Jacobi sums and Hecke Grössencharacters

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Abstract We give an extended abstract regarding our talk, and the associated Magma implementation of Jacobi sums and Hecke Grössencharacters. This builds upon seminal work of Weil (1952), and makes his construction explicitly computable, inherently relying on his upper bound for the conductor. Moreover, we can go slightly further than Weil by additionally allowing Kummer twists of the Jacobi sums. We also note the correspondence of these (twisted) Jacobi sums to tame prime information for hypergeometric motives.

Although our viewpoint and notation is derived from later work of Anderson, we do not use his formalism in any substantial way, and indeed the main thrust of all we do is already in Weil's work.

Let $\theta = \sum_j n_j \langle x_j \rangle \in \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]^0$ be an integral linear combination of nonzero elements $x_j \in \mathbf{Q}/\mathbf{Z}$ as a formal sum, with $\sum_j n_j x_j = 0$. We put m for the least common multiple of the denominators of the x_j, and write $K_\theta \subseteq \mathbf{Q}(\zeta_m)$ for the subfield corresponding by Galois theory to modding out $(\mathbf{Z}/m\mathbf{Z})^\star$ by those u for which the scaling $u \circ \theta = \sum_j n_j \langle u x_j \rangle$ is equal to θ. Letting α be a nontrivial additive character modulo p and recalling the Gauss sum of a multiplicative character ψ on $\mathbf{F}_\mathfrak{p}^\times$ as

$$G_\alpha(\psi) = -\sum_{x \in \mathbf{F}_\mathfrak{p}^\times} \psi(x)\,\alpha(\mathrm{Tr}\,x),$$

for ideals $\mathfrak{p}$ of $\mathbf{Q}(\zeta_m)$ we define the Jacobi sum

$$J_\theta(\mathfrak{p}) = \prod_j G_\alpha(\chi_\mathfrak{p}^{m x_j})^{n_j}$$

where this is independent of the choice of α and $\chi_\mathfrak{p}$ is the power residue symbol

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$$\chi_{\mathfrak{p}}(x) = \left(\frac{x}{\mathfrak{p}}\right)_m \equiv x^{(q-1)/m} \pmod{\mathfrak{p}}.$$

where q is the norm of $\mathfrak{p}$. One then has a partial L-function

$$L_\theta^\star(s) = \prod_{\mathfrak{p}} \left(1 - \frac{J_\theta(\mathfrak{p})}{q^s}\right)^{-1},$$

where the product is over $\mathfrak{p} \nmid m$ in K_θ.

In a 1952 paper [3], Weil associates a Grössencharacter to such a Jacobi sum L-function, and in particular gets an upper bound on the modulus. This gives us an algorithm in principle to compute said Grössencharacter, which has been implemented in the Magma computer algebra system [2, 1]. Briefly, one first determines the field of definition K_θ of the Grössencharacter as above, and then the ∞-type in a similar manner. The upper bound on the modulus then makes it a finite problem to recognize the correct twist in the Hecke character group (the dual of the ray class group), and by computing $J_\theta(\mathfrak{p})$ at sufficiently many primes of small norm we can isolate the desired twist. The possibility of including Kummer twists of the θ was not considered directly by Weil, but fits easily into the above framework.

The resulting Jacobi sum machinery also helps explain the tame prime behavior of hypergeometric motives, in particular giving the Euler factors when the inertia corresponding to such primes is in fact trivialized. As an example, for the quintic 3-fold at (say) $t = t_0 \cdot p^5$ with $p \equiv 1 \pmod 5$, the Euler factor corresponds to a Grössencharacter over $\mathbf{Q}(\zeta_5)$, with the precise twist varying with t_0.

This is joint work with David Roberts and Fernando Rodriguez Villegas.

References

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