Optimal transport with discrete mean field interaction

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Abstract In this note, we summarise some regularity results recently obtained for an optimal transport problem where the matter transported is either accelerated by an external force field, or self-interacting, at a given intermediate time.

1 Background

This note is a summary of an ongoing work [5]. The motivation comes from a previous work by the second author [6], where he studies the motion of a self-gravitating matter, classically described by the Euler-Poisson system. Letting ρ be the density of the matter, the gravitational field generated by a continuum of matter with density ρ is the gradient of a potential p linked to ρ by a Poisson coupling. The system is thus the following

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho v) = 0, \\ \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = -\rho\nabla p, \\ \Delta p = \rho. \end{cases} \tag{1}$$

A well known problem in cosmology, named the reconstruction problem, is to find a solution to (1) satisfying

$$\rho|_{t=0} = \rho_0, \qquad \rho|_{t=T} = \rho_T.$$

In [6], the reconstruction problem was formulated into a minimisation problem, minimising the action of the Lagrangian which is a convex functional. Through

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this variational formulation, the reconstruction problem becomes very similar to the time continuous formulation of the optimal transportation problem of Benamou and Brenier [1], and the existence, uniqueness of the minimiser was obtained by use of the Monge-Kantorovich duality. In the context of optimal transport as in [6], there holds $v = \nabla\phi$ for some potential ϕ, and the author obtained partial regularity results for ϕ and ρ, as well as the consistency of the minimiser with the solution of the Euler-Poisson system.

The optimal transport problem of [6] was formulated as finding minimisers of the action of the Lagrangian

$$I(\rho,v,p) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 + |\nabla p(t,x)|^2 dxdt, \tag{2}$$

over all ρ, p, v satisfying

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho v) &= 0, \\ \rho(0) = \rho_0,\ \ \rho(T) &= \rho_T, \\ \Delta p &= \rho, \end{aligned}$$

where $\mathbb{T}^d$ denotes the d-dimensional torus, as the study in [6] was performed in the space-periodic case.

In the work [5] we address the more general problem of finding minimisers for the action

$$I(\rho,v,p) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 + \mathscr{F}(\rho(t,x))dxdt, \tag{3}$$

for more general $\mathscr{F}$. The problem (2) falls in this class. In [3], Lee and McCann address the case where

$$\mathscr{F}(\rho) = -\int \rho(t,x)V(t,x)dx.$$

(Note that in the context of classical mechanics $\mathscr{F}$ would be the potential energy.) This Lagrangian corresponds to the case of a continuum of matter evolving in an external force field given by $\nabla V(t,x)$. We call this the non-interacting case for obvious reasons. This can be recast as a classic optimal transport problem, where the cost functional is given by

$$c(x,y) = \inf_{\substack{\gamma(0)=x,\gamma(T)=y \\ \gamma\in C^1([0,T],\mathbb{R}^d)}} \int_0^T \frac{1}{2}|\dot\gamma(t)|^2 - V(t,\gamma(t))dt. \tag{4}$$

For a small V satisfying some structure condition, they obtain that c satisfies the conditions found in [8] to ensure the regularity of the optimal map.

2 Time discretisation

In [5] we restrict ourselves to the case where the force field only acts at a single discrete time between 0 and T:

$$V(t,x) = \delta_{t=T/2}V(x).$$

We will call this case the "discrete" case. The minimisation problem therefore becomes

$$I(\rho,v) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2 dxdt + \int_{\mathbb{R}^d} \rho(T/2,x)Q(x)dx, \tag{5}$$

for some potential Q. This will allow to remove the smallness condition on V. Moreover, we will be able to extend our result to the mean-field case, where the force field is given by

$$\nabla V(x) = \int \rho(t,y)\nabla\kappa(x-y)dy. \tag{6}$$

This corresponds to the case where a particle located at x attracts or repels another particle located at y with a force equal to $\nabla\kappa(x-y)$. We will give a sufficient condition on κ to ensure a smooth transport map and intermediate density. Especially, we consider the gravitational case, which corresponds to the Coulomb kernel

$$\kappa(x-y) = \frac{c_d}{|x-y|^{d-2}},$$

that correponds to the potential energy

$$\begin{aligned}\mathscr{E}(t) = -\mathscr{F}(\rho(t)) &= -\frac{1}{2}\int \rho(t,x)\kappa(x-y)\rho(t,y)dxdy \\ &= -\frac{1}{2}\int \|\nabla p\|^2,\end{aligned}$$

where $\Delta p = \rho$.

