Stochastic Modeling with Randomized Markov Bridges and Conditioned Stochastic Differential Equations

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State Variable Y

- Commodities: Spot price: $(f(Y_t))_{t\geq 0}$, Foward price (martingale): $\mathbb{E}\left[f(Y_T)|\mathcal{F}_t\right]$,
- Dividend paying asset: $\sum_{i=j}^{N} \mathbb{E}\left[e^{-\rho(T_i-t)}\delta_i(Y_{T_i}) \,\middle|\, \mathcal{F}_t\right]$ ($T_i \geq t$ for $i = j, \ldots, N$),
- Interest-rate related, $(r(Y_t))_{t\geq 0}$: short rate, $\mathbb{E}\left[e^{-\int_t^T r(Y_u)du} g(Y_T) \,\middle|\, \mathcal{F}_t\right]$.

When Y is Markov, it generates the market information (filtration),

$$
\begin{aligned}
\mathbb{E}\left[f(Y_T)|\mathcal{F}_t\right] =& \mathbb{E}\left[f(Y_T)|\mathcal{F}_t^Y\right] \\
&= \mathbb{E}\left[f(Y_T)|Y_t\right] = \left\{\int_{\mathbb{R}^n} f(y)P(t,x,T,dy)\right\}\Big|_{x=Y_t}, \\
P(t,x,u,dy) :=& \mathbb{P}\left(Y_u \in dy | Y_t = x\right), \quad t < u, \\
\mathcal{F}_t^Y :=& \sigma(Y_s, 0 \leq s \leq t).
\end{aligned}
$$

Problem

Want to modify Y in such a way as to satisfy

$$\mathbb{Y} := (Y_{T_1}, \ldots, Y_{T_N}) \sim G(dy),$$

- $0 = T_0 < T_1 < \cdots < T_N = T$, fixed constant times,
- $G(dy)$: given law on E^N ="$(\subset \mathbb{R}^N)$, state space of $\mathbb{Y}$".

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Requirements:

- "Minimal" modification in a certain sense,
- It may be better to reserve
  - Markovian property, Martingale property, ...

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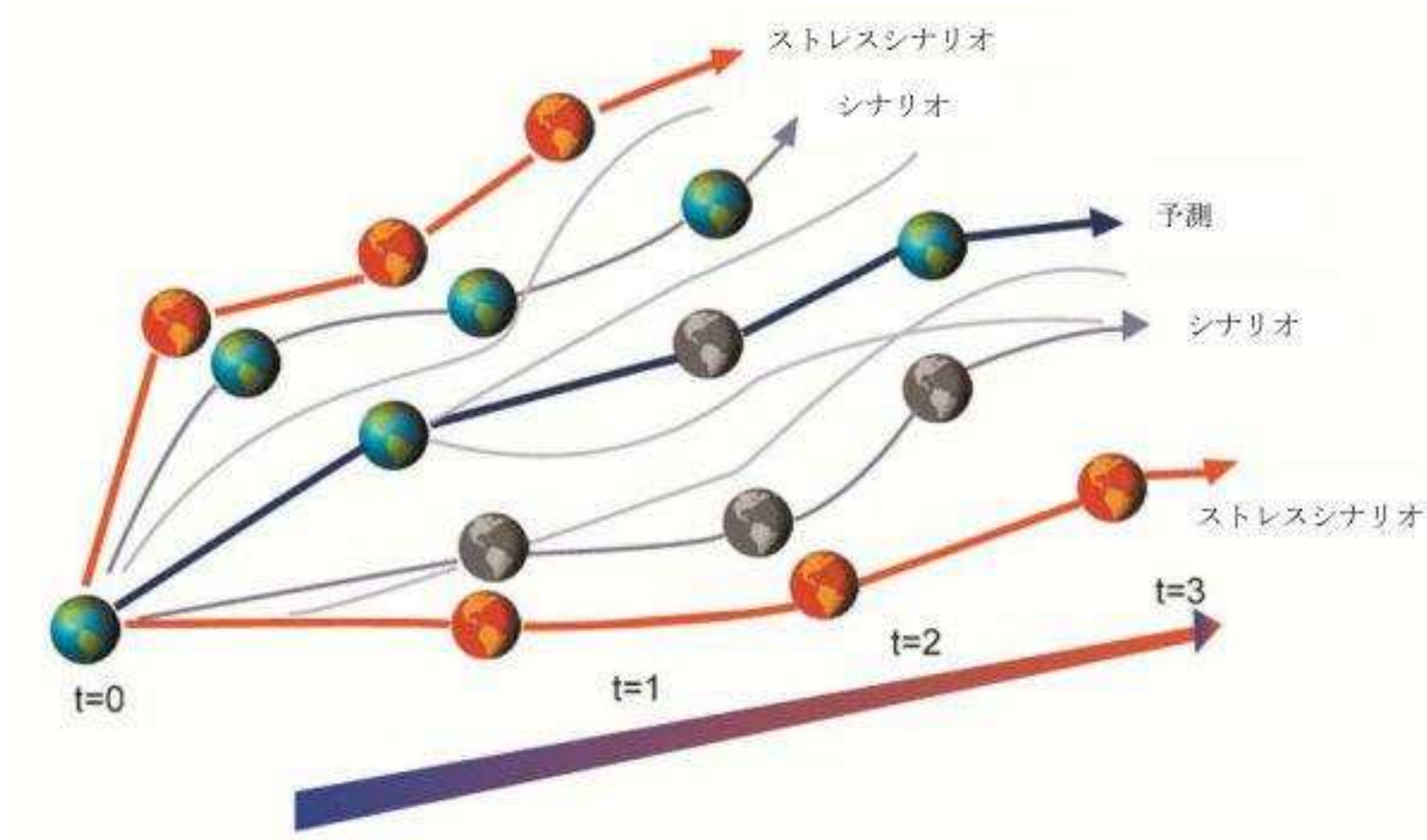
Approaches:

