

Diagonal form factors from non-diagonal ones

Zoltan Bajnok and Chao Wu

Abstract We prove the asymptotic large volume expression of diagonal form factors in integrable models by evaluating carefully the diagonal limit of a non-diagonal form factor in which we send the rapidity of the extra particle to infinity.

1 Introduction

Two dimensional integrable quantum field theories are useful toy models of statistical and particle physics as they provide many interesting observables, which can be calculated exactly [12]. These models are first solved in infinite volume, where the scattering matrix [21, 4], which connects asymptotic multiparticle states, are determined together with the form factors which are the matrix elements of local operators sandwiched between the same asymptotic states [19]. These form factors then can be used to build up the correlation functions, which define the theory in the Wightman sense [1].

In the relevant practical applications, however, quantum field theories are confined to a finite volume and the calculation of finite size corrections is unavoidable. Fortunately, all these finite size corrections can be expressed in terms of the infinite volume characteristics, such as masses, scattering matrices and form factors [10, 11, 16]. We can distinguish three domains in the volume according to the nature of the corrections. The leading finite size corrections are polynomial in the inverse power of the volume, while the sub-leading corrections are exponentially volume-suppressed.

Zoltan Bajnok

MTA Lendület Holographic QFT Group, Wigner Research Centre for Physics, Konkoly-Thege Miklós u. 29-33, 1121 Budapest, Hungary, e-mail: bajnok.zoltan@wigner.mta.hu

Chao Wu

MTA Lendület Holographic QFT Group, Wigner Research Centre for Physics, Konkoly-Thege Miklós u. 29-33, 1121 Budapest, Hungary, e-mail: chao.wu@wigner.mta.hu

Concerning the finite volume energy spectrum the domain when only polynomial corrections are kept is called the Bethe-Yang (BY) domain. We there merely need to take into account the finite volume quantization of the momenta, which originates from the periodicity requirement and explicitly includes the scattering phase-shifts [11]. The exponentially small corrections are due to virtual particles traveling around the world and the domain in which we keep only the leading exponential correction is called the Luscher domain [10]. In a small volume, when all exponentials contribute the same way, we have to sum them up leading to a description given by the Thermodynamic Bethe Ansatz (TBA) [20].

The situation for the form factors are not understood at the same level yet. The BY domain was investigated in [16, 17]. It was proven for non-diagonal form factors that all polynomial finite size effects come only from the finite volume (Kronecker-delta) normalization of states. The authors also conjectured the BY form of diagonal finite volume form factors, which they derived for two particle-states. The leading exponential finite size corrections for generic form factors are not known, except for the diagonal ones, for which exact conjectures exist. The LeClair-Mussardo (LM) conjecture expresses the exact finite volume/temperature one-point functions in terms of infinite volume diagonal connected form factors, and densities of mirror states determined by the TBA equation [9]. Actually it was shown in [13, 14] that the BY form of diagonal form factors implies the LM formula and vice versa. Using analytical continuation ala [5] Pozsgay extended the LM formula for finite volume diagonal matrix elements [15]. The aim of the present paper is to prove the conjectured BY form of diagonal form factors [18, 17] from the already proven non-diagonal BY form factors [16] by carefully calculating the diagonal limit, in which we send one particle's rapidity to infinity. By this way our result also leads to the prove of the LM formula. Here we focus on theories with one type of particles.

The paper is organized such that in the next section we summarize the known facts about the BY form of diagonal and non-diagonal form factors. We then in section 3 prove the diagonal conjecture and conclude in section 4.

2 The conjecture for diagonal large volume form factors

In this section we introduce the infinite volume form factors and their properties and use them later on to describe the finite volume form factors in the BY domain.