One sees straight away that between time 0 and $T/2$ we are solving the usual optimal transport problem in its "Benamou-Brenier" formulation [1], as well as between $T/2$ and T. More generally, as done in [6], one can consider multiple-steps time discretisation, where the potential energy term contributes only at time

$$t_i = \frac{iT}{N}, \quad i = 1,\cdots,N-1.$$

Between two time steps, the problem will be an optimal transport problem as in [1, 2] and [9]. Then at each time step, the gravitational effect will be taken into account, and the velocity will be discontinuous. From a Lagrangian point of view, the velocity of each particle will therefore be a piecewise constant function with respect to time. Then letting the time step go to 0, one will eventually recover the time continuous problem.

3 Main results

Let us consider a two-step time discretisation in the interval $[0,T]$: At $t = T/2$, the velocity is changed by an amount equal to ∇Q, the gradient of a potential Q. The initial density ρ_0 is supported on a bounded domain $\Omega_0 \subset \mathbb{R}^d$, and the final density ρ_T is supported on a bounded domain $\Omega_T \subset \mathbb{R}^d$, satisfying the balance condition

$$\int_{\Omega_0} \rho_0(x)\,dx = \int_{\Omega_T} \rho_T(y)\,dy. \tag{7}$$

As is always the case in solving problems of the form (3), the velocity v is the gradient of a potential, and we let ϕ be the velocity potential at time 0, i.e. $v(0,x) = \nabla\phi(x)$. At time $t = T/2$, v will be changed into $v + \nabla Q$ and one can see that for an initial point $x \in \Omega_0$, the final point $y = \mathbf{m}(x) \in \Omega_T$ is given by

$$\mathbf{m}(x) = x + T\nabla\phi + \frac{T}{2}\nabla Q\left(x + \frac{T}{2}\nabla\phi\right).$$

By computing the determinant of the Jacobian $D\mathbf{m}$ and noting that $\mathbf{m}$ pushes forward ρ_0 to ρ_T, one can derive the equation for ϕ. To be specific, define a modified potential

$$\tilde{\phi}(x) := \frac{T}{2}\phi(x) + \frac{1}{2}|x|^2, \qquad \text{for } x \in \Omega_0. \tag{8}$$

It is readily seen [1, 2, 9] that the modified potential $\tilde{\phi}$ is a convex function. Since $\mathbf{m}_{\#}\rho_0 = \rho_T$, we obtain that $\tilde{\phi}$ satisfies a Monge-Ampère type equation

$$\det\left[D^2\tilde{\phi} - \left(D^2\tilde{Q}(\nabla\tilde{\phi})\right)^{-1}\right] = \left(\frac{1}{\det D^2\tilde{Q}(\nabla\tilde{\phi})}\right)\frac{\rho_0}{\rho_T \circ \mathbf{m}}, \tag{9}$$

where $\tilde{Q}$ is a modified potential given by

$$\tilde{Q}(z) := \frac{T}{2}Q(z) + |z|^2, \tag{10}$$

with an associated natural boundary condition

$$\mathbf{m}(\Omega_0) = \Omega_T. \tag{11}$$

For regularity of the solution $\tilde{\phi}$ to the boundary value problem (9) and (11) (equivalently that of ϕ), it is necessary to impose certain conditions on the potential energy function $\tilde{Q}$ (equivalently on Q) and the domains Ω_0, Ω_T. In [5] we assume that $\tilde{Q}$ satisfies the following conditions:

(H0) The function $\tilde{Q}$ is smooth enough, say at least C^4,
(H1) The function $\tilde{Q}$ is uniformly convex, namely $D^2\tilde{Q} \geq \varepsilon_0 I$ for some $\varepsilon_0 > 0$,
(H2) The function $\tilde{Q}$ satisfies that for all $\xi, \eta \in \mathbb{R}^d$ with $\xi \perp \eta$,

$$\sum_{i,j,k,l,p,q,r,s} \left(D^4_{ijrs}\tilde{Q} - 2\tilde{Q}^{pq} D^3_{ijp}\tilde{Q} D^3_{qrs}\tilde{Q}\right) \tilde{Q}^{rk}\tilde{Q}^{sl}\xi_k\xi_l\eta_i\eta_j \leq -\delta_0|\xi|^2|\eta|^2, \quad (12)$$

where $\{\tilde{Q}^{ij}\}$ is the inverse of $\{\tilde{Q}_{ij}\}$, and δ_0 is a positive constant. When $\delta_0 = 0$, we call it **(H2w)**, a weak version of (H2).

Note that conditions (H0) and (H1) imply that the inverse matrix $(D^2\tilde{Q})^{-1}$ exists, and ensure that equation (9) well defined. Condition (H2) is an analogue of the Ma-Trudinger-Wang condition [8] in optimal transportation, which is necessary for regularity results. We also use the notion of Q-convexity of domains as in [8].

Our first main result is the following

Theorem 1. *Let ϕ be the velocity potential in the reconstruction problem. Assume the gravitational function $\tilde{Q}$ satisfies conditions (H0), (H1) and (H2), Ω_T is Q-convex with respect to Ω_0. Assume that $\rho_T \geq c_0$ for some positive constant c_0, $\rho_0 \in L^p(\Omega_0)$ for some $p > \frac{d+1}{2}$, and the balance condition* (7) *is satisfied. Then, the velocity potential ϕ is $C^{1,\alpha}(\overline{\Omega}_0)$ for some $\alpha \in (0,1)$.*

If furthermore, Ω_0, Ω_T are C^4 smooth and uniformly Q-convex with respect to each other, $\rho_0 \in C^2(\overline{\Omega}_0), \rho_T \in C^2(\overline{\Omega}_T)$, then $\phi \in C^3(\overline{\Omega}_0)$, and higher regularity follows from the theory of linear elliptic equations. In particular, if $\tilde{Q}, \Omega_0, \Omega_T, \rho_0, \rho_T$ are C^∞, then the velocity potential $\phi \in C^\infty(\overline{\Omega}_0)$.

The proof of Theorem 1 is done by linking the time discretisation problem to a transport problem, where the key observation is that the cost function $c(x,y)$ is given by $\tilde{Q}^*(x+y)$, where $\tilde{Q}^*$ is the Legendre transform of the gravitational function $\tilde{Q}$. Under this formulation, the regularity then follows from the established theory of optimal transportation, see for example [7, 4, 8, 10] and references therein.

Our second main result is the following:

Theorem 2. *Assume that Q is given by*

$$Q(x) = \frac{1}{2}\int_{\Omega_{T/2}} \rho(T/2, y)\kappa(x-y)dy, \quad (13)$$

where $\Omega_{T/2} = (\mathrm{Id} + \frac{\mathrm{T}}{2}\nabla\phi)(\Omega_0)$ is the intermediate domain at $t = \frac{T}{2}$, and that κ satisfies conditions (H0), (H1) and

(H2C) *for any $\xi, \eta \in \mathbb{R}^d$, $x, y \in \Omega_{T/2}$,*

$$\sum_{i,j,k,l,p,q,r,s} \left(D^4_{ijrs}\kappa(x-y)\right) \tilde{\kappa}^{rk}\tilde{\kappa}^{sl}\xi_k\xi_l\eta_i\eta_j \leq 0,$$

where $\{\tilde{\kappa}^{ij}\}$ is the inverse of $\{\kappa_{ij} + \frac{2}{T}I\}$, *We also assume some geometric conditions on the domains. Then the results of Theorem 1 remain true.*

The proof of Theorem 2 relies on the observation that (H2c) implies (H2), and is preserved under convex combinations, and therefore by convolution with the density $\rho(T/2)$, and on some a priori C^1 estimates on the potential. Full details and further remarks are contained in our work [5].

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