Remarks on $A_n^{(1)}$ face weights

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Abstract Elementary proofs are presented for the factorization of the elliptic Boltzmann weights of the $A_n^{(1)}$ face model, and for the sum-to-1 property in the trigonometric limit, at a special point of the spectral parameter. They generalize recent results obtained in the context of the corresponding trigonometric vertex model.

1 Introduction

In the recent work [8], the quantum R matrix for the symmetric tensor representation of the Drinfeld-Jimbo quantum affine algebra $U_q(A_n^{(1)})$ was revisited. A new factorized formula at a special value of the spectral parameter and a certain sum rule called sum-to-1 were established. These properties have led to vertex models that can be interpreted as integrable Markov processes on one-dimensional lattice including several examples studied earlier [7, Fig.1,2]. In this note we report analogous properties of the Boltzmann weights for yet another class of solvable lattice models known as IRF (interaction round face) models [2] or face models for short. More specifically, we consider the elliptic fusion $A_n^{(1)}$ face model corresponding to the symmetric tensor representation [6, 5]. For $n = 1$, it reduces to [1] and [4] when the fusion degree is 1 and general, respectively. There are restricted and unrestricted versions of the model. The trigonometric case of the latter reduces to the $U_q(A_n^{(1)})$ vertex model when the site variables tend to infinity. See Proposition 1. In this sense Theorem 1 and Theorem 2 given below, which are concerned with the unrestricted version, provide generalizations of [8, Th.2] and [8, eq.(30)] so as to include finite site variables (and also to the elliptic case in the former). In Section 3 we will also

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comment on the restricted version and difficulties to associate integrable stochastic models.

2 Results

Let $\theta_1(u) = \theta_1(u,p) = 2p^{\frac{1}{4}}\sin\pi u\prod_{k=1}^{\infty}(1-2p^{2k}\cos 2\pi u + p^{4k})(1-p^{2k})$ be one of the Jacobi theta function ($|p|<1$) enjoying the quasi-periodicity

$$\theta_1(u+1;e^{\pi i\tau}) = -\theta_1(u;e^{\pi i\tau}), \qquad \theta_1(u+\tau;e^{\pi i\tau}) = -e^{-\pi i\tau - 2\pi i u}\theta_1(u;e^{\pi i\tau}), \quad (1)$$

where $\mathrm{Im}\tau > 0$. We set

$$[u] = \theta_1(\frac{u}{L},p), \quad [u]_k = [u][u-1]\cdots[u-k+1], \qquad \begin{bmatrix} u \\ k \end{bmatrix} = \frac{[u]_k}{[k]_k} \qquad (k \in \mathbb{Z}_{\geq 0}), \quad (2)$$

with a nonzero parameter L. These are elliptic analogue of the q-factorial and the q-binomial:

$$(z)_m = (z;q)_m = \prod_{i=0}^{m-1}(1-zq^i), \qquad \binom{m}{l}_q = \frac{(q)_m}{(q)_l(q)_{m-l}}.$$

For $\alpha = (\alpha_1,\ldots,\alpha_k)$ with any k we write $|\alpha| = \alpha_1 + \cdots + \alpha_k$. The relation $\beta \geq \gamma$ or equivalently $\gamma \leq \beta$ means $\beta_i \geq \gamma_i$ for all i.

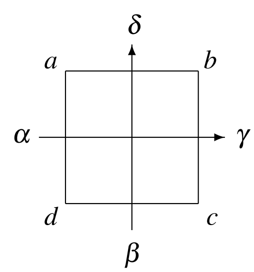
We take the set of local states as $\tilde{\mathscr{P}} = \eta + \mathbb{Z}^{n+1}$ with a generic $\eta \in \mathbb{C}^{n+1}$. Given positive integers l and m, let $a,b,c,d \in \tilde{\mathscr{P}}$ be the elements such that

$$\alpha = d - a \in B_l, \quad \beta = c - d \in B_m, \quad \gamma = c - b \in B_l, \quad \delta = b - a \in B_m, \quad (3)$$

where B_m is defined by

$$B_m = \{\alpha = (\alpha_1,\ldots,\alpha_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \mid |\alpha| = m\}. \quad (4)$$

The relations (3) imply $\alpha + \beta = \gamma + \delta$. The situation is summarized as