2.1 Infinite volume form factors

Infinite volume form factors are the matrix elements of local operators sandwiched between asymptotic states $\langle \theta'_1, \dots, \theta'_m | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle$. We use the rapidity θ to parametrize the momenta as $p = m \sinh \theta$. The crossing formula

$$\begin{aligned} \langle \theta'_1, \dots, \theta'_m | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle &= \langle \theta'_1, \dots, \theta'_{m-1} | \mathcal{O} | \bar{\theta}'_m - i\varepsilon, \theta_n, \dots, \theta_1 \rangle + \\ &\sum_{i=1}^n 2\pi\delta(\theta_m - \theta_i) \prod_{j=i+1}^n S(\theta_j - \theta_i) \langle \theta'_1, \dots, \theta'_{m-1} | \mathcal{O} | \theta_{n-1}, \dots, \theta_1 \rangle \end{aligned} \quad (1)$$

can be used to express every matrix element in terms of the elementary form factors

$$\langle 0 | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle = F_n(\theta_n, \dots, \theta_1) \quad (2)$$

where $\bar{\theta} = \theta + i\pi$ denotes the crossed rapidity and the two particle S-matrix satisfies $S(\theta) = S(i\pi - \theta) = S(-\theta)^{-1}$. Infinite volume states are normalized to Dirac δ -functions: as $\langle \theta' | \theta \rangle = 2\pi\delta(\theta - \theta')$. The elementary form factor satisfies the permutation and periodicity axiom

$$\begin{aligned} F_n(\theta_1, \theta_2, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) &= S(\theta_i - \theta_{i+1}) F_n(\theta_1, \theta_2, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) \\ &= F_n(\theta_2, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n, \theta_1 - 2i\pi) \end{aligned} \quad (3)$$

together with the kinematical singularity relation

$$-i\text{Res}_{\theta'=\theta} F_{n+2}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) = (1 - \prod_{i=1}^n S(\theta - \theta_i)) F_n(\theta_1, \dots, \theta_n) \quad (4)$$

For scalar operators, when properly normalized, the form factor also satisfies the cluster property

$$\lim_{\Lambda \rightarrow \infty} F_{n+m}(\theta_1 + \Lambda, \dots, \theta_n + \Lambda, \theta_{n+1}, \dots, \theta_{n+m}) = F_n(\theta_1, \dots, \theta_n) F_m(\theta_{n+1}, \dots, \theta_{n+m}) \quad (5)$$

which will be used to analyze the diagonal limit of $\langle \theta, \theta'_1, \dots, \theta'_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle$ via $\theta \rightarrow \infty$ in finite volume.

The diagonal form factors $\langle \theta_1, \dots, \theta_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle$ are singular due to the $\delta(\theta)$ terms coming from the normalization of the states and also from poles related to the kinematical singularity axiom. Actually, $F_{2n}(\bar{\theta}_1 + \varepsilon_1, \dots, \bar{\theta}_n + \varepsilon_n, \theta_n, \dots, \theta_1)$ is not singular when all ε_i go to zero simultaneously, but depends on the direction of the limit. The *connected* diagonal form factor is defined as the finite ε -independent part:

$$F_{2n}^c(\theta_1, \dots, \theta_k) = \text{Fp} (F_{2n}(\bar{\theta}_1 + \varepsilon_1, \dots, \bar{\theta}_n + \varepsilon_n, \theta_n, \dots, \theta_1)) \quad (6)$$

while the *symmetric* evaluation is simply

$$F_{2n}^s(\theta_1, \dots, \theta_k) = \lim_{\varepsilon \rightarrow 0} F_{2n}(\bar{\theta}_1 + \varepsilon, \dots, \bar{\theta}_n + \varepsilon, \theta_n, \dots, \theta_1) \quad (7)$$

In order to understand the singularity structure of the diagonal limit we note that the singular part can very nicely be visualized by graphs [17]:

