p-adic hypergeometrics

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Abstract We study classical hypergeometric series as a p-adic function of its parameters inspired by a problem in the Monthly [3] solved by D. Zagier.

§1. The classical generalized hypergeometric series is defined by

$$ {}_rF_{r-1}\left[\begin{matrix}\alpha_1 & \dots & \alpha_r \\ \beta_1 & \dots & \beta_{r-1}\end{matrix} \mid t\right] = \sum_{k\geq 0} \frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_{r-1})_k}\frac{t^k}{k!} \tag{1} $$

for $\alpha_j \in \mathbb{C}$ and $\beta_j \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$. If $\alpha_r = -n$ for n a non-negative integer the series terminates and we have

$$ {}_rF_{r-1}\left[\begin{matrix}\alpha_1 & \dots & -n \\ \beta_1 & \dots & \beta_{r-1}\end{matrix} \mid t\right] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\alpha_1)_k\cdots(\alpha_{r-1})_k}{(\beta_1)_k\cdots(\beta_{r-1})_k} t^k \,. \tag{2} $$

One can show that for fixed $\alpha_i \in \mathbb{Z}_p, \beta_i \in \mathbb{Z}_p\setminus\{0,-1,-2,\dots\}$ and $|t|_p < 1$ this yields a convergent Mahler series and hence a continuous function f of the variable $x := n$ in $\mathbb{Z}_p$

$$ f(x) := {}_rF_{r-1}\left[\begin{matrix}\alpha_1 & \dots & -x \\ \beta_1 & \dots & \beta_{r-1}\end{matrix} \mid t\right] = \sum_{k\geq 0} (-1)^k \binom{x}{k} \frac{(\alpha_1)_k\cdots(\alpha_{r-1})_k}{(\beta_1)_k\cdots(\beta_{r-1})_k} t^k \,. \tag{3} $$

These functions seem very interesting and worthy of further investigation.

§2. It turns out that a special case of these functions appears in the solution of an interesting Monthly problem [3] solved by D. Zagier. The problem is to prove that

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$$v_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right)=v_3\left(n^2\binom{2n}{n}\right), \tag{4}$$

where v_p denotes the p-adic valuation. Zagier does this by showing that there is a continuous function $f_1:\mathbb{Z}_3\longrightarrow -1+3\mathbb{Z}_3$ which interpolates the values

$$f_1(n)=\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}, \qquad n=1,2,\dots. \tag{5}$$

Considering the expansion

$$f_1(n)=A+Bn+Cn^2+\cdots$$

he goes further and conjectures, based on numerical evidence, that $B=0$; moreover, he mentions

Another interesting problem would be to evaluate in closed form the 3*-adic number A.*

We prove that in fact

$$A=-\frac{3}{2}\zeta_3(2)=2+3+2\cdot 3^2+2\cdot 3^6+3^7+2\cdot 3^8+2\cdot 3^9+O(3^{10}), \tag{6}$$

where $\zeta_3(s)$ is the Kubota-Leopoldt 3-adic zeta function.

§3. The connection with zeta values is perhaps to be expected: in general the Taylor coefficients of the functions of §1 involve multiple polylogarithms. In the specific case in question we have

$$f(n)=\sum_{k=1}^{n}\frac{1}{\binom{2k}{k}}\binom{n}{k}(-3)^{k-1}, \qquad f(n)=nf_1(n). \tag{7}$$

If we expand in general

$$f(x):=\sum_{k\geq 1}\frac{1}{\binom{2k}{k}}\binom{x}{k}t^{k-1}=\sum_{n\geq 0}b_n(t)x^n,$$

then

$$b_n(t)=\frac{1}{(t+4)}\sum_{0\leq j_1<j_2<\cdots<j_n}\frac{(\frac{t}{t+4})^{j_n}}{(j_1+\frac{1}{2})(j_2+\frac{1}{2})\cdots(j_n+\frac{1}{2})}. \tag{8}$$

These multiple polylogarithms can be expressed in terms of usual polylogarithms for small n. Trivially $b_0=0$. For $n=1$ we have the following identity of power series in $z=1-w$

$$b_1\left((w-w^{-1})^2\right)=(w^2-w^{-2})^{-1}\log\left(w^2\right), \qquad w=1-z. \tag{9}$$

By plugging in a primitive third root of unity $\zeta_3 \in \mathbb{C}_3$ for w it follows that 3-adically we have $b_1(-3) = 0$. This shows that in this case $f(x)$ is divisible by x and we may consider $f_1(x) := f(x)/x$ (see [3]).

