Schwarzian equations and equivariant functions

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Abstract In this review article we show how the theory of Schwarzian differential equations leads to an interesting class of meromorphic functions on the upper-half plane $\mathbb{H}$ named equivariant functions. These functions have the property that their Schwarz derivatives are weight 4 automorphic forms for a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$. It turns out that these functions must satisfy the relation

$$f(\gamma\tau) = \rho(\gamma)f(\tau)\,,\ \ \tau \in \mathbb{H}\,,\ \gamma \in \Gamma,$$

where ρ is a 2-dimensional complex representation of Γ and the matrix action on both sides is by linear fractional transformation. When ρ is the identity representation $\rho(\gamma) = \gamma$, the equivariant functions are parameterized by scalar automorphic forms, while if ρ is an arbitrary representation they are parameterized by vector-valued automorphic forms with multiplier ρ. If Γ is a modular subgroup we obtain important applications to modular forms for Γ as well as a description in terms of elliptic functions theory. We also prove the existence of equivariant functions for the most general case by constructing a vector bundle attached to the data (Γ,ρ) and applying the Kodaira vanishing theorem.

1 The Schwarz derivative

Let D be a domain in $\mathbb{C}$ and f a meromorphic function on D. The Schwarz derivative or the Schwarzian of f is defined by

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$$\{f,z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

It was named after Schwarz by Cayley, however, Schwarz himslef pointed out that it was discovered by Lagrange in 1781. It also appeared in a paper by Kummer (1836) [10]. The Schwarz derivative has many interesting properties which are given below. The functions involved are meromorphic functions on a domain D.

- Projective invariance:

$$\left\{\frac{af+b}{cf+d}, z\right\} = \{f,z\}\ ,\ a, b, c, d \in \mathbb{C}\ ,\ ad-bc \neq 0.$$

- Cocycle property: If w is a function of z, then

$$\{f,z\} = \{f,w\}(dw/dz)^2 + \{w,z\}.$$

- $\{f,z\} = 0$ if and only if $f(z) = \dfrac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$.
- If $w = \dfrac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, then

$$\{f,z\} = \{f,w\}\,\frac{(ad-bc)^2}{(cz+d)^4}.$$

- For two meromorphic functions f and g on D,

$$\{f,z\} = \{g,z\} \text{ if and only if } f(z) = \frac{ag(z)+b}{cg(z)+d},\ \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

- If $w(z)$ is a function of z with $w'(z_0) \neq 0$ for some $z_0 \in D$, then in a neighborhood of z_0, we have

$$\{z,w\} = \{w,z\}\,(dz/dw)^2.$$

Some of the properties are elementary and the rest follows from the following important connection with the theory of ordinary differential equations:

Let $R(z)$ be a meromorphic function on D and consider the second order differential equation

$$y'' + \frac{1}{2}R(z)y = 0$$

with two linearly independent solutions y_1 and y_2. Then $f = y_1/y_2$ is a solution to the Schwarz differential equation

$$\{f,z\} = R(z).$$

Conversely, if $f(z)$ is locally univalent and $\{f,z\} = R(z)$, then $y_1 = f/\sqrt{f'}$ and $y_2 = 1/\sqrt{f'}$ are two linearly independent solutions to $y'' + \frac{1}{2}R(z)y = 0$.

The Schwarz derivative plays an important role in the study of the complex projective line, univalent functions, conformal mapping , Teichmuller spaces and most importantly in the theory of modular forms and hypergeometric functions [1, 4, 6, 8, 9, 11].

We now look at the effect of the Schwarz derivative on automorphic functions for a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$, that is a Fuchsian group of the first kind acting on the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ by linear fractional transformation

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Proposition 1.1 *[8] If f is an automorphic function for a discrete group Γ, then $\{f,\tau\}$ is a weight 4 automorphic form for Γ that is holomorphic everywhere except at the points where f has a multiple zero or a multiple pole (including at the cusps). Moreover, if Γ is of genus zero and f is a Hauptmodul, then $\{f,\tau\}$ is modular for the normalizer of Γ in $PSL_2(\mathbb{R})$*

As an example, let λ be the Klein modular function for $\Gamma(2)$ given by

$$\lambda(\tau) = \left(\frac{\eta(\tau/2)}{\eta(2\tau)}\right)^8,$$

where η is the Dedekind eta-function given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n), \; q = \exp(2\pi i\tau),$$

then

$$\{\lambda,\tau\} = \frac{\pi^2}{2} E_4(\tau),$$

where E_4 is the weight 4 Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n,$$

with $\sigma_k(n)$ being the sum of the k-th powers of the positive divisors of n.