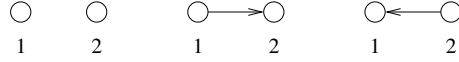
$$F_{2n}(\bar{\theta}_1 + \varepsilon_1, \dots, \bar{\theta}_n + \varepsilon_n, \theta_n, \dots, \theta_1) = \sum_{\text{allowed graphs}} F(\text{graph}) + O(\varepsilon_i) \quad (8)$$

where an allowed graph is an oriented tree-like (no-loop) graph in which at each vertex there is at most one outgoing edge. The contribution of a graph, $F(\text{graph})$, can be evaluated as follows: points (i_1, \dots, i_k) with no outgoing edges contribute a factor, $F_{2k}^c(\theta_{i_1}, \dots, \theta_{i_k})$, while for each edge from i to j we associate a factor $\frac{\varepsilon_j}{\varepsilon_i} \phi(\theta_i - \theta_j)$, where $\phi(\theta) = -i \partial_\theta \log S(\theta) = -i \frac{S'(\theta)}{S(\theta)}$. We recall the proof of (8) from [17] as similar argumentations will be used later on. The proof goes in induction in n and evaluates the residue at $\varepsilon_n = 0$ keeping all other ε s finite. Clearly such singular term can come only from graphs in which n has only an outgoing edge and no incoming one. The contributions of such terms are

$$\frac{1}{\varepsilon_n} (\varepsilon_1 \phi_{1n} + \dots + \varepsilon_{n-1} \phi_{n-1n}) F_{2n-2}(\bar{\theta}_1 + \varepsilon_1, \dots, \bar{\theta}_{n-1} + \varepsilon_{n-1}, \theta_{n-1}, \dots, \theta_1) \quad (9)$$

Now comparing this expression to the kinematical singularity axiom and using the definition of $\phi(\theta)$ together with the properties of the scattering matrix we can see that they completely agree. The formula (8) can be used to define connected form factors recursively by subtracting the singular terms and taking the diagonal limit. Observe also that taking all ε to be the same makes the lhs. of (8) the symmetric form factor, which is expressed by (8) in terms of the connected ones.

In particular, for the 2-particle form factor we have only three graphs:



which give

$$F_4(\bar{\theta}_1 + \varepsilon_1, \bar{\theta}_2 + \varepsilon_2, \theta_2, \theta_1) = F_4^c(\theta_1, \theta_2) + \frac{\varepsilon_1}{\varepsilon_2} \phi_{12} F_2^c(\theta_1) + \frac{\varepsilon_2}{\varepsilon_1} \phi_{21} F_2^c(\theta_2) + O(\varepsilon_i) \quad (10)$$

This equation on the one hand can be used to define $F_4^c(\theta_1, \theta_2)$, once $F_2^c(\theta)$ has been already defined, and on the other hand, it connects the symmetric form factor to the connected one:

$$F_4^s(\theta_1, \theta_2) = F_4^c(\theta_1, \theta_2) + \phi_{12} F_2^c(\theta_1) + \phi_{21} F_2^c(\theta_2) \quad (11)$$

2.2 Finite volume form factors in the BY domain

In the BY domain we drop the exponentially suppressed $O(e^{-mL})$ terms and keep only the $O(L^{-1})$ polynomial volume dependence. The quantization of the momenta is given by the BY equations

$$Q_j \equiv p(\theta_j)L - i \sum_{k:k \neq j} \log S(\theta_j - \theta_k) = 2\pi I_j \quad (12)$$

An n -particle state is labeled by the integers I_j , which can be traded for the momenta: $|I_1, \dots, I_n\rangle \equiv |\theta_1, \dots, \theta_n\rangle_L$. These states are normalized to Kronecker delta functions $\langle I' | I \rangle = \prod_j \delta_{I'_j, I_j}$. Since two point functions in finite and infinite volume are equal up to exponentially small $O(e^{-mL})$ terms, the finite and infinite volume form factors differ only in the normalization of states [16]. In particular, this implies the non-diagonal finite volume form factor formula

$$\langle \theta'_1, \dots, \theta'_m | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle_L = \frac{F_{n+m}(\bar{\theta}'_1, \dots, \bar{\theta}'_m, \theta_n, \dots, \theta_1)}{\sqrt{\rho_n \rho'_m}} + O(e^{-mL}) \quad (13)$$

where the densities of states are defined through the Bethe Ansatz equation via

$$\rho_n = \det |Q_{ij}| \quad ; \quad Q_{ij} = \partial_i Q_j \equiv \frac{\partial Q_j}{\partial \theta_i} \quad (14)$$

