergence to the unit root and Hodge gaps

The two principles stem from observations about common behavior of the examples in *CY3*. The first observation is that each supercongruence (1) seems to be part of a sequence. Dwork proved [2] that for $p \nmid m$

$$\frac{F_{s+1}(\alpha, 1^d | t)}{F_s(\alpha, 1^d | t^p)} \equiv \frac{F_s(\alpha, 1^d | t)}{F_{s-1}(\alpha, 1^d | t^p)} \pmod{p^s}, \quad s \geq 0. \quad (2)$$

Moreover, the rational functions $F_{s+1}(\alpha, 1^d | t)/F_s(\alpha, 1^d | t^p)$ converge as $s \rightarrow \infty$ to a Krasner analytic function which can be evaluated at a Teichmüller representative $\text{Teich}(\tau)$ which is not a zero of F_1 giving the *unit root* γ_p of the corresponding local L -series at p .

For $\alpha \in \text{CY3}$, computations suggest

$$\frac{F_s(\alpha, 1^d | 1)}{F_{s-1}(\alpha, 1^d | 1)} \equiv \gamma_p \pmod{p^{3s}}, \quad p \nmid ma_p, \quad s > 0, \quad (3)$$

where $\gamma_p \in \mathbb{Z}_p$ is the root of $T^2 - a_p T + p^3$ not divisible by p . Note that the case $s = 1$ reduces to (1) since $\gamma_p \equiv a_p \pmod{p^3}$.

Our second observation is that the appearance of a congruence to a power p^{3s} as opposed to the expected p^s is related to Hodge theory. Consider the hypergeometric family of motives $H(\alpha, 1^d | t)$ (see [1] for a computer implementation). For any $\tau \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ the motive $H(\alpha, 1^d | \tau)$ is defined over \mathbb{Q} , has rank d , weight $d - 1$ and its only non-zero Hodge numbers are $(h^{d-1,0}, \dots, h^{0,d-1}) = (1, \dots, 1)$. When $\tau = 1$ there is a mild degeneration and the rank drops to $d - 1$.

For $\alpha \in \text{CY3}$, the motive for $\tau = 1$ is the direct sum, up to semi-simplification, of a Tate motive $\mathbb{Q}(-1)$ and the motive $A = M(f)$ of the corresponding Hecke eigen-

form f of weight four. The Hodge numbers of A are $(1, 0, 0, 1)$. We view the gap of three between the initial 1 and the next 1 as explaining the supercongruences (3).

3 A congruence of depth five

To illustrate our two observations further, we use the decomposition established in [3, Cor. 2.1] for the case $\alpha = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$. We learned at the conference that this example was recently studied further by Osburn, Straub, and Zudilin [7], who proved (4) below for $s = 1$ modulo p^3 and report that Mortenson conjectured it modulo p^5 .

Again after semsimplifying, the motive $H(\alpha, 1^6 | 1)$ has a distinguished summand isomorphic to the Tate motive $\mathbb{Q}(-2)$ of rank 1 and weight 4. The complement of this $\mathbb{Q}(-2)$ breaks up into two pieces A and B . They are both rank 2 motives of weight 5. Namely, $A = M(f_6)$ is the motive associated to the unique normalized eigenform $f_6 = \sum_{n \geq 1} a_n q^n$ of level 8 and weight 6 and $B = M(f_4)(-1)$ is a Tate twist of the motive associated to the unique normalized eigenform $f_4 = \sum_{n \geq 1} b_n q^n$ of level 8 and weight 4. The LMFDB [5] conveniently gives data on modular forms, including the a_n and b_n here.

The trace of Frob_p on the full rank 5 motive $H(\alpha, 1^6 | 1)$ is given by

$$a_p + b_p p + p^2.$$

Numerically, we observe the following supercongruences

$$\frac{F_s(\alpha, 1^d | 1)}{F_{s-1}(\alpha, 1^d | 1)} \equiv \gamma_p \pmod{p^{5s}}, \quad p \nmid 2a_p, \quad s \geq 1, \quad (4)$$

where $\gamma_p \in \mathbb{Z}_p$ is the root of $T^2 - a_p T + p^5$ not divisible by p .

The Hodge numbers for A and B are $(1, 0, 0, 0, 0, 1)$ and $(1, 0, 0, 1)$ respectively, with the gap of five in the Hodge numbers for A nicely matching the exponent of the supercongruences.

4 A summarizing conjecture

We now state a conjecture that generalizes the situations discussed so far.

Conjecture 1 *For fixed $\tau = \pm 1$, let A be the unique submotive of $H(\alpha, 1^d | \tau)$ with $h^{0, d-1}(A) = 1$ and let r the smallest positive integer such that $h^{r, d-1-r}(A) = 1$. For $p \nmid m$ such that $F_1(\alpha, 1^d | \tau) \in \mathbb{Z}_p^\times$, let γ_p be the unit root of A . Then*

$$\frac{F_s(\alpha, 1^d | \tau)}{F_{s-1}(\alpha, 1^d | \tau)} \equiv \gamma_p \pmod{p^{rs}}, \quad s \geq 1. \quad (5)$$

In particular, for $s = 1$ we have

$$F_1(\alpha, 1^d | \tau) \equiv a_p \pmod{p^r}, \quad (6)$$

where a_p is the trace of Frob_p acting on A .

i) For generic α, τ we expect $r = 1$ and (5) follows (see (2) and the subsequent paragraph). For the conjecture to predict $r > 1$, the motive has to split appropriately.

ii) For $\alpha = (1/2, \dots, 1/2)$ and $\tau = (-1)^d$ the motive $H(\alpha, 1^d | \tau)$ acquires an involution and we expect $r = 2$ for any $d \geq 7$; all numerical evidence is consistent with this assertion.

iii) For large d the unit roots involved are not in general related to classical modular forms since the motives A will typically have degrees greater than two.

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