If $\Gamma = \Gamma_0(8)$ and we consider the Hauptmodul f_8 for Γ given by

$$f_8(\tau) = \frac{\eta(4\tau)^{12}}{\eta(2\tau)^4\eta(8\tau)^8}$$

then

$$\frac{1}{2\pi^2}\{f_8,\tau\} = \frac{1}{4}(\theta_3^4+\theta_4^4)^2 = \theta_{D_4\oplus D_4}(2\tau),$$

where $\theta_{D_4\oplus D_4}(2\tau)$ is the theta function of two copies of the root lattice D_4 and θ_3 and θ_4 are the Jacobi theta-functions

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}, \ \ \theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} .$$

For later use, we also give

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+1/2)^2} .$$

Finally, if $\Gamma = \Gamma_1(5)$ and f_5 is the Hauptmodul given by

$$f_5(\tau) = q\prod_{n\geq 1}(1-q^n)^{5\left(\frac{n}{5}\right)}$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol, then

$$\frac{1}{2\pi^2}\{f_5,\tau\} = \theta_{Q_8(1)}$$

where $Q_8(1)$ is the Icosian or Maass lattice which is the 8-dimensional 5-unimodular lattice with determinant 625 and minimal norm 4.

Notice that in the above three examples, the Schwarz derivatives are all holomorphic as the the groups involved are torsion-free and thus their Hauptmoduls do not have multiple zeros or poles.

One may ask if the converse of the above properties is true: Suppose that the Schwarz derivative $\{f,\tau\}$ of a meromorphic function on $\mathbb{H}$ is a weight 4 automorphic, what can be said about f? Does it have any automorphic properties? The following sections will be devoted to elucidate this question.

2 Equivariant functions

Suppose that F is a weight 4 automorphic form for a discrete group Γ and f is a meromorphic function on $\mathbb{H}$ such that $\{f,\tau\} = F(\tau)$. Then using the properties of the Schwarz derivative from the previous section we have, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$
\begin{aligned}
(c\tau+d)^4 F(\tau) &= F\left(\frac{a\tau+b}{c\tau+d}\right) \\
&= \left\{ f\left(\frac{a\tau+b}{c\tau+d}\right), \frac{a\tau+b}{c\tau+d} \right\} \\
&= (c\tau+d)^4 \left\{ f\left(\frac{a\tau+b}{c\tau+d}\right), \tau \right\}.
\end{aligned}
$$

Therefore,

$$
\{f,\tau\} = \left\{ f\left(\frac{a\tau+b}{c\tau+d}\right), \tau \right\}.
$$

Hence, there exists $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ such that

$$
f\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{Af(\tau)+B}{Cf(\tau)+D}.
$$

This defines a 2-dimensional representation ρ of Γ in $\mathrm{GL}_2(\mathbb{C})$ such that

$$
f(\gamma\tau) = \rho(\gamma)\, f(\tau) \tag{1}
$$

where on both sides the action of the matrices is by linear fractional transformation.

We will distinguish three cases:

1. $\rho = 1$ a constant , in which case f is an automorphic function.
2. $\rho = \mathrm{Id}$, the embedding of Γ in $\mathrm{GL}_2(\mathbb{C})$ or the defining representation of Γ, providing a meromorphic function commuting with the action of Γ which we call an equivariant function for Γ.
3. ρ is a general representation not equal to one in the above cases giving a function f called a $\rho-$equivariant function for Γ.

We will be interested in the last two cases. A trivial example of an equivariant function for a discrete group Γ is $f(\tau) = \tau$. We will see in the next section that there are infinitely many examples parametrized by automorphic forms for Γ. Furthermore, the set of equivariant functions will have a structure of an infinite dimensional vector space isomorphic to the space of meromorphic sections of the canonical bundle of the compact Riemann surface $X(\Gamma) = (\Gamma\backslash\mathbb{H})^*$ where the star indicates that we have added the cusps to the quotient space, in other words, the space of meromorphic differential forms on $X(\Gamma)$.

In the general case, we establish the existence of $\rho-$equivariant functions for an arbitrary representation ρ of Γ. This will include the case when $\rho : \Gamma \longrightarrow \mathbb{C}^*$ is a character. Of course, when this character is unitary, then we recover the classical automorphic functions with a character.