The conjectured formula for diagonal form factors takes the form [18]:

$$\langle \theta_1, \dots, \theta_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle_L = \frac{\sum_{\alpha \cup \bar{\alpha}} F_{\alpha}^c \rho_{\bar{\alpha}}}{\rho_n} + O(e^{-mL}) \quad (15)$$

where the index set $I = \{1, \dots, n\}$ is split in all possible ways $I = \alpha \cup \bar{\alpha}$, $F_{\alpha}^c = F_{2k}^c(\theta_{\alpha_1}, \dots, \theta_{\alpha_k})$ with $|\alpha| = k$ and $\rho_{\bar{\alpha}}$ is the shorthand for $\rho_{n-k}(\theta_{\bar{\alpha}_1}, \dots, \theta_{\bar{\alpha}_{n-k}})$, which denotes the sub-determinant of the matrix, Q_{ij} , with indices only from $\bar{\alpha}$. There is an analogous expression in terms of the symmetric form factors [17]

$$\langle \theta_1, \dots, \theta_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle_L = \frac{\sum_{\alpha \cup \bar{\alpha}} F_{\alpha}^s \rho_{\bar{\alpha}}^s}{\rho_n} + O(e^{-mL}) \quad (16)$$

where now ρ_{α}^s is the density of states corresponding to the variables with labels in α . The equivalence of the two formulas was shown in [17]. Let us note that for $L = 0$ the sum reduces to one single term $\sum_{\alpha \cup \bar{\alpha}} F_{\alpha}^s \rho_{\bar{\alpha}}^s \rightarrow F_n^s$ as all other ρ^s factor vanish.

Let us spell out the details for two particles. The diagonal finite volume form factor up to exponential correction is

$$\langle \theta_1, \theta_2 | \mathcal{O} | \theta_2, \theta_1 \rangle_L = \frac{F_4^c(\theta_1, \theta_2) + \rho_1(\theta_1) F_2^c(\theta_2) + \rho_1(\theta_2) F_2^c(\theta_1) + \rho_2(\theta_1, \theta_2) F_0}{\rho_2(\theta_1, \theta_2)} \quad (17)$$

where

$$\rho_2(\theta_1, \theta_2) = \begin{vmatrix} E_1 L + \phi_{12} & -\phi_{12} \\ -\phi_{21} & E_2 L + \phi_{21} \end{vmatrix} \quad ; \quad \rho_1(\theta_i) = E_i L + \phi_{i3-i}$$

The analogous formula with the symmetric evaluation reads as

$$\langle \theta_1, \theta_2 | \mathcal{O} | \theta_2, \theta_1 \rangle_L = \frac{F_4^s(\theta_1, \theta_2) + \rho_1^s(\theta_1) F_2^s(\theta_2) + \rho_1^s(\theta_2) F_2^s(\theta_1) + \rho_2^s(\theta_1, \theta_2) F_0^s}{\rho_2(\theta_1, \theta_2)} \quad (18)$$

where

$$\rho_2^s(\theta_1, \theta_2) = \rho_2(\theta_1, \theta_2) \quad ; \quad \rho_1^s(\theta_i) = E_i L$$

3 The proof for diagonal large volume form factors

The idea of the proof follows from the large θ behaviour of the scattering matrix, namely $S(\theta) \rightarrow 1$, for $\theta \rightarrow \infty$. This also lies behind the cluster property of the form factors. Thus by taking the non-diagonal form factor $\langle \theta, \theta'_1, \dots, \theta'_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle_L$ and sending $\theta \rightarrow \infty$, the extra particle decouples and we can approach the diagonal form factor. This can be achieved by choosing the same quantization numbers for both the θ_j and θ'_j particles:

$$Q'_j \equiv p(\theta'_j)L - i \sum_{k:k \neq j} \log S(\theta'_j - \theta'_k) - i \log S(\theta'_j - \theta) = 2\pi I_j \quad (19)$$

Indeed, by sending (the quantization number of) θ to infinity the BA equations, Q'_j , reduce to the Bethe Ansatz equations, Q_j . This means that in the limit considered $\theta'_i = \theta_i + \varepsilon_i$ and ε_i goes to zero. In principle, ε_i depends on $\{\theta_i\}$ and on the way how θ goes to infinity.