- Madan and Yor (2003): Three constructions of Markovian martingale,
- Obloj and Spoida (2013): Iterated Azéma-Yor embedding,
- Fan, Hamza and Klebaner (2015): Two constructions for self-similar Markovian martingales

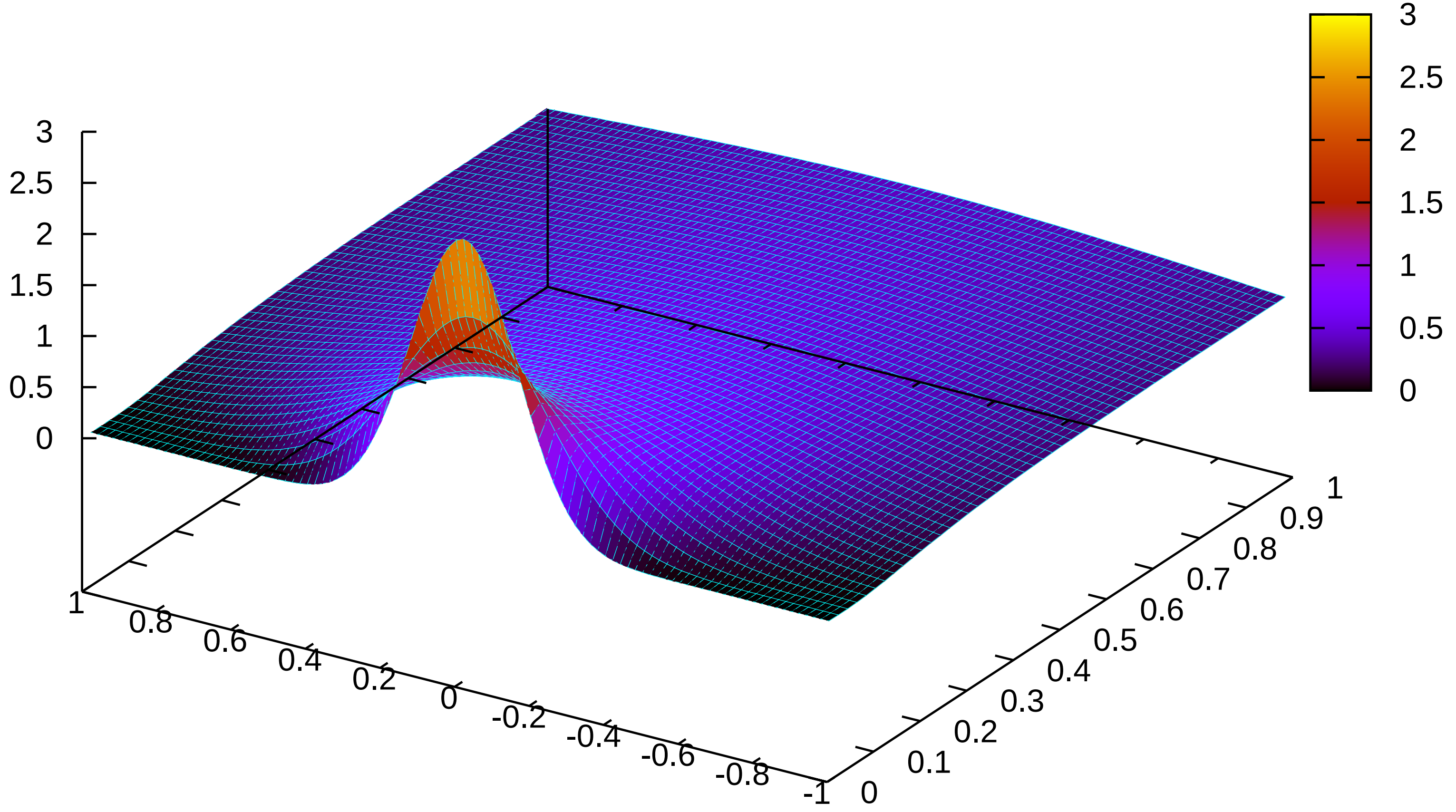
Motivations (for financial studies)

- Flexible modeling of cashflows,
- Fitting implied volatility curves (for option pricing models),
  - Breeden-Litzenberger (1978): can recover the marginal pdf of the law of the underling asset.
- Stochastic stress test model,
  - Hand-pick (deterministic) scenarios ???
  - Randomized (i.e., the law of) stress scenarios: Breuer and Csiszar (2013) $\leftrightarrow$ Entropic VaR (Föllmer and Knispel, 2011),
  - Insurance: long-term, multi-period (e.g., several years) stress testing,
    - Want to "stochastically interpolate" the randomized scenarios from a dynamic valuation point of view.