3 The automorphic and modular aspects

In tis section we focus solely on the equivariant functions, that is when ρ is the defining representation. We have already seen that $f(\tau)=\tau$ is equivariant for every discrete group. It turns out that there are many more nontrivial equivariant functions.

Theorem 3.1 *[5] Let Γ be a discrete group. We have*

1. *Let f a nonzero automorphic form of weight k for Γ (even with a character), then*

$$h_f(\tau) := \tau + k\,\frac{f(\tau)}{f'(\tau)} \tag{1}$$

is an equivariant function for Γ.

2. *Let h be an equivariant function for Γ. Then $h(\tau)=h_f(\tau)$ for some automorphic form with a character for Γ if and only if the poles of $1/(h(\tau)-\tau)$ are simple with rational residues. Moreover, if Γ has genus 0, then we can omit the character from this statement.*

If k is a nonzero integer and c is a nonzero constant, then $h_f=h_{f^k}=h_{cf}$ and so the correspondence $f\mapsto h_f$ is not one-to-one. Because of the second part of the theorem, an equivariant functions that arises from an automorphic form as in (1) is called a rational equivariant form. It turns out that for such an equivariant function $h(\tau)$, the residues of $1/(h(\tau)-\tau)$ have bounded denominators, and any common multiple of these denominators can be the weight for an automorphic form f such that $h=h_f$. Moreover, not all equivariant functions are rational. An example of a non-rational equivariant function is given by

$$h(\tau) = \tau + 4\,\frac{E_4(\tau)}{E_4'(\tau)+E_6(\tau)},$$

where E_6 is the weight six Eisenstein series

$$E_6(\tau) = 1-504\sum_{n\geq 1}\sigma_5(n)q^n\,.$$

Indeed, one can show that $1/(h(\tau)-\tau)$ has a simple pole at the cubic root of unity, but with residue $\dfrac{1}{4}+\dfrac{\pi i}{6}$.

The following theorem provides two important applications of equivariant functions to modular forms.

Theorem 3.2 *[12] Let f be a modular form for a finite index subgroup of $SL_2(\mathbb{Z})$ of nonzero weight, then*

1. *The derivative of f has infinitely many non-equivalent zeros in $\mathbb{H}$, all but a finite number are simple zeros.*

2. *The $q-$expansion of f, where q is the uniformizer at ∞ for Γ, cannot have only a finite number of nonzero coefficients.*

This theorem follows from the properties of the equivariant function h_f attached to f as in (1); the most important of which is that h_f takes always real values. This is clear if h_f has a pole in $\mathbb{H}$ as it will take rational values at the orbit of this pole, but if h_f is holomorphic, then we have to apply the theorem of Denjoie-Wolfe applied to the iterates of h_f. Then we prove that h_f has infinitely many non-equivalent poles in $\mathbb{H}$. To prove that the zeros are all simple except for a finite number of them requires the use the Rankin-Cohen brackets. The second statement is usually proven using the $L-$function of the modular form, but here it is a simple consequence of the first statement.

We end this section with an interesting connection with the cross-ration which is defined for four distinct complex numbers z_i, $1 \leq i \leq 4$ by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}\,.$$

As it is projectively invariant, the cross-ration of four distinct equivariant functions for a discrete group Γ is an automorphic function for Γ. As examples, we have

$$[\tau, h_{\theta_2}, h_{\theta_3}, h_{\theta_4}] = \lambda,$$

and

$$[\tau, h_{E_4}, h_{\Delta}, h_{E_6}] = \frac{1}{1728}\, j,$$

where $\Delta = \eta^{24}$ is the discriminant cusp form and the Dedekind $j-$ function is given by $j = E_4^3/\Delta$.

