Quasilinear parabolic and elliptic equations with singular potentials

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Abstract In this paper we describe the asymptotic behavior of the solutions to quasilinear parabolic equations with a Hardy potential. We prove that all the solutions have the same asymptotic behavior: they all tend to the solution of the original problem which satisfies a zero initial condition. Moreover, we derive estimates on the "distance" between the solutions of the evolution problem and the solutions of elliptic problems showing that in many cases (as for example the autonomous case) these last solutions are "good approximations" of the solutions of the original parabolic PDE.

1 Introduction

Let us consider the following nonlinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \lambda \dfrac{u}{|x|^2} + f(x,t) & \text{in } \ \Omega_T \equiv \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain containing the origin and λ and T are positive constants.

Here $a(x,t,\xi) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function[1] satisfying

$$a(x,t,\xi)\xi \geq \alpha |\xi|^2, \qquad \alpha > 0, \tag{2}$$

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[1] i.e., it is continuous with respect to ξ for almost every $(x,t) \in \Omega_T$, and measurable with respect to (x,t) for every $\xi \in \mathbb{R}^N$

$$|a(x,t,\xi)| \leq \beta[|\xi| + \mu(x,t)], \qquad \beta > 0, \qquad \mu \in L^2(\Omega_T), \tag{3}$$

$$(a(x,t,\xi) - a(x,t,\xi')) \cdot (\xi - \xi') \geq \alpha|\xi - \xi'|^2, \tag{4}$$

and the data satisfy (for example)

$$u_0 \in L^2(\Omega) \qquad f \in L^2(\Omega_T). \tag{5}$$

The model problem we have in mind is the following

$$\begin{cases} u_t - \Delta u = \lambda \dfrac{u}{|x|^2} + f(x,t) & \text{in } \Omega_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{6}$$

We recall that if the data f and u_0 are nonnegative and not both identically zero, there exists a dimension dependent constant Λ_N such that (6) has no solution for $\lambda > \Lambda_N$ (see [10]). More in details, the constant Λ_N is the optimal constant (not attained) in the Hardy's inequality

$$\Lambda_N \int_\Omega \frac{u^2}{|x|^2} dx \leq \int_\Omega |\nabla u|^2 dx \qquad \text{for every } u \in H_0^1(\Omega) \quad \text{where} \quad \Lambda_N \equiv \left(\frac{N-2}{2}\right)^2, \tag{7}$$

(see [24] and [20]).
Hence, here, in order to guarantee the existence of solutions, we assume $\lambda < \Lambda_N$ in the model case (6) and its generalization

$$\lambda < \alpha \Lambda_N. \tag{8}$$

in the general case (1).
The main aim of this paper is the study of the asymptotic behavior of the solutions of (1).

The peculiarity of these problem is the presence of the singular Hardy potential, also called in literature "inverse-square" potential. This kind of singular potential arises, for example, in the context of combustion theory (see [11], [49] and the references therein) and quantum mechanics (see [10], [44], [49] and the references therein).

There is an extensive literature on problems with Hardy potentials both in the stationary and evolution cases and it is a difficult task to give a complete bibliography. In the elliptic case, more related to our framework are [2]-[5], [14], [41], [35], [45], [46] and [7]. In the parabolic case, a mile stone is the pioneer paper [10] which revealing the surprising effects of these singular potentials on the solutions stimulated the study of these problems. More connected to our results are [6], [49], [47], [19], [25], [26], [1] [43], and [38].

In particular, in [43] it is studied the influence of the regularity of the data f and u_0 on the regularity of the solutions of (1), while in [49] and [38], among other results, there is a description of the behavior (in time) of the solutions when $f \equiv 0$.

Hence, here we want to complete these results studying what is the asymptotic behavior of the solutions when f is not identically zero.

We point out that the presence of a singular potential term has a strong influence not only, as recalled above, on the existence theory, but also on the regularity and on the asymptotic behavior, even when the datum f is zero. As a matter of fact, it is well known that if $\lambda \equiv 0 = f$ and the initial datum u_0 is bounded then also the solution of (1) is bounded; moreover, this result remains true in the more general case of non zero data f belonging to $L^r(0,T;L^q(\Omega))$ with r and q satisfying

$$\frac{1}{r} + \frac{N}{2q} < 1 \tag{9}$$

(see [9] and the references therein).

Surprisingly, the previous L^∞-regularity fails even in the model case (6) as soon as λ becomes positive if f and u_0 are not both identically zero (otherwise $u \equiv 0$ is a bounded solution) since every solution (for nonnegative initial data u_0) satisfies[2]

$$u(x,t) \geq \frac{C}{|x|^{\alpha_1}} \qquad \text{for almost every } \ (x,t) \in \Omega' \times [\varepsilon, \hat{T}], \tag{10}$$

for every $\varepsilon \in (0,\hat{T})$, $0 < \hat{T} < T$ and $\Omega' \subset\subset \Omega$, where the constant C depends only on ε, $\hat{T}$, Ω' and λ, while α_1 is the smallest root of $z^2 - (N-2)z + \lambda = 0$.

Indeed, the singular potential term influences the solutions also when the summability coefficients r and q of f do not satisfy (9). As a matter of fact, again the regularity of the solutions in presence of the Hardy potential is different from the classical semilinear case $\lambda = 0$ (see [43] if $\lambda > 0$ and [31], [13], [33], [34], [27], [16] and the references therein if $\lambda = 0$).