To the above configuration round a face we assign a function of the spectral parameter u called Boltzmann weight. Its unnormalized version, denoted by $\overline{W}_{l,m}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\Big| u\right)$, is constructed from the $l=1$ case as follows:

$$\overline{W}_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\Big| u\right) = \sum \prod_{i=0}^{l-1} \overline{W}_{1,m}\left(\begin{matrix} a^{(i)} & b^{(i)} \\ a^{(i+1)} & b^{(i+1)} \end{matrix}\Big| u-i\right), \tag{5}$$

$$\overline{W}_{1,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\Big| u\right) = \frac{[u+b_\nu - a_\mu]\prod_{j=1\,(j\neq\mu)}^{n+1}[b_\nu - a_j + 1]}{\prod_{j=1}^{n+1}[c_\nu - b_j]} \quad (d = a+\mathbf{e}_\mu,\ c = b+\mathbf{e}_\nu),$$

where $\mathbf{e}_i = (0,\dots,0,\overset{ith}{1},0,\dots,0)$. In (5), $a^{(0)},\dots,a^{(l)} \in \tilde{\mathscr{P}}$ is a path form $a^{(0)} = a$ to $a^{(l)} = d$ such that $a^{(i+1)} - a^{(i)} \in B_1\,(0 \le i < l)$. The sum is taken over $b^{(1)},\dots,b^{(l-1)} \in \tilde{\mathscr{P}}$ satisfying the conditions $b^{(i+1)} - b^{(i)} \in B_1\,(0 \le i < l)$ with $b^{(0)} = b$ and $b^{(l)} = c$. It is independent of the choice of $a^{(1)},\dots,a^{(l-1)}$ (cf. [4, Fig.2.4]). We understand that $\overline{W}_{l,m}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\Big| u\right) = 0$ unless (3) is satisfied for some $\alpha,\beta,\gamma,\delta$.

The normalized weight is defined by

$$W_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\Big| u\right) = \overline{W}_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\Big| u\right)\frac{[1]^l}{[l]_l}\begin{bmatrix} m \\ l \end{bmatrix}^{-1}. \tag{6}$$

It satisfies [5] the (unrestricted) star-triangle relation (or dynamical Yang-Baxter equation) [2]:

$$\begin{aligned} &\sum_g W_{k,m}\left(\begin{matrix} a & b \\ f & g \end{matrix}\Big| u\right) W_{l,m}\left(\begin{matrix} f & g \\ e & d \end{matrix}\Big| v\right) W_{k,l}\left(\begin{matrix} b & c \\ g & d \end{matrix}\Big| u-v\right) \\ &= \sum_g W_{k,l}\left(\begin{matrix} a & g \\ f & e \end{matrix}\Big| u-v\right) W_{l,m}\left(\begin{matrix} a & b \\ g & c \end{matrix}\Big| v\right) W_{k,m}\left(\begin{matrix} g & c \\ e & d \end{matrix}\Big| u\right), \end{aligned} \tag{7}$$

where the sum extends over $g \in \tilde{\mathscr{P}}$ giving nonzero weights. Under the same setting (3) as in (6), we introduce the product

$$S_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\right) = \begin{bmatrix} m \\ l \end{bmatrix}^{-1} \prod_{1\le i,j\le n+1} \frac{[c_i - d_j]_{c_i - b_i}}{[c_i - b_j]_{c_i - b_i}}. \tag{8}$$

Note that $S_{l,m}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right) = 0$ unless $d \le b$ because of the factor $\prod_{i=1}^{n+1}[c_i - d_i]_{c_i - b_i}$. The following result giving an explicit factorized formula of the weight $W_{l,m}$ at special value of the spectral parameter is the elliptic face model analogue of [8, Th.2].

Theorem 1. *If $l \le m$, the following equality is valid:*

$$W_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\Big| u=0\right) = S_{l,m}\left(\begin{matrix} a & b \\ d & c \end{matrix}\right). \tag{9}$$

Proof. We are to show

$$\overline{W}_{l,m}\begin{pmatrix} a & b \\ d & c \end{pmatrix}\Big|0\Big) = \frac{[l]_l}{[1]^l}\prod_{i,j}\frac{[c_i-d_j]_{c_i-b_i}}{[c_i-b_j]_{c_i-b_i}}. \tag{10}$$