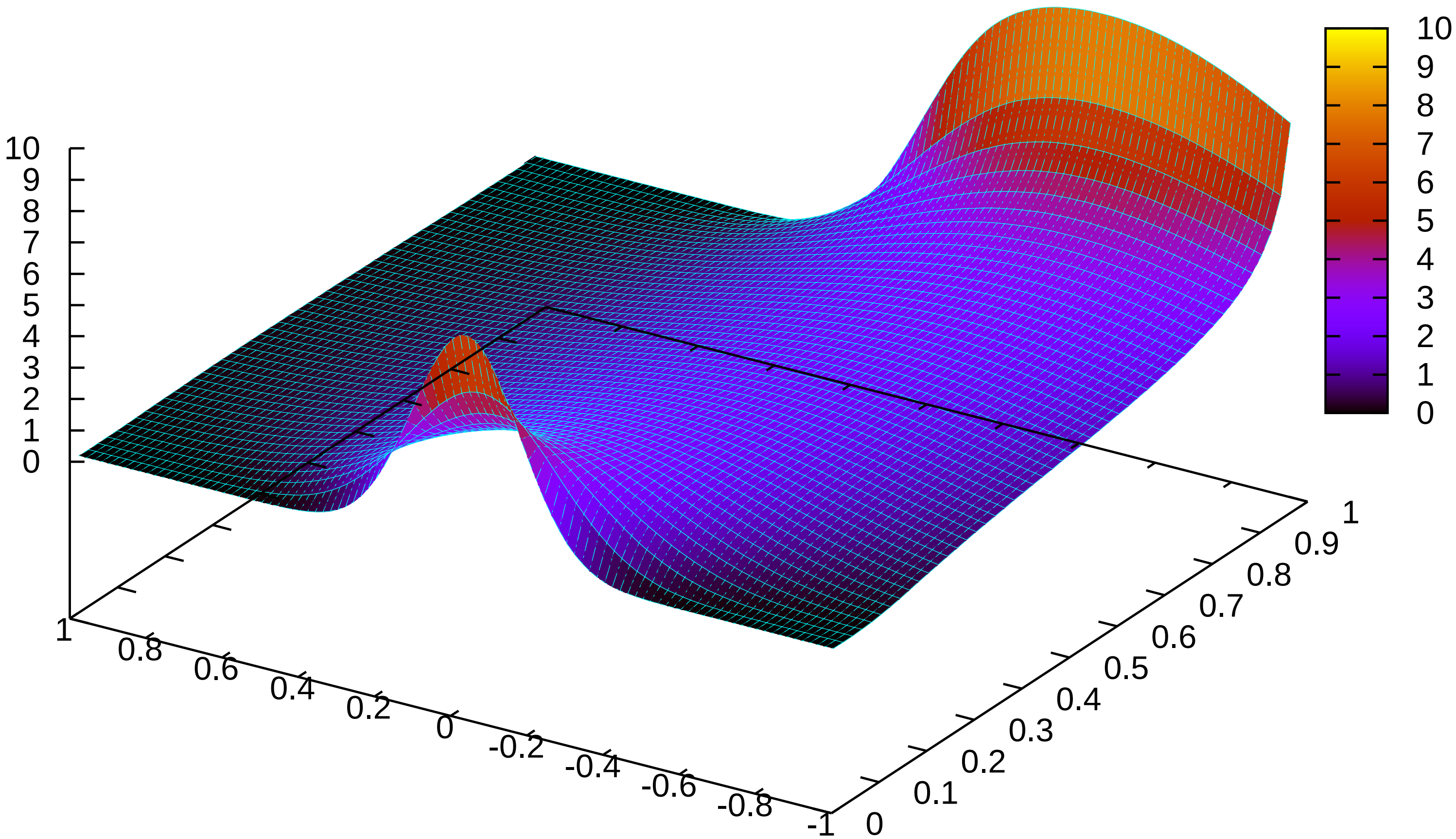
IAA: Stress Testing and Scenario Analysis (2013)



Marginal Densities of BM



Marginal Densities of CSDE: $X_1 \sim \mathrm{Exp}(1)$



Plan

- "Strong" approach: RMB and filtering.
  - Ref. Brody, Hughston and Macrina (2008): RBB, Hoyle, Hughston and Macrina (2011): RLB, Filipovic, Hughston and Macrina (2012): attempt to get a local-vol, .., Macrina and S (2017).

- "Weak" approach: CSDE (Baudoin, 2002).

- Single marginal to multiple marginals

- Skewed class (skew-normal diffusions)

“Strong” Approach ($N = 1$ Case for simplicity)

Setup:

- $(Y_t^{(y)})_{t\in[0,T]}$: (time-homogeneous) Markov process, $Y_0^{(y)} = y$. Let

$$\tilde{P}_t(x, dy) := \tilde{p}_t(x, y)dy := \mathbb{P}\left(Y_t \in dy | Y_0 = x\right).$$

- $(Y_t^{(y,T,z)})_{t\in[0,T]}$: Markov bridge, $Y_0^{(y,T,z)} = y$, $Y_T^{(y,T,z)} = z$ a.s.
- X: r.v., $\sim G$, indep. of Y, (system).
- $Z_t := Y_t^{(y,T,X)}$, $t \in [0, T]$, (noisy observation).
- $\mathcal{F}_t^Z := \sigma(Z_u; u \in [0, t]) \vee \mathcal{N}$, $t \in [0, T]$.

Targets:

- Filtering: $\pi_t(dx) := \mathbb{P}\left(X \in dx | \mathcal{F}_t^Z\right)$.
- Dynamics of $\mathbb{E}[f(X)|\mathcal{F}_t^Z] = \mathbb{E}[f(Z_T)|\mathcal{F}_t^Z]$.
- Dynamics of Z wrt $(\mathcal{F}_t^Z)_{t\in[0,T]}$.
- “Transition probability”: $q_{s,t}(dx) := \mathbb{P}\left(Z_t \in dx | \mathcal{F}_s^Z\right)$ $(s \le t)$.

