Special values of hypergeometric functions and periods of CM elliptic curves

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Abstract Let $X = X_0^6(1)/W_6$ be the the quotient of the Shimura curve $X_0^6(1)$ by all the Atkin-Lehner involutions. By realizing modular forms on X in two ways, one in terms of hypergeometric functions and the other in terms of Borcherds forms, and using Schofers formula for values of Borcherds forms at CM-points, we obtain special values of certain hypergeometric functions in terms of periods of elliptic curves over $\overline{Q}$ with complex multiplication.

Let $X_0^D(N)$ be the Shimura curve associated to an Eichler order of level N in an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. When $D = 1$, the Shimura curve $X_0^1(N)$ is just the classical modular curve $X_0(N)$ and there are many different constructions of modular forms on $X_0(N)$ in literature, such as Eisenstein series, Dedekind eta functions, Poincare series, theta series, and etc. These explicit constructions provide practical tools for solving problems related to classical modular curves. On the other hand, when $D \neq 1$, because of the lack of cusps, most of the methods for classical modular curves cannot possibly be extended to the case of general Shimura curves. However, in recent years, there have been several methods for Shimura curve emerging in literature, such as the method of Yang [9] realizing modular forms on a Shimura curve of genus zero in terms of solutions of its Schwarzian differential equation, the method of Voight and Willis [7] for computing power series expansions of modular forms, the method of Nelson [4] for computing values of modular forms using explicit Shimizu lifting [8], and the method of Elkies [1] via $K3$ surfaces. Finally, there is a powerful method that realizes modular forms on Shimura curves as Borcherds forms. To make the method of Borcherds forms useful in practice, one would employ Schofer's formula [5] for values of Borcherds

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forms at CM-points (see [2] for sample computation). In [3], we developed a systematic method to construct Borcherds forms and determined the equations of all hyperelliptic Shimura curves $X_0^D(N)$ using Schofer's formula.

In [11], by combining the method of Schwarzian differential equations and the method of Borcherds forms, we obtain some intriguing evaluations of hypergeometric functions, such as

$$ {}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};-\frac{3^7\cdot 7^4}{2^{10}\cdot 5^6}\right)=\frac{1}{2}\sqrt[4]{10}\sqrt{7+\sqrt{43}}\,\frac{\omega_{-43}}{\omega_{-4}} \tag{1}$$

and

$$ {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};-\frac{3^7\cdot 7^4}{2^{10}\cdot 5^6}\right)=\frac{100}{21}\omega_{-43}^2, \tag{2}$$

where for a negative fundamental discriminant d, we let

$$\omega_d=\frac{1}{\sqrt{|d|}}\prod_{a=1}^{|d|-1}\Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d}$$

be the Chowla-Selberg period. Here χ_d is the Kronecker character associated to $\mathbb{Q}(\sqrt{d})$, μ_d is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, and h_d is the class number of $\mathbb{Q}(\sqrt{d})$. Note that if E is an elliptic curve over $\overline{\mathbb{Q}}$ with complex multiplication by $\mathbb{Q}(\sqrt{d})$, then its periods are algebraic multiples of $\sqrt{\pi}\omega_d$. We now explain the origin of such evaluations.

Assume that $t(\tau)$ is a modular function on $X_0^D(N)$ that takes algebraic values at all CM-points. Then according to Shimura [6, Theorem 7.1] and Yoshida [12, Theorem 1.2 and (1.4) of Chapter 3], the value of $t'(\tau)$ at a CM-point of discriminant d is an algebraic multiple of ω_d^2. Here we choose $t(\tau)$ to be the Hauptmodul of $X=X_0^6(1)/W_6$, the quotient of $X_0^6(1)$ by all the Atkin-Lehner involutions, that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, and -3, respectively. Now the Schwarzian differential equation of X is essentially a hypergeometric differential equation (see [9]), which means that all (meromorphic) modular forms on X can be expressed in terms of hypergeometric functions. In particular, we have

$$t'(\tau)=\frac{2t^{1/4}(1-t)^{1/2}}{Ci}\left({}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};t\right)-C\,{}_2F_1\left(\frac{7}{24},\frac{11}{24};\frac{5}{4};t\right)\right)^2,$$

where $C=-1/\sqrt[4]{12}\omega_{-4}^2$ (see Lemma 8 of [10]). Manipulating this identity and recalling the result of Shimura and Yoshida above, we find that at a CM-point τ_d of discriminant d, we have

$${}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};t(\tau_d)\right)\in\frac{\omega_d}{\omega_{-4}}\cdot\overline{\mathbb{Q}}$$

and

$${}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};t(\tau_d)\right)\in\omega_d^2\cdot\overline{\mathbb{Q}}.$$

This explains the algebraicity of values of hypergeometric functions at singular moduli. To determine actual values, we use theory of Borcherds forms.

In [9], we find that the one-dimensional space of modular forms of weight 8 on X is spanned by

$$\left({}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};t\right)+\frac{1}{\sqrt[4]{12}\omega_{-4}^2}t^{1/4}{}_2F_1\left(\frac{7}{24},\frac{11}{24};\frac{5}{4};t\right)\right)^8.$$

On the other hand, we can construct a Borcherds form Ψ of weight 8. As the space of modular forms has dimension 1, these two modular forms must be scalar multiples of each other. Evaluating Ψ at the CM-point of discriminant -4 using Schofer's formula, we can determine the ratio of the two modular forms. Then evaluating at other CM-points, we obtain the special values of hypergeometric functions.

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