4 The elliptic aspect

The ideas in this section first started in [3] and were developed further in [15] and more recently in full generalization in [2]. Let $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ be a lattice in $\mathbb{C}$ with $\Im\,\omega_2/\omega_1 > 0$. Its Weierstrass $\wp-$function is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L\setminus\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2},$$

and the Weierstrass $\zeta-$function is given by

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in L\setminus\{0\}} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}.$$

Notice that $\zeta'(z) = -\wp(z)$, and while $\wp$ is $L-$periodic, ζ is quasi-periodic with respect to L in the sense that for $\omega \in L$ and $z \in \mathbb{C}$, we have

$$\zeta(z+\omega) = \zeta(z) + H_L(\omega)$$

where the quasi-period map depends on the Lattice L. It is $\mathbb{Z}-$linear and so it is determined by the quasi-periods $\eta_1 = H_L(\omega_1)$ and $\eta_2 = H_L(\omega_2)$. Moreover, H_L is homogeneous of weight -1 in the sense that if $\alpha \in \mathbb{C}^\times$, then

$$H_{\alpha L}(\alpha\omega) = \alpha^{-1} H_L(\omega).$$

The quasi-periods satisfy the Legendre relation

$$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i.$$

We now suppose that that $\omega_1 = 1$ and $\omega_2 = \tau \in \mathbb{H}$. Using the fact that $\mathrm{SL}_2(\mathbb{Z})$ acts on L by isomorphisms (by a change of basis) and using the homogeneity of the quasi period map H_L, it is easy to see that

$$h_0(\tau) = \frac{\eta_2}{\eta_1}$$

is equivariant for $\mathrm{SL}_2(\mathbb{Z})$ [3].

In fact, from the expression of the Weierstrass $\zeta-$function one can prove that

$$\eta_1 = \frac{\pi^2}{3} E_2(\tau)$$

where E_2 is the weight 2 Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)}.$$

Therefore, using the Legendre relation, we get

$$h_0(\tau) = \tau + \frac{6}{i\pi E_2(\tau)} = \tau + 12\frac{\Delta(\tau)}{\Delta'(\tau)},$$

and thus $h_0 = h_\Delta$ is a rational equivariant function.

Let us put

$$M_\tau = \begin{pmatrix} \tau & \eta_2 \\ 1 & \eta_1 \end{pmatrix}.$$

Then M_τ is invertible as $\det M_\tau = -2\pi i$ by the Legendre relation.

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and denote By $Eq(\Gamma)$ the set of all equivariant functions for $\mathrm{SL}_2(\mathbb{Z})$ excluding the trivial one $h(\tau) = \tau$. Also denote by

$M_2(\Gamma)$ be the set of all weight two meromorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$. We have

Theorem 4.1 *The map from $M_2(\Gamma)$ to $Eq(\Gamma)$*

$$f \mapsto M_\tau f$$

is a bijection where $M_\tau f$ is the linear fraction of f given by M_τ.

The above map sends the zero modular form to h_0 which is equivariant for $\mathrm{SL}_2(\mathbb{Z})$ and hence for every subgroup. In the meantime, h_0 was built using the quasi-periods of the Weierstrass $\zeta-$function. One might ask: what about the remaining equivariant functions? can they arise also from elliptic objects in the same way h_0 does? In the paper [2], this question is fully answered, and indeed for each equivariant function for Γ, one can construct a generalizations of the Weierstrass $\zeta-$function called elliptic zeta functions which are quasi-periodic maps on the set of lattices such the quotient of two fundamental quasi-periods is an equivariant function. The interesting aspect is that there is a triangular commutative correspondence between the set of these elliptic zeta functions, $M_2(\Gamma)$ and $Eq(\Gamma)$ which encompasses the modular and elliptic nature of equivariant functions.

As for the geometric aspect, because the weight 2 meromorphic modular forms are identified with the meromorphic differential forms on the Riemann surface $X(\Gamma)$, we can thus view the equivariant functions as the global meromorphic sections of the canonical bundle of $X(\Gamma)$.

5 The general case

In this section, we consider the case of a general discrete group and an arbitrary representation $\rho : \Gamma \longrightarrow \mathrm{GL}_2(\mathbb{C})$ and investigate the existence of $\rho-$equivariant functions for Γ, that is, the meromorphic functions on $\mathbb{H}$ such that

$$f(\gamma\tau) = \rho(\gamma)f(\tau).$$

We will denote the set of such functions by $Eq(\Gamma,\rho)$. Let us first recall the notion of vector-valued automorphic forms for the data (Γ,ρ). A meromorphic function $F = (f_1, f_2)^t : \mathbb{H} \longrightarrow \mathbb{C}^2$ where f_1 and f_2 are tow meromorphic functions on $\mathbb{H}$ is called a 2-dimensional vector-valued automorphic form for Γ of multiplier ρ and weight $k \in \mathbb{Z}$ if

$$(c\tau+d)^{-k}F(\gamma\tau) = \rho(\gamma)F(\tau),\ \tau \in \mathbb{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

in addition to the usual growth behavior at the cusps. Denote by $V_k(\Gamma,\rho)$ the space of all such forms. They were fairly studied in the last two decades by various authors in different contexts from algebraic, arithmetic, analytic, geometric and theoretical physics points of view, see [14] and the extensive list of references therein. Their existence is well established in the literature for a unitary representation ρ and for Γ being a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ or a genus zero discrete group among other cases. The existence for an arbitrary data (Γ,ρ) has been recently proved in [14] even for Γ being a Fuchsian group of the second kind.