Constructing MB (Fitzsimmons et al, 1993)

- On the canonical path-space Ω_T^1 endowed with the natural filtration $(\mathcal{F}_t^1)_{t\in[0,T]}$, denote the law of $Y^{(y)}$ by $\tilde{\mathbb{P}}_y^1$.
- Assume the duality, i.e., $^\exists\big(\hat{Y},(\hat{P}_t)_{t\geq 0}\big)$, so that

$$\int_E (\tilde{P}_t f_1)(x) f_2(x) m(dx) = \int_E f_1(x)(\hat{P}_t f_2)(x) m(dx),$$

$(\tilde{P}_t f_1)(x) := \int_E f_1(y)\tilde{P}_t(x,dy)$, $(\hat{P}_t f_2)(x) := \int_E f_2(y)\hat{P}_t(x,dy)$.

- The law of MB $(Y_t^{(y,T,z)})_{t\in[0,T)}$ on $(\Omega_T^1, \mathcal{F}_{T-}^1)$ is $\mathbb{P}_{y,z}^1$, where

$$d\mathbb{P}_{y,z}^1 \mid_{\mathcal{F}_t^1} := \frac{\tilde{p}_{T-t}(Y_t, z)}{\tilde{p}_T(y,z)} d\tilde{\mathbb{P}}_y^1 \mid_{\mathcal{F}_t^1},$$

- On $(\Omega, \mathcal{F}_{T-}, \mathbb{P}_{y,z}^1)$ the Markov bridge process that satisfies $\mathbb{P}_{y,z}^1(Y_0 = y, Y_{T-} = z) = 1$ is determined.
- $\mathbb{P}_{y,z}^1$ is a (regular) version of $\tilde{\mathbb{P}}_y^1(\cdot|Y_{T-} = z)$ and $T-$ can be change to T.

Constructing RMB

- Define X on another probability space $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$. Recall $G = \mathbb{P}^2 \circ X^{-1}$.
- On $(\Omega, \mathcal{F}_T, \mathbb{P}_y) := (\Omega^1 \times \Omega^2, \mathcal{F}_T^1 \otimes \mathcal{F}^2, \tilde{\mathbb{P}}_y^1 \otimes \mathbb{P}^2)$, we (want to) construct

$$Z_t(\omega_1, \omega_2) := Y_t^{(y,T,X(\omega_2))}(\omega_1), \quad t \in [0, T].$$

- Starting with $(Y_t^{(y)})_{t\in[0,T]}$ the solution to a Markovian SDE, we can construct the associated MB $(Y_t^{(y,T,z)})_{t\in[0,T]}$ as the solution to an SDE. Then, we can discuss the regularity (measurability) $z \mapsto Y^{(y,T,z)}$.
- Chaumont and Uribe Bravo (2011): weak continuity of $\mathbb{P}_{y,z}^1$ wrt z.

Filtering Results (1)

- $\mathbb{P}(X \in dz | \mathcal{F}_t^Z) = \pi_t(dz|Z_t, Z_0) = \dfrac{\dfrac{\tilde{p}_{T-t}(Z_t, z)}{\tilde{p}_T(Z_0, z)} G(dz)}{\displaystyle\int_E \dfrac{\tilde{p}_{T-t}(Z_t, z')}{\tilde{p}_T(Z_0, z')} G(dz')}$ $(0 \le t < T)$.
- $\mathbb{P}(Z_t \in dz | \mathcal{F}_s^Z) = q_{s,t}(dz|Z_s, Z_0) = q(s, Z_s; t, z|Z_0)dz$ $(0 \le s < t < T)$, where

$$q(s, x; t, y|z_0) := \frac{\tilde{p}_{t-s}(x, y) \left\{ \displaystyle\int_E \frac{\tilde{p}_{T-t}(y, z)}{\tilde{p}_T(z_0, z)} G(dz) \right\}}{\displaystyle\int_E \frac{\tilde{p}_{T-s}(x, z')}{\tilde{p}_T(z_0, z')} G(dz')}.$$

For a fixed z_0, the Chapman-Kolmogorov identity holds:

$$q(s, x; u, z|z_0) = \int_E q(s, x; t, y|z_0) q(t, y; u, z|z_0) dy$$

for $0 < s < t < u < T$ and $x, y, z \in E$.

Filtering Results (2): Kushner-Stratonovich equation

Consider $dY_t = b(Y_t)dt + \sigma(Y_t)dW_t$. For

$$S_t := \pi_t(f) := \int_E f(y)\pi_t(dy) = \mathbb{E}\left[f(X)|\mathcal{F}_t^Z\right],$$

we obtain

$$\pi_t(f) = \pi_0(f) + \int_0^t \{\pi_s(f\nabla\ell_s) - \pi_s(f)\pi_s(\nabla\ell_s)\}^\top (\sigma\sigma^\top)^{1/2}(Z_s)dU_s,$$

where

- $\ell_s(z) := \ln \tilde{p}_{T-s}(Z_s, z)$,
- $U_t := \int_0^t (\sigma\sigma^\top)^{-1/2}(Z_s)\left[dZ_s - \left\{b(Z_s) + (\sigma\sigma^\top)(Z_s)\pi_s(\nabla\ell_s)\right\} ds\right]$, $\mathcal{F}_t^Z$-Brownian motion ("innovation" process)
- SDE for $(Z, \mathcal{F}_t^Z)$ can be written down, too.

Filtering Results (2): Kushner-Stratonovich equation

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- SDE for $(Z, \mathcal{F}_t^Z)$ can be written down, too.
- SV model s.t. $f^{-1}(S_T) \sim G$.

