On optimal periodic dividend strategies for Lévy risk processes [1]

Kazutoshi Yamazaki

Department of Mathematics, Kansai University

Mathematics of Risk, Melbourne, 2017

[1] joint work with K. Noba, J. L. Pérez and K. Yano

Classical optimal dividend problems

- We are given a spectrally negative Lévy process X that models the surplus of an insurance company.
- A strategy $\pi := (L^\pi(t); t \geq 0)$ is given by a non-decreasing, right-continuous, and adapted process such that

$$\Delta L^\pi(t) \leq U^\pi(t-), \quad t \geq 0,$$

  where $U^\pi(t) := X(t) - L^\pi(t)$.
- The objective is to find the optimal strategy that maximizes

$$\mathbb{E}_x\Big(\int_{[0,\eta_0^\pi]} \mathrm{e}^{-qt} \mathrm{d}L^\pi(t) \Big),$$

  where

$$\eta_0^\pi := \inf\{t > 0 : U^\pi(t) < 0\}.$$

Periodic case

- We are given an independent Poisson process N^r with parameter $r > 0$.
- A strategy $\pi := (L^\pi(t); t \geq 0)$ is given by a non-decreasing and adapted process of the form

$$L^\pi(t) = \int_{[0,t]} \nu^\pi(s) \mathrm{d} N^r(s), \qquad t \geq 0$$

such that

$$\Delta L^\pi(t) \leq U^\pi(t-), \quad t \geq 0,$$

where $U^\pi(t) := X(t) - L^\pi(t)$.

- The objective is to find the optimal strategy that maximizes

$$\mathbb{E}_x\Big(\int_{[0,\eta_0^\pi]} \mathrm{e}^{-qt} \mathrm{d} L^\pi(t) \Big),$$

where $\eta_0^\pi := \inf\{t > 0 : X(t) - L^\pi(t) < 0\}$.

Known results for the classical case and conjectures

- By Avram, Pistorius, Palmowski (2007), under the barrier strategy π_b that reflects the process from above at b,

$$\mathbb{E}_x\Big(\int_{[0,\sigma_0^{\pi_b}]} e^{-qt}\mathrm{d}L^{\pi_b}(t)\Big) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(b)} & x \leq b, \\ \frac{W^{(q)}(b)}{W^{(q)\prime}(b)} + x - b & x > b. \end{cases}$$

- Suppose the Lévy measure has a completely monotone density. The Lévy measure $\overline{\Pi}$ of the dual process $-X$ has a density π such that

$$(-1)^n \pi^{(n)}(x) \geq 0, \quad x > 0.$$

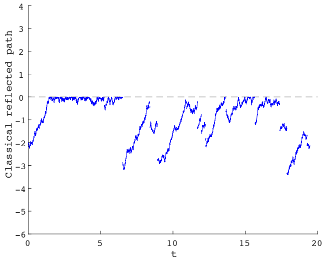
  - We have $W^{(q)\prime\prime\prime}(x) > 0$ for all $x > 0$ and hence there exists $\bar{b} \in [0, \infty)$ such that $W^{(q)\prime\prime} < 0$ on $(0, \bar{b})$ and $W^{(q)\prime\prime} > 0$ on $(\bar{b}, \infty)$.
  - As shown in Loeffen (2009), it is optimal to reflect at $\bar{b}$.

- Analogous barrier strategy is expected to be optimal?

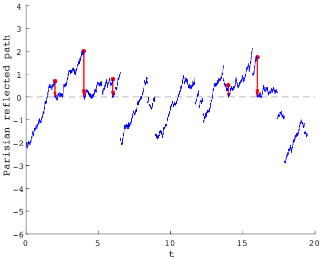
Parisian reflection

- Classical reflection:
  - the process is observed continuously;
  - $L(t)$ so that $X(t) - L(t)$ is a reflected process with barrier b;
  - minimum amount required to push down so that it stays below b.
- Parisian reflection:
  - the process is observed only at exponential times;
  - it is pushed down to b whenever the process is above b at exponential interarrival times.