With some effort one can prove that as power series in z, with $w = 1 - z$, we have

$$b_2\left((w - w^{-1})^2\right) = (w^2 - w^{-2})^{-1}[\mathrm{Li}_2(1 - w^2) - \tfrac{1}{2}\mathrm{Li}_2(1 - w^4) - \mathrm{Li}_2(1 - w^{-2}) + \tfrac{1}{2}\mathrm{Li}_2(1 - w^{-4})], \tag{10}$$

where Li_2 is the standard dilogarithm function.

Plugging in $w = \zeta_3 \in \mathbb{C}_3$ into (10) and using a result of Coleman [1] we obtain (6). The identity is the special case $p = 3, r = 1$ of the following. Given a prime $p > 2$ fix $\zeta_p \in \mathbb{C}_p$ a primitive p-th root of unity.

Theorem 1. *i) The following limit exists*

$$A(\zeta_p) := \lim_{s\to\infty} \frac{1}{\binom{2p^s}{p^s} p^{2s}} \sum_{k=0}^{p^s-1} \binom{2k}{k} (\zeta_p + \zeta_p^{-1})^{2(p^s-1-k)} \tag{11}$$

ii) Let $\omega : \mathbb{F}_p^\times \to \mathbb{C}_p^\times$ be the Teichmüller character. For $0 < r < p-1$ we have

$$\frac{1}{(\omega^r(4) - 2\omega^r(2))} \sum_{i=1}^{p-1} \omega(i)^{-r} (\zeta_p^{2i} - \zeta_p^{-2i}) A(\zeta_p^i) = L_p(2, \omega^{r-1}), \tag{12}$$

where L_p is Kubota-Leopoldt's p-adic L-function.

We note in passing that

$$\lim_{s\to\infty} \binom{2p^s}{p^s} = 2 \prod_{k\geq 1} \frac{\Gamma_p(2p^k)}{\Gamma_p(p^k)^2}$$

(see [2, §6.3.4, ex. 16]), where Γ_p denotes the p-adic gamma function.

The beauty of the expressions (9) and (10) is that though their proof were obtained working over the complex numbers they are identities of power series with rational coefficients and hence also hold p-adically in an appropriate domain. Fortunately, this domain includes the point were need to evaluate for Zagier's questions ($w = \zeta_3 \in \mathbb{C}_3$).

For $n = 3$ there is an expression for $b_3(t)$ in terms of polylogarithms valid over the complex numbers, which is much more difficult to obtain. For $n > 3$ we do not expect $b_n(t)$ to reduce to polylogarithms.

However, to apply this expression for b_3 to our p-adic setting requires some form of analytic continuation. This we will achieve by delicate manipulations using Coleman's integration but the details have not yet been fully carried out.

The expectation nevertheless is that for $p = 3$ we should have that $b_3(-3)$ is a simple multiple of $L_3(3, \chi_{-3})$. But $L_3(s, \chi_{-3})$ is identically zero since χ_{-3} is odd! Hence the constant B of Zagier should vanish because it is a special value of an L-function which happens to be identically zero.

§4. We tested numerically to see if there are any other relations for $b_n(t)$ and p-adic L-values and found only the following likely identities:

$$\mathbb{Q}_3 : \begin{cases} b_4(-3) = -\frac{27}{8}\zeta_3(4) \\ b_6(-3) = -\frac{297}{32}\zeta_3(6) \end{cases} \qquad \mathbb{Q}_5 : \begin{cases} b_2(-5) = 0 \\ b_3(-5) = -\frac{25}{12}\zeta_5(3) \end{cases} \tag{13}$$

but we did not attempt to prove these. We pointed out above that $b_4(t)$ and $b_6(t)$ are not expected to be expressible in terms of polylogarithms. Hence the connection of the observed identities for $b_4(-3)$ and $b_6(-3)$ in $\mathbb{Q}_3$ appear to be less obvious than the others.

References

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