Triangular modular curves

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Abstract We consider certain generalizations of modular curves arising from congruence subgroups of triangle groups.

1 Triangle groups

Let $a,b,c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ satisfy $a \leq b \leq c$. Consider the triangle T with angles $\pi/a, \pi/b, \pi/c$ (with $\pi/\infty = 0$) in the space H, where H is the sphere, Euclidean plane, or hyperbolic plane according as the quantity $\chi(a,b,c) = 1/a + 1/b + 1/c - 1$ is positive, zero, or negative. Let τ_a, τ_b, τ_c be reflections in the sides of T and let $\Delta = \Delta(a,b,c)$ be the subgroup of orientation-preserving isometries in the group generated by the reflections: then Δ is generated by

$$\delta_a = \tau_b \tau_c, \quad \delta_b = \tau_c \tau_a, \quad \delta_c = \tau_a \tau_b$$

and has a presentation

$$\Delta = \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_a \delta_b \delta_c = 1 \rangle.$$

We call Δ a *triangle group*. The quotient

$$X = X(a,b,c;1) = \Delta(a,b,c) \backslash H$$

is a complex Riemannian 1-orbifold of genus zero; it has as many punctures as occurrences of ∞ among a,b,c.

Example 1. We have $\Delta(2,3,3) \simeq A_4$, and the other spherical triangle groups (i.e., those with $\chi(a,b,c) > 0$) correspond to the Platonic solids. The Euclidean triangle

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groups are the familiar tesselations of the plane by triangles. We have $\Delta(2,3,\infty) \simeq \mathrm{PSL}_2(\mathbb{Z})$; and $\Delta(\infty,\infty,\infty) \simeq \Gamma(2)$, the free abelian group on two generators.

A uniformizer for X is expressed by an explicit ratio of ${}_2F_1$-hypergeometric functions, with parameters given in terms of a,b,c. As a consequence, containments of triangle groups imply relations between ${}_2F_1$-hypergeometric functions, with arguments given by Belyi maps. Moreover, the quotient is a moduli space for certain abelian varieties, often called *hypergeometric abelian varieties*: the values of the hypergeometric functions are periods of the *generalized Legendre curve*

$$y^N = x^A(1-x)^B(1-tx)^C$$

for certain integers A,B,C,N again given explicitly in terms of a,b,c.

The triangle group Δ is *arithmetic* if and only if it is commensurable with the units of reduced norm 1 in an order in a quaternion algebra over a number field (necessarily defined over a totally real field and ramified at all but one real place). There are only 85 arithmetic triangle groups, the list given by Takeuchi [4]; for these groups, the corresponding curve X is a Shimura curve.

2 Triangular modular curves

For the remaining nonarithmetic triangle groups, there is still a quaternion algebra! This observation was used by Cohen–Wolfart [2] in their work on transcendence of values of hypergeometric functions. This relationship can be interpreted geometrically: there is a finite map $X \to V$ where V is a quaternionic Shimura variety, a moduli space for abelian varieties with quaternionic multiplication, suitably interpreted. The dimension $\mathrm{adim}(a,b,c)$ of V is given in terms of a,b,c; we call it the *arithmetic dimension* of (a,b,c). Nugent–Voight [3] have proven that for every t, the set $\{(a,b,c) : \mathrm{adim}(a,b,c) = t\}$ is finite and effectively computable. For example, there are $148+16 = 164$ triples with arithmetic dimension 2.

Like with the modular curves, we now add level structure: we take a congruence subgroup $\Gamma(\mathfrak{P}) \leq \Gamma$ of the uniformizing group Γ for V, and we intersect

$$\Delta(\mathfrak{p}) = \Gamma(\mathfrak{P}) \cap \Delta.$$

By pullback, this gives a cover

$$\phi : X(\mathfrak{p}) = \Delta(\mathfrak{p})\backslash H \to X(1);$$

this corresponds geometrically to adding level structure to the family of hypergeometric abelian varieties. Clark–Voight [1] have proven that the cover ϕ has Galois group $\mathrm{PSL}_2(\mathbb{F}_\mathfrak{p})$ or $\mathrm{PGL}_2(\mathbb{F}_\mathfrak{p})$ (cases distinguished by a Legendre symbol); moreover, the minimal field of definition of ϕ is explicitly given as an at most quadratic extension of an explicitly given totally real abelian number field with controlled ramification.

We call these curves $X(\mathfrak{p})$ *triangular modular curves* as generalizations of the classical modular curves, and we expect that their study will be as richly rewarding for arithmetic geometers as the classical case.

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References

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