Sample paths (upper barrier case)



classical reflection



Parisian reflection

Lévy processes $X_r(t)$ with Parisian reflection below

- $\mathcal{T}_r = \{T(i); i \geq 1\}$: dividend payment opportunities.
- We have

$$X_r(t) = X(t), \quad 0 \leq t < T_b^+(1)$$

where

$$T_b^+(1) := \inf\{T(i) : \ X(T(i)) > b\}.$$

- The process then jumps down by $|X(T_b^+(1)) - b|$ s.t. $X_r(T_b^+(1)) = b$.
- For $T_b^+(1) \leq t < T_b^+(2) := \inf\{T(i) > T_b^+(1) : \ X_r(T(i)-) > b\}$,

$$X_r(t) = X(t) - |X(T_b^+(1)) - b|.$$

Solution Techniques

- Compute the expected NPV of dividends under a periodic barrier strategy $\pi^b := (L_r^b(t); t \geq 0)$, for $b \geq 0$:

$$v_b(x) := \mathbb{E}_x\Big(\int_{[0,\sigma_0^b]} e^{-qt} \mathrm{d}L_r^b(t) \Big), \quad x \geq 0,$$

where

$$\sigma_0^b := \inf\{t > 0 : U_r^b(t) < 0\}.$$

- Choose the candidate barrier b^* so that v_{b^*} gets "smoother".
- Verify the optimality.

Spectrally Negative Lévy Processes

- Let X be a spectrally negative Lévy process with a Laplace exponent:

$$\psi(s) := \log \mathbb{E}\left[e^{sX(1)}\right] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} (e^{sz} - 1 - sz1_{\{-1<z<0\}})\nu(\mathrm{d}z),$$

  such that $\int_{(-\infty,0)}(1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} z\,\nu(\mathrm{d}z) < \infty$.
- We exclude the case X is the negative of a subordinator.

Scale Functions

- Recall that X is a spectrally negative Lévy process with Laplace exponent $\psi(s) = \log \mathbb{E}\left[e^{sX(1)}\right]$.
- Fix any $q \geq 0$, there exists a function called the q-scale function

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(s) - q}, \qquad s > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Scale Functions (Cont'd)

Let us define the first down- and up-crossing times, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X(t) < a\}$$
$$\tau_b^+ := \inf\{t \geq 0 : X(t) > b\}.$$

Then we have for any $b > 0$

$$\mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)},$$
$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}}\right] = Z^{(q)}(x) - Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)},$$

where

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\mathrm{d}y,$$
$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x).$$

More notations

For $q, r > 0$ and $x \in \mathbb{R}$,

$$\overline{\overline{W}}^{(q)}(x) := \int_0^x \overline{W}^{(q)}(y)\mathrm{d}y,$$

$$Z^{(q)}(x, \Phi(q+r)) := e^{\Phi(q+r)x}\left(1 - r\int_0^x e^{-\Phi(q+r)z}W^{(q)}(z)\mathrm{d}z\right)$$

$$= r\int_0^\infty e^{-\Phi(q+r)z}W^{(q)}(z+x)\mathrm{d}z > 0,$$

$$Z^{(q)\prime}(x, \Phi(q+r)) := \frac{\partial}{\partial x}Z^{(q)}(x, \Phi(q+r))$$

$$= \Phi(q+r)Z^{(q)}(x, \Phi(q+r)) - rW^{(q)}(x).$$

Expression of v_b

Lemma (Pérez & Yamazaki (2016))