For finite θ , the form factor is non-diagonal and we can use

$$\langle \theta, \theta'_1, \dots, \theta'_n | \mathcal{O} | \theta_n, \dots, \theta_1 \rangle_L = \frac{F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1)}{\sqrt{\rho'_{n+1} \rho_n}} + O(e^{-mL}) \quad (20)$$

The numerator is a finite quantity for any θ and has a finite $\theta \rightarrow \infty$ limit accordingly. We can see in the limit that $\rho'_{n+1}(\theta, \theta'_1, \dots, \theta'_n)$ goes to $\rho_1(\theta)\rho_n(\theta_1, \dots, \theta_n)$. Similarly, for the form factors $F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1)$ the cluster property guaranties the factorization $F_{2n}(\bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1)F_1(\bar{\theta})$, where additionally $\theta'_i \rightarrow \theta_i$. Actually the expression depends in the direction we take the limit in which all ε_i go to zero and our main task is to calculate this limit explicitly. Fortunately, the direction is dictated by the difference of the BA equations:

$$Q'_j - Q_j = E_j L \varepsilon_j + \sum_{k:k \neq j} \phi_{jk}(\varepsilon_j - \varepsilon_k) - \delta_j = \sum_k Q_{jk} \varepsilon_k - \delta_j = 0 \quad (21)$$

where we have used the notations:

$$E_j = \partial_j p(\theta_j) \quad ; \quad \phi_{jk} = \phi(\theta_j - \theta_k) = -i \partial_j \log S(\theta_j - \theta_k) \quad ; \quad \delta_j = i \log S(\theta_j - \theta) \quad (22)$$

Clearly δ_j s are small and so are the ε_j s. In the following we analyze the ε and δ dependence of the form factor $F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1)$. Similarly to the diagonal limit of form factors we can describe the δ and ε dependence by graphs. We claim that

$$F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_k, \theta_k, \dots, \theta_1) = \sum_{\text{allowed graphs, colorings}} F(\text{graph}) + O(\varepsilon_i, \delta) \quad (23)$$

where, additionally to the previous graphs in (8), we should allow the coloring of those vertices, which do not have any outgoing edge, i.e. they can be either black or white. For each black dot with label i we associate a factor $\frac{\delta_i}{\varepsilon_i}$. Note that in the $\theta \rightarrow \infty$ limit we will have an overall $F_1(\bar{\theta})$ factor, which we factor out.

Let us see how it works for $n = 1$: The single dot can be either black or white:



thus the two contributions are

$$F_3(\bar{\theta}, \bar{\theta}'_1, \theta_1) F_1(\bar{\theta})^{-1} = \frac{\delta_1}{\varepsilon_1} + F_2^c(\theta_1) + \dots \quad (24)$$

where ellipsis represents terms vanishing in the $\delta, \varepsilon \rightarrow 0$ limit. Let us show that $F_2^c(\theta_1)$ is not singular, i.e. the singularity of the lhs. is exactly $\frac{\delta_1}{\varepsilon_1}$. The kinematical residue equation tells us that

$$F_3(\bar{\theta}, \bar{\theta}'_1, \theta_1) = \frac{i}{\varepsilon_1} (1 - S(\theta'_1 - \theta + i\pi)) F_1(\bar{\theta}) + O(1) = \frac{\delta_1}{\varepsilon_1} F_1(\bar{\theta}) + O(1) \quad (25)$$

Thus, once the singularity is subtracted, we can safely take the $\varepsilon_1 \rightarrow 0$ and the $\delta \rightarrow 0$ limits leading to

$$\lim_{\delta, \varepsilon_1 \rightarrow 0} (F_3(\bar{\theta}, \bar{\theta}'_1, \theta_1) - \frac{\delta_1}{\varepsilon_1} F_1(\bar{\theta})) = F_2^c(\theta_1) F_1(\bar{\theta}) \quad (26)$$

where we used the cluster property of form factors and the fact that the two particle diagonal connected form factor is non-singular.