Here and in what follows unless otherwise stated, the sums and products are taken always over $1,\dots,n+1$ under the condition (if any) written explicitly. We invoke the induction on l. It is straightforward to check (10) for $l=1$. By the definition (5) the $l+1$ case is expressed as

$$\overline{W}_{l+1,m}\Big(\begin{matrix} a & b \\ d & c \end{matrix}\Big|0\Big) = \sum_\nu \overline{W}_{l,m}\Big(\begin{matrix} a & b \\ d' & c' \end{matrix}\Big|0\Big)\overline{W}_{1,m}\Big(\begin{matrix} d' & c' \\ d & c \end{matrix}\Big|-l\Big) \qquad (d'=d-\mathbf{e}_\mu, c'=c-\mathbf{e}_\nu)$$

for some fixed $\mu\in[1,n+1]$. Due to the induction hypothesis on $\overline{W}_{l,m}$, the equality to be shown becomes

$$\begin{aligned}&\sum_\nu \frac{[l]_l}{[1]^l}\Big(\prod_{i,j}\frac{[c'_i-d'_j]_{c'_i-b_i}}{[c'_i-b_j]_{c'_i-b_i}}\Big)\frac{[-l+c'_\nu-d'_\mu]\prod_{k\neq\mu}[c'_\nu-d'_k+1]}{\prod_k[c_\nu-c'_k]}\\ &=\frac{[l+1]_{l+1}}{[1]^{l+1}}\prod_{i,j}\frac{[c_i-d_j]_{c_i-b_i}}{[c_i-b_j]_{c_i-b_i}}.\end{aligned} \tag{11}$$

After removing common factors using $c'_i=c_i-\delta_{i\nu}, d'_i=d_i-\delta_{i\mu}$, one finds that (11) is equivalent to

$$\sum_\nu [c_\nu-d_\mu-l]\prod_{i\neq\nu}\frac{[c_i-d_\mu+1]}{[c_\nu-c_i]}\prod_j[c_\nu-b_j]=[l+1]\prod_i[b_i-d_\mu+1]$$

with l determined by $l+1=\sum_j(c_j-b_j)$. One can eliminate d_μ and rescale the variables by $(b_j,c_j)\to(Lb_j+d_\mu, Lc_j+d_\mu)$ for all j. The resulting equality follows from Lemma 1.

Lemma 1. *Let $b_1,\dots,b_n,c_1,\dots,c_n\in\mathbb{C}$ be generic and set $s=\sum_{i=1}^n(c_i-b_i)$. Then for any $n\in\mathbb{Z}_{\geq 1}$ the following identity holds:*

$$\sum_{i=1}^n\theta_1(z+c_i-s)\prod_{j=1\,(j\neq i)}^n\frac{\theta_1(z+c_j)}{\theta_1(c_i-c_j)}\prod_{j=1}^n\theta_1(c_i-b_j)=\theta_1(s)\prod_{i=1}^n\theta_1(z+b_i).$$

Proof. Denote the LHS − RHS by $f(z)$. From (1) we see that $f(z)$ satisfies (12) with $B=\frac{n}{2}$, $A_1=\frac{n(1+\tau)}{2}+\sum_{j=1}^n b_j$ and $A_2=n$. Moreover it is easily checked that $f(z)$ possesses zeros at $z=-c_1,\dots,-c_n$. Therefore Lemma 2 claims $-(c_1+\cdots+c_n)-(B\tau+\frac{1}{2}A_2-A_1)\equiv 0 \bmod \mathbb{Z}+\mathbb{Z}\tau$. But this gives $s\equiv 0$ which is a contradiction since b_j,c_j can be arbitrary. Therefore $f(z)$ must vanish identically.

Lemma 2. *Let* $\mathrm{Im}\tau>0$. *Suppose an entire function $f(z)\not\equiv 0$ satisfies the quasi-periodicity*

$$f(z+1)=e^{-2\pi iB}f(z), \qquad f(z+\tau)=e^{-2\pi i(A_1+A_2z)}f(z). \tag{12}$$

Then $A_2\in\mathbb{Z}_{\geq 0}$ holds and $f(z)$ has exactly A_2 zeros $z_1,\dots,z_{A_2} \mod \mathbb{Z}+\mathbb{Z}\tau$. Moreover $z_1+\cdots+z_{A_2}\equiv B\tau+\frac{1}{2}A_2-A_1 \mod \mathbb{Z}+\mathbb{Z}\tau$ holds.