“Weak” Approach (Baudoin, 2002)

- Y: sol. of (Markovian) SDE on the canonical space $C([0,T],E)$, ($E \subset \mathbb{R}^n$).
- $\mathbb{Y} := (Y_{T_1}, \ldots, Y_{T_N})$: E^N-valued r.v. (or some segment of the trajectory Y).
- G: law on $\mathbb{R}^N$.
- Minimal probability:

$$\mathbb{P}^G_{y_0}(A) := \int_E \mathbb{P}(A|\mathbb{Y} = y, Y_0 = y_0)G(dy), \quad A \in \mathcal{F}_T,$$

which satisfies i) $\mathbb{E}^G[H|\mathbb{Y}] = E[H|\mathbb{Y}]$ for $^\forall H$, ii) $\mathbb{Y} \sim G$ under $\mathbb{P}^G$.
If $G(dy) = g(y)dy$, then,

$$d\mathbb{P}^G_{y_0} = \left(\frac{g}{f}\right)(\mathbb{Y}|y_0)d\mathbb{P}_{y_0},$$

$$f(y|y_0)dy := \mathbb{P}\left(\mathbb{Y} \in dy | Y_0 = y_0\right) = \prod_{j=1}^N \tilde{p}_{T_j - T_{j-1}}(y_{j-1}, y_j)dy.$$

$\mathbb{P}^G$-Dynamics

Suppose $N = 1$, for example.

- For $t \in [0, T)$,

$$\frac{d\mathbb{P}^G_{y_0}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathbb{E}\left[\frac{g(X_T)}{\tilde{p}_T(y_0, Y_T)}\,\bigg|\,\mathcal{F}_t\right] = \int_E \frac{\tilde{p}_{T-t}(Y_t, y)}{\tilde{p}_T(y_0, y)} G(dy).$$

- $(\mathbb{P}^G, \mathcal{F}_t)$-dynamics of Y:

$$dY_t = \left\{b(Y_t) + (\sigma\sigma^\top)(Y_t)\nabla\Lambda(t, Y_t|Y_0)\right\} dt + \sigma(Y_t)dW_t^G, \quad t \in [0, T),$$

$$\Lambda(t, y|y_0) := \log\left\{\int_E \frac{\tilde{p}_{T-t}(y, z)}{\tilde{p}_T(y_0, z)} G(dz)\right\}.$$

- Transition/conditional probability formulas same as those of "strong" approach.

“Minimality” of $\mathbb{P}^G$

$$\min_{\mathbb{Q}\in\mathcal{P}_{G,h}} \mathbb{E}\left[h\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \mathbb{E}\left[h\left(\frac{d\mathbb{P}^G}{d\mathbb{P}}\right)\right],$$

where $h : [0,\infty) \to \mathbb{R}$: convex and set

$$\mathcal{P}_{G,h} := \left\{ \mathbb{Q} \,\middle|\, \begin{array}{c} \text{prob. measure on } (\Omega, \mathcal{F}_T),\ \mathbb{Q} \ll \mathbb{P}, \\ \mathbb{Y} \sim G \text{ under } \mathbb{Q},\ \mathbb{E}\left|h\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right| < \infty \end{array} \right\},$$

assuming $\int_E \left|h\left(\frac{dG}{d\mathbb{P}_\mathbb{Y}}\right)\right| d\mathbb{P}_\mathbb{Y} < \infty$, $\mathbb{P}_\mathbb{Y}(dy) := \mathbb{P}(\mathbb{Y} \in dy)$.

- $h(x) = x\log x$: relative entropy,
- $h(x) = x^2 - 1$: variance,

Skew-Normal Diffusion (1)

Let

$$dY_t = (k + KY_t)dt + \Sigma dW_t, \quad Y_0 \in \mathbb{R}^n,$$

$k \in \mathbb{R}^n$, $K, \Sigma \in \mathbb{R}^{n\times n}$ s.t. $\Sigma\Sigma^\top > 0$, and W: BM. Then,

$$\mathbb{P}\left(Y_t \in dz | Y_s = y\right) = \tilde{p}_{t-s}(y,z)dz = \phi_n\left(z - \mu_t(y), V_t\right) \quad 0 \leq s < t,$$

where

$$\begin{aligned}
\phi_n\left(z; V_t\right) :=& \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(V_t)}} \exp\left(-\frac{1}{2} z^\top V_t^{-1} z\right), \\
\mu_t(y) :=& e^{tK} y + \left(\int_0^t e^{sK} ds\right) k, \\
V_t :=& \int_0^t e^{sK} \Sigma\Sigma^\top e^{sK^\top} ds.
\end{aligned}$$

First, consider $N = 1$ case. Choose

$$G(dy) := f_{\text{GMSN}}\left(y; a, b, A, B, C\right) dy,$$

$$f_{\text{GMSN}}(x; a, b, A, B, C) := \frac{1}{\Phi_m\left(Cb - a, A + CBC^\top\right)} \phi_n(x - b, B) \Phi_m\left(Cx - a, A\right),$$

$a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $A \in \mathbb{S}^m_{++}$, $B \in \mathbb{S}^n_{++}$, $C \in \mathbb{R}^{m \times n}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, and

$$\Phi_m(x; A) := \int_{\prod_{i=1}^m (-\infty, x_i]} \phi_m(y; A) dy,$$

Skew-Normal Diffusion (2)

First, consider $N = 1$ case. Choose

$$G(dy) := f_{\mathrm{GMSN}}\left(y; a, b, A, B, C\right) dy,$$

$$f_{\mathrm{GMSN}}(x; a, b, A, B, C) := \frac{1}{\Phi_m\left(Cb - a, A + CBC^\top\right)} \phi_n(x - b, B)\Phi_m\left(Cx - a, A\right),$$

$a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $A \in \mathbb{S}^m_{++}$, $B \in \mathbb{S}^n_{++}$, $C \in \mathbb{R}^{m\times n}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, and

$$\Phi_m(x; A) := \int_{\prod_{i=1}^m(-\infty, x_i]} \phi_m(y; A) dy,$$

Introduced by Gupta et al. (2004). Actually,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} A + CBC^\top & -CB \\ -BC^\top & B \end{pmatrix}\right),$$

then, $X_2 | \{X_1 \leq Cb\} \sim \mathrm{GMSN}(a, b, A, B, C)$.