For all $b \geq 0$ *and* $x \in \mathbb{R}$,

$$\begin{aligned} v_b(x) &= \mathbb{E}_x\Big(\int_{[0,\sigma_0^b]} e^{-qt} \mathrm{d}L_r^b(t) \Big) \\ &= \frac{r}{\Phi(q+r) Z^{(q)\prime}(b, \Phi(q+r))} \\ &\times \Big(W^{(q)}(x) - rW^{(q)}(b) \overline{W}^{(q+r)}(x-b) \\ &\qquad + r \int_0^{x-b} W^{(q+r)}(x-b-y) W^{(q)}(y+b) \mathrm{d}y \Big) \\ &- r \overline{\overline{W}}^{(q+r)}(x-b). \end{aligned}$$

Selection of b^*

Usually the following three selection criteria are equivalent:

- Get b^* such that v_{b^*} is smoother at b^*.
- Get b^* such that v_{b^*} such that $\frac{\partial}{\partial b} v_b(x)|_{b=b^*} = 0$.
- Get b^* such that $v'_{b^*}(b^*) = 1$.

Using the smoothness criteria

For any choice of b,

- v_b is C^2 on $(0,\infty)\backslash\{b\}$ for the case X is of bounded variation.
- v_b is C^3 on $(0,\infty)\backslash\{b\}$ for the case X is of unbounded variation.

Suppose

$$\mathfrak{C}_b : W^{(q)\prime}(b) = \frac{\Phi(q+r)}{r} Z^{(q)\prime}(b, \Phi(q+r)),$$

holds, then

- v_b is C^2 on $(0,\infty)$ for the case X is of bounded variation.
- v_b is C^3 on $(0,\infty)$ for the case X is of unbounded variation.

Existence of a candidate barrier

- We have

$$\mathfrak{C}_b : W^{(q)\prime}(b) = \frac{\Phi(q+r)}{r} Z^{(q)\prime}(b, \Phi(q+r))$$
$$\iff h(b) = 0$$

where

$$h(b) = -r \int_b^{\infty} e^{-\Phi(q+r)y} W^{(q)\prime\prime}(y) \mathrm{d}y, \quad b > 0.$$

- h has a nice form if the Lévy measure has a completely monotone density.

Completely monotone case

1. For some finite measure $\mu^{(q)}$,
$$W^{(q)}(x) = \Phi'(q)e^{\Phi(q)x} - \int_0^\infty e^{-xt}\mu^{(q)}(\mathrm{d}t), \qquad x > 0.$$
2. We have $W^{(q)\prime\prime\prime}(x) > 0$ for all $x > 0$.
3. There exists $\bar{b} \in [0, \infty)$ such that $W^{(q)\prime\prime} < 0$ on $(0, \bar{b})$ and $W^{(q)\prime\prime} > 0$ on $(\bar{b}, \infty)$.
4. $\bar{b}$ is the optimal barrier in the classical case.

Hence
$$h(b) = -r\int_b^\infty e^{-\Phi(q+r)y} W^{(q)\prime\prime}(y)\mathrm{d}y, \quad b > 0.$$
must (1) first decrease and then increases or (2) increase monotonically. It also converges to 0.

Definition of b^*

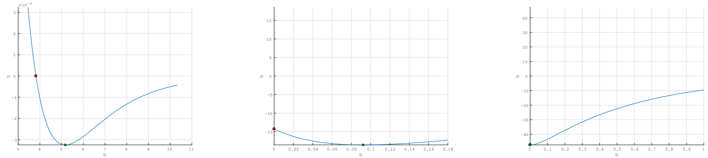


Figure: Plots of h and the points b^* and $\bar{b}$ indicated by circles and squares, respectively.

- If $h(0+) > 0$, then there must exist b^* such that $h(b^*) = 0$.
- Otherwise we set $b^* = 0$.