The first result of importance to us is

Theorem 5.1 *[13] Let $F=(f_1,f_2)^t$ be a 2-dimensional vector-valued automorphic form of multiplier ρ and arbitrary weight for Γ, then $h_F=f_1/f_2$ is a $\rho-$equivariant function for Γ.*

This settles the question of the existence of $\rho-$equivariant functions which is then a consequence of the existence of vector valued automorphic forms. A more interesting result is that every $\rho-$equivariant functions arises in this way

Theorem 5.2 *[13] The map from $V_k(\Gamma,\rho)$ to $Eq(\Gamma,\rho)$ given by*

$$F=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \mapsto h_F = f_1/f_2 \tag{1}$$

is surjective.

Surprisingly, the proof uses almost all the properties of the Schwarz derivative which lead to the next theorem. If D is a domain in $\mathbb{C}$, and $R(z)$ is a holomorphic function on D, then we cannot guarantee the existence of two linearly independent global solutions to the differential equation

$$y''+R(z)y=0$$

when D is not simply connected, and all we can hope for are local solutions. However, when $R(z)$ comes from a Schwarz derivative, then we have a different outcome.

Theorem 5.3 *Let D be a domain and f be a meromorphic function on D such that $R(z)=\{f,z\}$ is holomorphic on D. Then the differential equation $y''+R(z)y=0$ has two linearly independent global solutions on D.*

It is this important result and the use of the Bol identity that lead to the surjectivity of the map (1).

So far we have established this close connection between $\rho-$equivariant functions for Γ and 2-dimensional vector valued automorphic forms. All we need is to prove, for arbitrary data (Γ,ρ), the existence of such automorphic forms. To this end we associate to (Γ,ρ) a vector bundle $\mathscr{E}=\mathscr{E}_{\Gamma,\rho}$ over $X=X(\Gamma)$ constructed as follows:
We choose a covering $\mathscr{U}=(U_i)_{i\in I}$ where I is the set of cusps and elliptic fixed points on X. We then construct holomorphic maps $\psi_i : U_i \longrightarrow \mathrm{GL}_2(\mathbb{C})$ having ρ as a factor

of automorphy [7]. This is carried out by solving the Riemann-Hilbert problem over U_i with the monodromy ρ. These maps yield a cocycle $(F_{ij}) \in \mathscr{Z}^1(\mathscr{U}, \mathrm{GL}(2,\mathscr{O}))$ to which is associated a rank two holomorphic vector bundle $\mathscr{E}$ over X whose transition functions are the maps F_{ij} on $U_i \cap U_j$.

Now if P is a given point (that can be a cusp) and $\mathscr{L}$ is the line bundle over X corresponding to the divisor $[P]$, then using the Kodaira vanishing theorem, there exists an integer $\mu \geq 0$ such that

$$\dim H^0(X, \mathscr{O}(\mathscr{L}^\mu \otimes \mathscr{E})) \geq 2$$

where $\mathscr{O}(\mathscr{L}^\mu \otimes \mathscr{E})$ is the set of holomorphic sections of the sheaf $\mathscr{L}^\mu \otimes \mathscr{E}$ which can be seen as sections in $H^0(X \setminus \{P\}, \mathscr{O}(\mathscr{E}))$ having a pole at P of order at most μ. Thus we have two linearly independent meromorphic sections of $\mathscr{E}$ with a single pole at P. When lifted to $H \cup \{\text{cusps}\}$ these sections yield two linearly independent vector-valued automorphic functions (of weight 0) attached to (Γ, ρ) with poles at the fiber of P. The full details of the proof can found for a higher dimension of the representation in [14].

We have therefore established the following:

Theorem 5.4 *For every discrete group Γ and every 2-dimensional representation ρ of Γ, vector-valued $\Gamma-$automorphic functions of multiplier ρ exist and so do $\rho-$equivariant functions for Γ.*

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