Now we adapt the proof in the induction step in (8) by noticing that the ε_n^{-1} singularity can come either from terms with only one outgoing edge or from being black. Thus the residue is

$$\frac{1}{\varepsilon_n} (\delta_n + \varepsilon_1 \phi_{1n} + \dots + \varepsilon_{n-1} \phi_{n-1n}) F_{2n-1}(\bar{\theta}, \bar{\theta}_1 + \varepsilon_1, \dots, \bar{\theta}_{n-1} + \varepsilon_{n-1}, \theta_{n-1}, \dots, \theta_1) \quad (27)$$

Let us calculate the analogous term from the kinematical residue axiom:

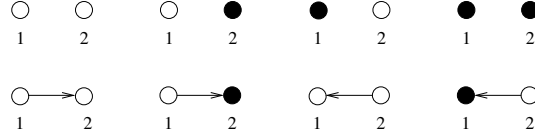
$$F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1) \rightarrow \frac{i}{\varepsilon_n} \left(1 - \frac{S(\theta'_n - \theta_{n-1}) \dots S(\theta'_n - \theta_1)}{S(\theta'_n - \theta'_{n-1}) \dots S(\theta'_n - \theta'_1)} \frac{1}{S(\theta'_n - \theta)} \right) \times F_{2n-1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_{n-1}, \theta_{n-1}, \dots, \theta_1) \quad (28)$$

The bracket can be expanded as

$$() = -i(\delta_n + \phi_{n-1}\varepsilon_{n-1} + \cdots + \phi_{n1}\varepsilon_1) \quad (29)$$

which completes the induction.

In particular, for two particles we have the following diagrams:



which lead to the formula

$$F_5(\bar{\theta}, \bar{\theta}'_1, \bar{\theta}'_2, \theta_2, \theta_1)F_1^{-1} = F_4^c(\theta_1, \theta_2) + \frac{\varepsilon_2}{\varepsilon_1}\phi_{21}\frac{\delta_2}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2}\phi_{12}\frac{\delta_1}{\varepsilon_1} + \frac{\delta_1}{\varepsilon_1}\frac{\delta_2}{\varepsilon_2} \quad (30)$$

$$+ \frac{\varepsilon_1}{\varepsilon_2}\phi_{12}F_2^c(\theta_1) + \frac{\delta_2}{\varepsilon_2}F_2^c(\theta_1) + \frac{\varepsilon_2}{\varepsilon_1}\phi_{21}F_2^c(\theta_2) + \frac{\delta_1}{\varepsilon_1}F_2^c(\theta_2)$$

It is interesting to check the coefficient of $F_2^c(\theta_1)$:

$$\frac{\varepsilon_1\phi_{12} + \delta_2}{\varepsilon_2} = E_2L + \phi_{21} = \rho_1(\theta_2) \quad (31)$$

where we used the BA equations. Similarly

$$\frac{\delta_1}{\varepsilon_1}\frac{\delta_2}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2}\phi_{12}\frac{\delta_1}{\varepsilon_1} + \frac{\varepsilon_2}{\varepsilon_1}\phi_{21}\frac{\delta_2}{\varepsilon_2} = \rho_2(\theta_1, \theta_2) \quad (32)$$

which leads to the sought for formula for $n = 2$:

$$F_5(\bar{\theta}, \bar{\theta}'_1, \bar{\theta}'_2, \theta_2, \theta_1)F_1^{-1} = F_4^c(\theta_1, \theta_2) + \rho_1(\theta_2)F_2^c(\theta_1) + \rho_1(\theta_1)F_2^c(\theta_2) + \rho_2(\theta_1, \theta_2) \quad (33)$$