Verification lemma

Let

$$\mathcal{L}f(x) := \gamma f'(x) + \frac{1}{2}\eta^2 f''(x) + \int_{(-\infty,0)} \left[f(x+z) - f(x) - f'(x) z 1_{\{-1<z<0\}} \right] \Pi(\mathrm{d}z).$$

Lemma (Verification lemma)

Suppose $\hat{\pi} \in \mathcal{A}$ is such that $v_{\hat{\pi}}$ is $C^1(0,\infty)$ (resp. $C^2(0,\infty)$) for the case X is of bounded (resp. unbounded) variation, and satisfies

$$(\mathcal{L} - q) v_{\hat{\pi}}(x) + r \max_{0 \leq l \leq x} \{ l + v_{\hat{\pi}}(x - l) - v_{\hat{\pi}}(x) \} \leq 0, \quad x > 0.$$

Then $v_{\hat{\pi}}(x) = v(x)$ for all $x \geq 0$ and hence $\hat{\pi}$ is an optimal strategy.

Verification

Easy to show the following.

Lemma

For $b^* \geq 0$, we have

$$(\mathcal{L} - q)v_{b^*}(x) = \begin{cases} 0 & \text{if } x \in (0, b^*], \\ -r\left\{(x - b^*) + v_{b^*}(b^*) - v_{b^*}(x)\right\} & \text{if } x \in (b^*, \infty). \end{cases}$$

Hard to show: for $x > b^*$,

$$\max_{0 \leq l \leq x} \left\{l + v_{\hat{\pi}}(x - l) - v_{\hat{\pi}}(x)\right\} = \begin{cases} 0 & x \in (0, b^*] \\ (x - b^*) + v_{b^*}(b^*) - v_{b^*}(x) & x \in (b^*, \infty) \end{cases}.$$

Verification (cont'd)

Lemma

For $b^* \geq 0$, we have $v'_{b^*}(x) \geq 1$ for $x \in (0, b^*)$ and $0 \leq v'_{b^*}(x) \leq 1$ for $x \in (b^*, \infty)$.

For the proof, we use

$$W^{(q)}(x) = \Phi'(q)e^{\Phi(q)x} - \int_0^\infty e^{-xt}\mu^{(q)}(\mathrm{d}t), \qquad x > 0.$$

Convergence results

Lemma

1. *The optimal periodic barrier* b_r^* *is increasing in* r.
2. *We have* $b_r^* \to \bar{b}$ *as* $r \to \infty$.
3. *When* $W^{(q)\prime}(0+) < \infty$, b_r^* *is zero for sufficiently small* r. *When* $W^{(q)\prime}(0+) = \infty$, $b_r^* \to 0$ *as* $r \to 0$.

Theorem

As $r \to \infty$, $v^{(r)}(x)$ *converges to* $\bar{v}(x)$ *for all* $x \geq 0$.

Numerical results

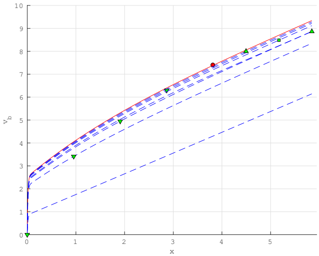
Assume

$$X(t) - X(0) = ct + \sigma B(t) - \sum_{n=1}^{N(t)} Z_n, \quad 0 \leq t < \infty.$$

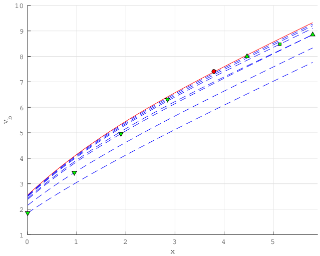
Here,

- $B = (B(t); t \geq 0)$ is a standard Brownian motion;
- $N = (N(t); t \geq 0)$ is a Poisson process with arrival rate κ;
- $Z = (Z_n; n = 1, 2, \ldots)$ is an i.i.d. sequence of exponential random variables with parameter λ;
- The processes B, N, and Z are assumed mutually independent.