Skew-Normal Distribution

Many sub(super)-classes of GMSN are studied by (mainly) statisticians.

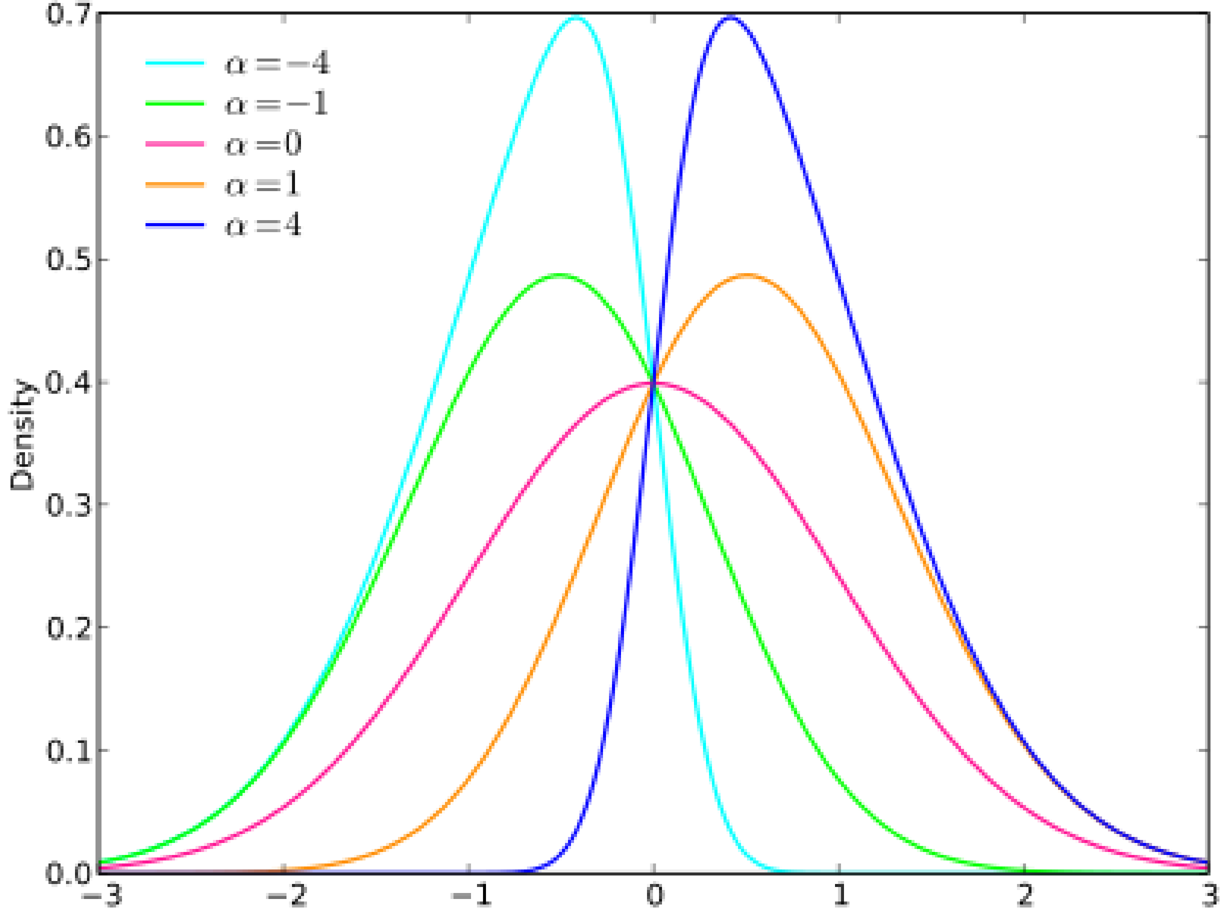
- Can refer to Monograph by Azzalini and Capitanio (2013).
- $\exists$ Several ways of constructions.
- The univariate SN density: letting $m = n = 1$, and

$$\begin{aligned} f_{\mathrm{USN}}(x;\mu,\sigma,\alpha) :=& f_{\mathrm{GMSN}}\left(x;\frac{\alpha\mu}{\sigma},\mu,1,\sigma,\frac{\alpha}{\sigma}\right) \\ =& \frac{2}{\sigma}\phi_1\left(\frac{y-\mu}{\sigma};1\right)\Phi_1\left(\alpha\left(\frac{y-\mu}{\sigma}\right);1\right). \end{aligned}$$

- The distribution function of USN:

$$\begin{aligned} F_{\mathrm{USN}}(x;\mu,\sigma,\alpha) =& \Phi_1\left(\frac{x-\mu}{\sigma};1\right) - 2T\left(\frac{x-\mu}{\sigma};\alpha\right), \\ T(x;\alpha) :=& \frac{1}{2\pi}\int_0^\alpha \frac{e^{-\frac{1}{2}x^2(1+y^2)}}{1+y^2}dy \quad : \text{Owen's T-function.} \end{aligned}$$

Density Plots (by Wikipedia)



Conditional Probability

- Filtered conditional probabilities are *conditionally skew-normal:*

$$\pi_t(dz) = f_{\mathrm{GMSN}}\left(z; a, \tilde{b}(t, Z_0, Z_t), A, \tilde{B}_t, C\right) dz, \quad t \in [0, T),$$

where

$$\begin{aligned}\tilde{b}(t, Z_0, Z_t) &:= \left(V_{T-t}^{-1} - V_T^{-1} + B^{-1}\right)^{-1} \left\{V_{T-t}^{-1}\mu_{T-t}(Z_t) - V_T^{-1}\mu_T(Z_0) + B^{-1}b\right\}, \\ \tilde{B}_t &:= \left(V_{T-t}^{-1} - V_T^{-1} + B^{-1}\right)^{-1}.\end{aligned}$$