In the following we prove the form of the diagonal form factors in the general case by induction. First we notice that once we use the BA equations to express δ_i in terms of ε_k then all denominators of ε_s disappear. Focus on ε_n^{-1} and observe that

$$\delta_n + \varepsilon_1\phi_{1n} + \cdots + \varepsilon_{n-1}\phi_{n-1n} = \varepsilon_n(E_nL + \phi_{n-1n} + \cdots + \phi_{1n}) \quad (34)$$

This implies that the diagonal finite volume form factor is a polynomial in L and linear in each E_kL . We first check the $L = 0$ piece and then calculate the derivative wrt. E_nL as the full expression is symmetric in all variables. Note that the would be singular term in ε_n at $L = 0$:

$$\frac{1}{\varepsilon_n}\varepsilon_n(E_nL + \phi_{n-1n} + \cdots + \phi_{1n})|_{L=0} = \frac{1}{\varepsilon}(\varepsilon\phi_{n-1n} + \cdots + \varepsilon\phi_{1n}) \quad (35)$$

is exactly the same we would obtain if we had calculated the diagonal limit of the form factor in the symmetric evaluation, i.e. for $L = 0$ we obtain the symmetric

n -particle form factor. We now check the linear term in $E_n L$. In doing so we differentiate the expression (23) wrt. $E_n L$:

$$\partial_{E_n L} F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1) = F_{2n-1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_{n-1}, \theta_{n-1}, \dots, \theta_1) \quad (36)$$

since the term $E_n L$ can come only through the singularity at $\varepsilon_n = 0$. Note that on the rhs. θ_k satisfies the original BA and not the one where θ_n is missing. Let us now take a look at the expression we would like to prove:

$$F_{2n+1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_n, \theta_n, \dots, \theta_1) F_1^{-1} = \sum_{\alpha \cup \bar{\alpha} = I} F_\alpha^c \rho_{\bar{\alpha}} = \sum_{\alpha \cup \bar{\alpha} = I} F_\alpha^s \rho_{\bar{\alpha}}^s \quad (37)$$

where $I = \{1, \dots, n\}$. Clearly the rhs. is also a polynomial in L , which is linear in each $E_k L$. To finish the proof, we note that the $L = 0$ constant part of the rhs. is the symmetric form factor. Using that $\partial_{E_n L} \rho_\alpha = \rho_{\alpha \setminus \{n\}}$ if $n \in \alpha$ and 0 otherwise we can see that

$$\partial_{E_n L} \sum_{\alpha \cup \bar{\alpha} = I} F_\alpha^c \rho_{\bar{\alpha}} = \sum_{\beta \cup \bar{\beta} = I \setminus \{n\}} F_\beta^c \rho_{\bar{\beta}} = F_{2n-1}(\bar{\theta}, \bar{\theta}'_1, \dots, \bar{\theta}'_{n-1}, \theta_{n-1}, \dots, \theta_1) F_1^{-1} \quad (38)$$

by the induction hypothesis, which completes the proof.

4 Conclusion

In this paper we proved the large volume expression for the diagonal form factors by taking carefully the limit of a nondiagonal form factor. Our result completes the proof of the LM formula, which describes exactly the one-point function in a finite volume.

Diagonal finite volume form factors are relevant in the AdS/CFT correspondence as they conjectured to describe the Heavy-Heavy-Light (HHL) type three point functions of the maximally supersymmetric 4D gauge theory [3]. This conjecture was first proved at weak coupling [6] then at strong coupling [2], finally for all couplings in [8, 7]. We have profited from all of these proofs in and used them in the present paper.

There is a natural extension of our results for diagonal form factors in non-diagonal theories. Clearly the same idea of adding one more particle and sending its rapidity to infinity can be applied there too and we have an ongoing research into this direction.