Proof. Let C be a period rectangle $(\xi,\xi+1,\xi+1+\tau,\xi+\tau)$ on which there is no zero of $f(z)$. From the Cauchy theorem the number of zeros of $f(z)$ in C is equal to $\int_C \frac{f'(z)}{f(z)}\frac{dz}{2\pi i}$. Calculating the integral by using (12) one gets A_2. The latter assertion can be shown similarly by considering the integral $\int_C \frac{zf'(z)}{f(z)}\frac{dz}{2\pi i}$.

From Theorem 1 and (7) it follows that $S_{l,m}\begin{pmatrix} a & b\\ d & c\end{pmatrix}$ also satisfies the (unrestricted) star-triangle relation (7) without spectral parameter. The discrepancy of the factorizing points $u=0$ in (9) and "$u=l-m$" in [8, Th.2] is merely due to a conventional difference in defining the face and the vertex weights.

Since (6) and (8) are homogeneous of degree 0 in the symbol $[\cdots]$, the trigonometric limit $p\to 0$ may be understood as replacing (2) by $[u]=q^{u/2}-q^{-u/2}$ with generic $q=\exp\frac{2\pi i}{L}$. Under this prescription the elliptic binomial $\begin{bmatrix} m\\ l\end{bmatrix}$ from (2) is replaced by $q^{l(l-m)/2}\binom{m}{l}_q$, therefore the trigonometric limit of (8) becomes

$$S_{l,m}\begin{pmatrix} a & b\\ d & c\end{pmatrix}_{\mathrm{trig}}=\binom{m}{l}_q^{-1}\prod_{1\leq i,j\leq n+1}\frac{(q^{b_i-d_j+1})_{c_i-b_i}}{(q^{b_i-b_j+1})_{c_i-b_i}}. \tag{13}$$

The following result is a trigonometric face model analogue of [8, Th.6].

Theorem 2. *Suppose $l\leq m$. Then the sum-to-1 holds in the trigonometric case:*

$$\sum_b S_{l,m}\begin{pmatrix} a & b\\ d & c\end{pmatrix}_{\mathrm{trig}}=1, \tag{14}$$

where the sum runs over those b satisfying $c-d\in B_m$ and $d-a\in B_l$.

Proof. The relation (14) is equivalent to

$$\binom{m}{l}_q=\sum_{\gamma\in B_l,\gamma\leq\beta}\ \prod_{1\leq i,j\leq n+1}\frac{(q^{c_{ij}-\gamma_i+\beta_j+1})_{\gamma_i}}{(q^{c_{ij}-\gamma_i+\gamma_j+1})_{\gamma_i}} \qquad (c_{ij}=c_i-c_j) \tag{15}$$

for any fixed $\beta=(\beta_1,\dots,\beta_{n+1})\in B_m$, $l\leq m$ and the parameters $c_1,\dots,c_{n+1}$, where the sum is taken over $\gamma\in B_l$ (4) under the constraint $\gamma\leq\beta$. In fact we are going to show

$$\frac{(w_1^{-1}\dots w_n^{-1}q^{-l+1})_l}{(q)_l}=\sum_{|\gamma|=l}\ \prod_{1\leq i,j\leq n}\frac{\left(q^{-\gamma_i+1}z_i/(z_jw_j)\right)_{\gamma_i}}{\left(q^{\gamma_j-\gamma_i+1}z_i/z_j\right)_{\gamma_i}} \qquad (l\in\mathbb{Z}_{\geq 0}), \tag{16}$$

where the sum is over $\gamma\in\mathbb{Z}_{\geq 0}^n$ such that $|\gamma|=l$, and $w_1,\dots,w_n,z_1,\dots,z_n$ are arbitrary parameters. The relation (15) is deduced from $(16)|_{n\to n+1}$ by setting

$z_i = q^{c_i}, w_i = q^{-\beta_i}$ and specializing β_i's to nonnegative integers. In particular, the constraint $\gamma \le \beta$ automatically arises from the $i = j$ factor $\prod_{i=1}^{n}(q^{-\gamma_i+1+\beta_i})_{\gamma_i}$ in the numerator. To show (16) we rewrite it slightly as

$$q^{\frac{l^2}{2}}\frac{(w_1\dots w_n)_l}{(q)_l} = \sum_{|\gamma|=l}\prod_{i=1}^{n} q^{\frac{\gamma_i^2}{2}}\frac{(w_i)_{\gamma_i}}{(q)_{\gamma_i}}\prod_{1\le i\ne j\le n}\frac{(z_jw_j/z_i)_{\gamma_i}}{(q^{-\gamma_j}z_j/z_i)_{\gamma_i}}. \qquad (17)$$