Let

$$G(dx) := \mathbb{P}(X \in dx) = f_{\mathrm{GMSN}}\left(x; a, \mu_T(Z_0), A, V_T, C\right) dx.$$

- Filtered conditional probability:

$$\pi_t(dz) = f_{\mathrm{GMSN}}\left(z; a, \mu_{T-t}(Z_t), A, V_{T-t}, C\right) dz, \quad t \in [0, T),$$

- Transition probability:

$$q(s, x; t, y | Z_0) = q(s, x; t, y) = f_{\mathrm{GMSN}}\left(y; \tilde{a}_t, \mu_{t-s}(x), \tilde{A}_{T-t}, V_{t-s}, \tilde{C}_{T-t}\right),$$

where $\tilde{a}_t := a - C\left(\int_0^t e^{uK} du\right) k$, $\tilde{A}_t := A + CV_tC^\top$, $\tilde{C}_t := Ce^{tK}$.

Conditional/Transition Probability

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- Initial value *independent*! $(Z, \mathcal{F}_t^Z)$ and $(Y, \mathbb{P}^G)$ become (*completely*) Markov.

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- Initial value *independent*! $(Z, \mathcal{F}_t^Z)$ and $(Y, \mathbb{P}^G)$ become (*completely*) Markov.
- Totally different approach: Corns and Satchell (2007), inspired by Itô-Mckean's skew-BM.

Lemma

For

$$f_{\mathrm{GMSN}}(x; a, b, A, B, C) := \frac{1}{\Phi_m\left(Cb - a, A + CBC^\top\right)} \phi_n(x - b, B) \Phi_m\left(Cx - a, A\right),$$

it holds that

$$\begin{aligned} &\exp\left(-\frac{1}{2} y^\top P y + p^\top y\right) f_{\mathrm{GMSN}}(y; a, b, A, B, C) \\ &= f_{\mathrm{GMSN}}\left(y; a, \tilde{b}, A, \tilde{B}, C\right) \\ &\quad \times \frac{1}{\sqrt{\det(I + BP)}} \frac{\Phi_m\left(C\tilde{b} - a; A + C\tilde{B}C^\top\right)}{\Phi_m\left(Cb - a; A + CBC^\top\right)} \exp\left\{\frac{1}{2}\left(\tilde{b}^\top \tilde{B}^{-1} \tilde{b} - b^\top B^{-1} b\right)\right\}, \end{aligned}$$

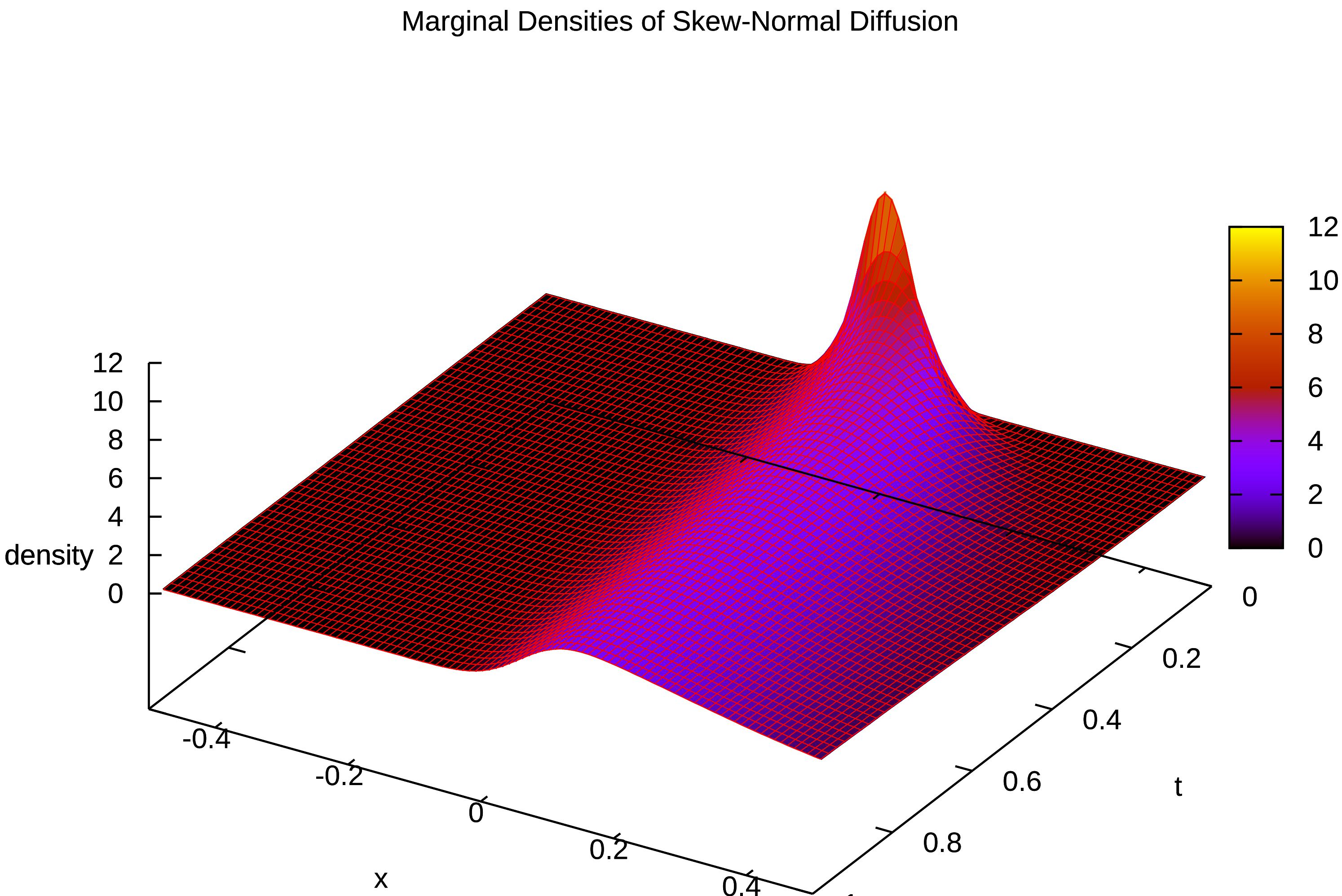
where $\tilde{b} := (P + B^{-1})^{-1}(p + B^{-1}b)$ and $\tilde{B} := (P + B^{-1})^{-1}$.