Value functions: $b^* > 0$

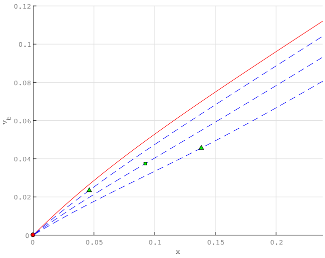


Unbounded var

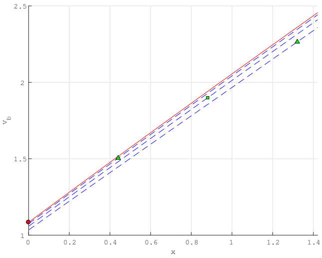


Bounded var

Value functions: $b^* = 0$

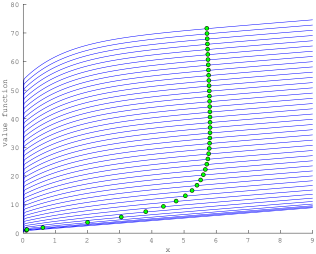


Unbounded var

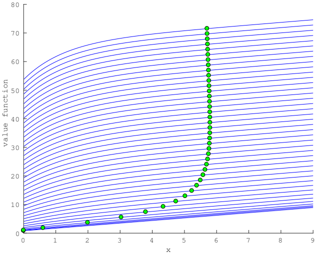


Bounded var

With respect to c

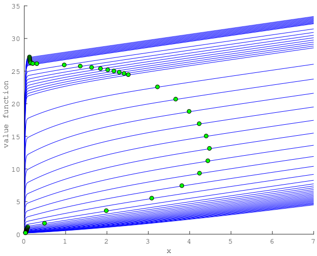


Unbounded var

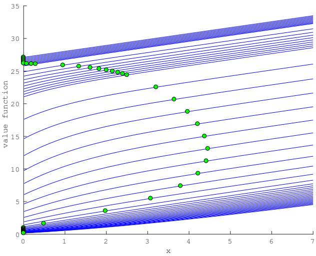


Bounded var

With respect to κ

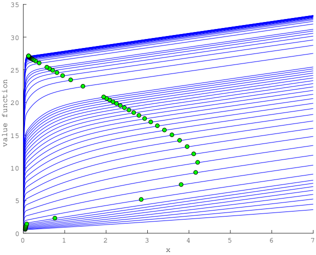


Unbounded var

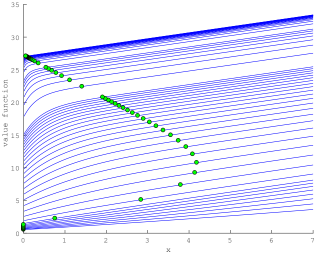


Bounded var

With respect to λ

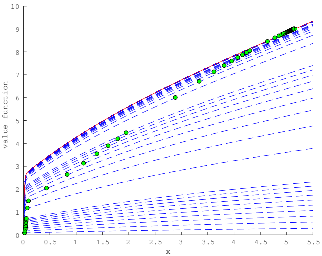


Unbounded var

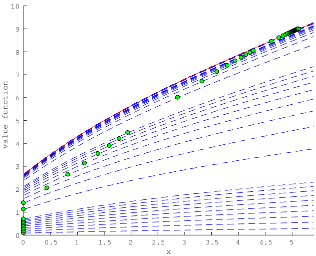


Bounded var

With respect to r



Unbounded var



Bounded var

References

- Avram, F., Palmowski, Z., and Pistorius, M.R. On the optimal dividend problem for a spectrally negative Lvy process. *The Annals of Applied Probability*, 17.1, 156-180, (2007).
- Loeffen, R. L. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lýy processes. *The Annals of Applied Probability*, 18.5, 1669-1680, (2008).
- Pérez, J.L., and Yamazaki, K. Mixed Periodic-classical barrier strategies for Lvy risk processes. *arXiv:*1609.01671, (2016).
- Pérez, J.L. and Yamazaki, K. Periodic barrier strategies for a spectrally positive Lévy process. *Insurance: Mathematics and Economics* (forthcoming).
- Noba, K., Pérez, J.L., Yamazaki, K, and Yano, K. On optimal periodic dividend strategies for Levy risk processes. *arXiv:* 1708.01678, (2017).