Denote the RHS by $F_n(w_1,\dots,w_n|z_1,\dots,z_n)$. We will suppress a part of the arguments when they are kept unchanged in the formulas. It is easy to see

$$F_n(w_1,w_2|z_1,z_2) = F(w_2,w_1|z_2,z_1) = F_n(\frac{z_2w_2}{z_1},\frac{z_1w_1}{z_2}|z_1,z_2).$$

Thus the coefficients in the expansion $F_n(w_1,w_2|z_1,z_2) = \sum_{0\le i,j\le l} C_{i,j}(z_1,z_2)w_1^iw_2^j$ are rational functions in $z_1,\dots,z_n$ obeying $C_{i,j}(z_1,z_2) = C_{j,i}(z_2,z_1) = \left(\frac{z_1}{z_2}\right)^{i-j}C_{j,i}(z_1,z_2)$. On the other hand from the explicit formula (17), one also finds that any $C_{i,j}(z_1,z_2)$ remains finite in the either limit $\frac{z_1}{z_2},\frac{z_2}{z_i}\to\infty$ or $\frac{z_1}{z_2},\frac{z_2}{z_i}\to 0$ for $i\ge 3$. It follows that $C_{i,j}(z_1,z_2) = 0$ unless $i = j$, hence

$$F_n(w_1,w_2,\dots,w_n|z_1,\dots,z_n) = F_n(1,w_1w_2,w_3,\dots,w_n|z_1,\dots,z_n).$$

Moreover it is easily seen

$$F_n(1,w_1w_2,w_3,\dots,w_n|z_1,z_2,\dots,z_n) = F_{n-1}(w_1w_2,w_3,\dots,w_n|z_2,\dots,z_n).$$

Repeating this we reach $F_1(w_1\cdots w_n|z_n)$ giving the LHS of (17).

We note that the sum-to-1 (14) does not hold in the elliptic case. Remember that our local states are taken from $\tilde{\mathscr{P}} = \eta + \mathbb{Z}^{n+1}$ with a generic $\eta \in \mathbb{C}^{n+1}$. So we set $a = \eta + \tilde{a}$ with $\tilde{a} \in \mathbb{Z}^{n+1}$ etc in (4), and assume that it is valid also for $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$. It is easy to check

Proposition 1. *Assume $l \le m$ and $|q| < 1$. Then the following equality holds:*

$$\lim_{\eta\to\infty} S_{l,m}\begin{pmatrix}\eta+\tilde{a} & \eta+\tilde{b}\\ \eta+\tilde{d} & \eta+\tilde{c}\end{pmatrix}_{\mathrm{trig}} = q^{\sum_{i<j}(\beta_i-\gamma_i)\gamma_j}\binom{m}{l}_q^{-1}\prod_{i=1}^{n+1}\binom{\beta_i}{\gamma_i}_q, \qquad (18)$$

where the limit means $\eta_i - \eta_{i+1} \to \infty$ for all $1 \le i \le n$, and the RHS is zero unless $0 \le \gamma_i \le \beta_i, \forall i$.

The limit reduces the unrestricted trigonometric $A_n^{(1)}$ face model to the vertex model at a special value of the spectral parameter in the sense that the RHS of $(18)|_{q\to q^2}$ reproduces [8, eq.(23)] that was obtained as the special value of the quantum R matrix associated with the symmetric tensor representation of $U_q(A_n^{(1)})$.

3 Discussion

Since the weights $W_{l,m}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\Big| u\right)$ remain unchanged by shifting $a,b,c,d \in \tilde{\mathscr{P}}$ by $\mathrm{const}\cdot(1,\dots,1)$, we regard them as elements from $\mathscr{P} := \tilde{\mathscr{P}}/\mathbb{C}(1,\dots,1)$ in the sequel. Given $l, m_1,\dots,m_M \in \mathbb{Z}_{\geq 1}$ and $u, w_1,\dots,w_M \in \mathbb{C}$, the transfer matrix $T_l(u) = T_l\left(u \,\Big|\, \begin{smallmatrix} m_1,\dots,m_M \\ w_1,\dots,w_M \end{smallmatrix}\right)$ of the unrestricted $A_n^{(1)}$ face model with periodic boundary condition is a linear map on the space of independent row configurations on length M row $\bigoplus \mathbb{C}|a^{(1)},\dots a^{(M)}\rangle$ where the sum is taken over $a^{(1)},\dots a^{(M)} \in \mathscr{P}$ such that $a^{(i+1)} - a^{(i)} \in B_{m_i}\,(a^{(M+1)} = a^{(1)})$. Its action is specified as $T_l(u)|b^{(1)},\dots b^{(M)}\rangle = \sum_{a^{(1)},\dots a^{(M)}} T_l(u)^{a^{(1)},\dots a^{(M)}}_{b^{(1)},\dots b^{(M)}} |a^{(1)},\dots a^{(M)}\rangle$ in terms of the matrix elements