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References

1. Babujian, H., Karowski, M.: Towards the construction of Wightman functions of integrable quantum field theories. *Int. J. Mod. Phys. A* **19S2**, 34–49 (2004). DOI 10.1142/S0217751X04020294
2. Bajnok, Z., Janik, R.A.: Classical limit of diagonal form factors and HHL correlators. *JHEP* **01**, 063 (2017). DOI 10.1007/JHEP01(2017)063
3. Bajnok, Z., Janik, R.A., Wereszczynski, A.: HHL correlators, orbit averaging and form factors. *JHEP* **09**, 050 (2014). DOI 10.1007/JHEP09(2014)050
4. Dorey, P.: Exact S matrices pp. 85–125 (1996)
5. Dorey, P., Tateo, R.: Excited states by analytic continuation of TBA equations. *Nucl. Phys. B* **482**, 639–659 (1996). DOI 10.1016/S0550-3213(96)00516-0
6. Hollo, L., Jiang, Y., Petrovskii, A.: Diagonal Form Factors and Heavy-Heavy-Light Three-Point Functions at Weak Coupling. *JHEP* **09**, 125 (2015). DOI 10.1007/JHEP09(2015)125
7. Jiang, Y.: Diagonal Form Factors and Hexagon Form Factors II. Non-BPS Light Operator. *JHEP* **01**, 021 (2017). DOI 10.1007/JHEP01(2017)021
8. Jiang, Y., Petrovskii, A.: Diagonal form factors and hexagon form factors. *JHEP* **07**, 120 (2016). DOI 10.1007/JHEP07(2016)120
9. Leclair, A., Mussardo, G.: Finite temperature correlation functions in integrable QFT. *Nucl. Phys. B* **552**, 624–642 (1999). DOI 10.1016/S0550-3213(99)00280-1
10. Luscher, M.: Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States. *Commun. Math. Phys.* **104**, 177 (1986). DOI 10.1007/BF01211589
11. Luscher, M.: Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States. *Commun. Math. Phys.* **105**, 153–188 (1986). DOI 10.1007/BF01211097
12. Mussardo, G.: *Statistical field theory*. Oxford Univ. Press, New York, NY (2010)
13. Pozsgay, B.: Mean values of local operators in highly excited Bethe states. *J. Stat. Mech.* **1101**, P01,011 (2011). DOI 10.1088/1742-5468/2011/01/P01011
14. Pozsgay, B.: Form factor approach to diagonal finite volume matrix elements in Integrable QFT. *JHEP* **07**, 157 (2013). DOI 10.1007/JHEP07(2013)157
15. Pozsgay, B., Szecsenyi, I.M., Takacs, G.: Exact finite volume expectation values of local operators in excited states. *JHEP* **04**, 023 (2015). DOI 10.1007/JHEP04(2015)023
16. Pozsgay, B., Takacs, G.: Form-factors in finite volume I: Form-factor bootstrap and truncated conformal space. *Nucl. Phys. B* **788**, 167–208 (2008). DOI 10.1016/j.nuclphysb.2007.06.027
17. Pozsgay, B., Takacs, G.: Form factors in finite volume. II. Disconnected terms and finite temperature correlators. *Nucl. Phys. B* **788**, 209–251 (2008). DOI 10.1016/j.nuclphysb.2007.07.008
18. Saleur, H.: A Comment on finite temperature correlations in integrable QFT. *Nucl. Phys. B* **567**, 602–610 (2000). DOI 10.1016/S0550-3213(99)00665-3
19. Smirnov, F.: Form-factors in completely integrable models of quantum field theory. *Adv.Ser.Math.Phys.* **14**, 1–208 (1992)
20. Zamolodchikov, A.B.: Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models. *Nucl. Phys. B* **342**, 695–720 (1990). DOI 10.1016/0550-3213(90)90333-9
21. Zamolodchikov, A.B., Zamolodchikov, A.B.: Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. *Annals Phys.* **120**, 253–291 (1979). DOI 10.1016/0003-4916(79)90391-9