Sketch for Transition Density

$$q(s,x;t,y|z_0) = \frac{p_{t-s}(x,y)\displaystyle\int_{\mathbb{R}^n} J_{t,T}(z;z_0,y)dz}{\displaystyle\int_{\mathbb{R}^n} J_{s,T}(z;z_0,x)dz},$$

$$\begin{aligned}
J_{t,T}(z;z_0,y) :=& \frac{p_{T-t}(y,z)}{p_T(y_0,z)} f_{\mathrm{GMSN}}(z;a,\mu_T(z_0),A,V_T,C) \\
=& C_{t,T}\exp\left(\text{``quadratic in } z\text{''}\right) f_{\mathrm{GMSN}}(z;a,\mu_T(z_0),A,V_T,C) \\
=& \tilde{C}_{t,T} f_{\mathrm{GMSN}}(z;a,\tilde{\mu}_T(z_0),A,\tilde{V}_T,C)
\end{aligned}$$

Marginals of Skew-Normal Diffusion



Condition Multiple Marginals: $\mathbb{Y} := (Y_{T_1}, \dots, Y_{T_N})$ ($N \geq 2$)

- In "strong" approach, concatenate RMBs.
- In "weak" approach, assuming $G(dy) = g(y)dy$, compute the martingale density process

$$L_t := \frac{d\mathbb{P}^G_{y_0}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathbb{E}\left[H(\mathbb{Y}|y_0) \mid \mathcal{F}_t\right], \quad t \in [0,T],$$

$$H(y_1, \dots, y_N | y_0) := \frac{g(y_1, \dots, y_n)}{\prod_{j=1}^N \tilde{p}_{T_j - T_{j-1}}(y_{j-1}, y_j)}.$$

$(Y, \mathbb{P}^G)$-dynamics is obtained by Girsanov.

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Notes:

- $(Y, \mathbb{P}^G)$ and $(Z, \mathcal{F}^Z_t)$ are NON-Markov in general.
- Calculations of L_t ($t \in [0, T]$) are complicated (including $N \times n$-dim integrations).

Markov Example

$H(y_1, \ldots, y_N | y_0) = h_0(y_0) h_1(y_1) \cdots h_N(y_N)$, i.e.,

$$g(y_1, \ldots, y_N | y_0) = h_0(y_0) h_1(y_1) \cdots h_N(y_N) \prod_{j=1}^{N} \tilde{p}_{T_j - T_{j-1}}(y_{j-1}, y_j).$$

- Markov chain structure: $g(y_1, \ldots, y_n | y_0) = \prod_{k=1}^{N} g_k(y_k | y_{k-1})$, where

$$g_k\left(y_k | Y_{T_{k-1}}\right) = g_k\left(y_k | Y_{T_1}, \ldots, Y_{T_{k-1}}\right)$$

  is the conditional pdf of Y_{T_k} given $Y_{T_{k-1}}$ (or given $(Y_{T_1}, \ldots, Y_{T_{k-1}})$) under $\mathbb{P}^G$.
  - Can employ arbitrary conditional marginals, selecting $h_1, \ldots, h_N$.

Skew-Normal Example

- Choose arbitrary GMSN (conditional) densities s.t.

$$g_k(y_k|y_{k-1}) := f_{\text{GMSN}}\left(y_k; a_k, \mu_{T_k-T_{k-1}}(y_{k-1}), A_k, V_{T_k-T_{k-1}}, C_k\right), \quad k = 1, \ldots, N,$$

$a_k \in \mathbb{R}^m$, $b_k \in \mathbb{R}^n$, $A_k \in \mathbb{S}^m_{++}$, $B_k \in \mathbb{S}^n_{++}$, $C_k \in \mathbb{R}^{m\times n}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$. i.e.,

$$h_k(y_k) = \frac{\Phi_m\left(C_k y_k - a_k, A_k\right)}{\Phi_m\left(C_{k+1}\mu_{T_{k+1}-T_k}(y_k) - a_{k+1}, A_{k+1} + C_{k+1}V_{T_{k+1}-T_k}C_{k+1}^\top\right)},$$

recalling

$$\mu_{T_k-T_{k-1}}(Y_{T_{k-1}}) = \mathbb{E}[Y_{T_k}|Y_{T_{k-1}}], \quad V_{T_k-T_{k-1}} = \mathbb{V}[Y_{T_k}|Y_{T_{k-1}}].$$

  - $(Y, \mathbb{P}^G)$ is a (time-inhomogeneous) skew-normal diffusion.
  - Explicit formulas of transition/conditional probabilities.