$$T_l(u)^{a^{(1)},\dots a^{(M)}}_{b^{(1)},\dots b^{(M)}} = \prod_{i=1}^{M} W_{l,m_i}\left(\begin{matrix} a^{(i)} & a^{(i+1)} \\ b^{(i)} & b^{(i+1)} \end{matrix}\Big| u - w_i\right) \qquad (a^{(M+1)} = a^{(1)}, b^{(M+1)} = b^{(1)}). \tag{19}$$

Theorem 1 tells that $S_l := T_l(u)_{u=w_1=\cdots=w_M}$ has a simple factorized matrix elements. We write its elements as $S_{l,b^{(1)},\dots b^{(M)}}^{\;a^{(1)},\dots a^{(M)}}$. The star-triangle relation (7) implies the commutativity $[T_l(u), T_{l'}(u')] = [S_l, S_{l'}] = 0$.

Let us consider whether $X = T_l(u)$ or S_l admits an interpretation as a Markov matrix of a discrete time stochastic process. The related issue was treated in [3] for $n = 1$ and mainly when $\min(l, m_1,\dots,m_M) = 1$. One needs (i) sum-to-1 property $\sum_{a^{(1)},\dots a^{(M)}} X^{a^{(1)},\dots a^{(M)}}_{b^{(1)},\dots b^{(M)}} = 1$ and (ii) nonnegativity $\forall X^{a^{(1)},\dots a^{(M)}}_{b^{(1)},\dots b^{(M)}} \geq 0$. We concentrate on the trigonometric case in what follows. From Theorem 1 and the fact that $S_{l,m}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_{\mathrm{trig}}$ in (13) is independent of a, (i) indeed holds for S_l. On the other hand (13) also indicates that (ii) is not valid in general without confining the site variables in a certain range. A typical such prescription is *restriction* [4, 6, 5], where one takes $L = \ell + n + 1$ in (2) with some $\ell \in \mathbb{Z}_{\geq 1}$ and lets the site variables range over the finite set of level ℓ dominant integral weights $\{(L + a_{n+1} - a_1 - 1)\Lambda_0 + \sum_{i=1}^{n}(a_i - a_{i+1} - 1)\Lambda_i \mid L + a_{n+1} > a_1 > \cdots > a_{n+1}, a_i - a_j \in \mathbb{Z}\}$. They are to obey a stronger adjacency condition [5, p546, (c-2)] than (3) which is actually the fusion rule of the WZW conformal field theory. (The formal limit $\ell \to \infty$ still works to restrict the site variables to the positive Weyl chamber and is called "classically restricted".) Then the star-triangle relation remains valid by virtue of nontrivial cancellation of unwanted terms. However, discarding the contribution to the sum (14) from those b not satisfying the adjacency condition spoils the sum-to-1 property. For example when $(n,l,m) = (2,1,2), a = (2,1,0), c = (4,2,0), d = (3,1,0)$ and ℓ is sufficiently large, the unrestricted sum (14) consists of two terms $S_{l,m}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)_{\mathrm{trig}} = \binom{2}{1}_q^{-1} \frac{(q^{-1};q)_1}{(q^{-2};q)_1}$ for $b = (4,1,0)$ and $S_{l,m}\left(\begin{smallmatrix} a & b' \\ d & c \end{smallmatrix}\right)_{\mathrm{trig}} = \binom{2}{1}_q^{-1} \frac{(q^3;q)_1}{(q^2;q)_1}$ for $b' = (3,2,0)$ summing up to 1, but b' must be discarded in the restricted case since $a \overset{m=2}{\Rightarrow} b'$ [5, (c-2)] does not hold.

Thus we see that in order to satisfy (i) and (ii) simultaneously one needs to resort to a construction different from the restriction.

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