Great changes appear also in the behavior in time of the solutions. As a matter of fact, if $\lambda = 0 = f(x,t)$ it is well known that the solutions of (1) become immediately bounded also in presence of unbounded initial data u_0 belonging only to $L^{r_0}(\Omega)$ ($r_0 \geq 1$) and satisfy the same decay estimates of the heat equation

$$\|u(t)\|_{L^\infty(\Omega)} \leq c \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{\frac{N}{2r_0}} e^{\sigma t}} \quad \text{for almost every } \ t \in (0,T), \tag{11}$$

where $\sigma = \frac{c}{|\Omega|^{\frac{2}{N}}}$ is a constant depending on the measure of Ω (see [36] and the references therein). The previous bound, or more in general estimates of the type

$$\|u(t)\|_{L^\infty(\Omega)} \leq c \frac{\|u_0\|^{h_0}_{L^{r_0}(\Omega)}}{t^{h_1}} \qquad h_0, h_1 > 0, \tag{12}$$

are often referred as ultracontractive estimates and hold for many different kinds of parabolic PDE (degenerate or singular) like, for example, the p-Laplacian equation, the fast diffusion equation, the porous medium equation etc. These estimates are

[2] The proof of (10) can be easily obtained following the outline of the proof of (2.5) of Theorem 2.2 in [10]

widely studied because they describe the behavior in time of the solutions and often imply also further important properties like, for example, the uniqueness (see for example [50], [18], [8], [30], [12], [29], [22], [23], [48], [21], [17], [42], [36], [28], [37] and the references therein).

Unfortunately, by estimate (10) above it follows that estimate (11) together with (12) fail in presence of a Hardy potential term. Anyway, in [38] it is proved that if $f \equiv 0$, $\lambda > 0$ and $u_0 \in L^2(\Omega)$, then there exists a solution that satisfies

$$\|u(t)\|_{L^{2\gamma}(\Omega)} \leq c\frac{\|u_0\|_{L^2(\Omega)}}{t^{\delta}e^{\sigma t}} \quad \text{for almost every } t \in (0,T), \qquad \delta = \frac{N(\gamma-1)}{4\gamma}, \tag{13}$$

for every $\gamma > 1$ satisfying

$$\gamma \in \left(1, \frac{1+\sqrt{1-\theta}}{\theta}\right) \quad \text{where} \quad \theta = \frac{\lambda}{\alpha \Lambda_N}.$$

Hence an increasing of regularity appears (depending on the “size” λ of the singular potential), but according with (10), there is not the boundedness of the solutions.

As said above, aim of this paper is to describe what happens when f is not identically zero.

We will show that under the previous assumptions on the operator a and on the data f and u_0, there exists only one “good” global solution u of (1). Moreover, if v is the global solution of

$$\begin{cases} v_t - \operatorname{div}(a(x,t,\nabla v)) = \lambda \dfrac{v}{|x|^2} + f(x,t) & \text{in } \Omega \times (0,+\infty), \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{14}$$

i.e., v satisfies the same PDE of u (with the same datum f) but verifies the different initial condition $v(x,0) = v_0 \in L^2(\Omega)$, then the following estimate holds

$$\|u(t) - v(t)\|_{L^2(\Omega)} \leq \frac{\|u_0 - v_0\|_{L^2(\Omega)}}{e^{\sigma t}} \quad \text{for every } t > 0, \tag{15}$$

where σ is a positive constant which depends on λ (see formula (26) below). In particular, it results

$$\lim_{t\to+\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0. \tag{16}$$

Hence, for t large, the initial data do not influence the behavior of the solutions since by (16) it follows that all the global solutions tend to the solution which assumes the null initial datum.

We recall that in absence of the singular potential term we can replace the L^2-norm in the left-hand side of (15) with the L^∞-norm and, consequently, together with (16), the following stronger result holds true

$$\lim_{t\to+\infty} \|u(t) - v(t)\|_{L^\infty(\Omega)} = 0. \tag{17}$$

(see [39]). Thus, the presence of the Hardy potential provokes again a change in the behavior of the solutions since generally the difference of two solutions u and v cannot be bounded if $\lambda > 0$. As a matter of fact, it is sufficient to notice that choosing $f = 0$ and $v = 0$ (which corresponds to the choice $v_0 = 0$) the boundedness of u-v becomes the boundedness of u which by (10) we know to be false in presence of a Hardy potential.

Moreover, in the autonomous case

$$a(x,t,\xi) = a(x,\xi) \qquad f(x,t) = f(x)$$

we prove that all the global solutions of (1) (whatever is the value of the initial datum u_0) tend to the solution $w \in H_0^1(\Omega)$ of the associate elliptic problem

$$\begin{cases} -\text{div}(a(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + f(x) \text{ in } \Omega, \\ w(x) = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega\,. \end{cases}$$

Indeed, we estimate also the difference $u - v$ between the global solutions of (1) and the global solution v of the different evolution problem (not necessarily of parabolic type)

$$\begin{cases} v_t - \text{div}(b(x,t,\nabla v)) = \lambda \dfrac{v}{|x|^2} + F(x,t) \text{ in } \Omega \times (0,+\infty), \\ v(x,t) = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) \qquad\qquad\qquad\qquad \text{in } \Omega, \end{cases}$$

looking for conditions which guarantee that this difference goes to zero (letting $t \to +\infty$).

Finally, we estimate also the difference $u - w$ between a global solution of (1) and the solutions w of the stationary problem

$$\begin{cases} -\text{div}(b(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + F(x) \text{ in } \Omega, \\ w(x) = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega\,, \end{cases} \tag{18}$$

showing that in the non autonomous case, under suitable “proximity” conditions on the operators a and b and the data f and F, the global solution of (1) tends to the solution w of the stationary problem (18).

The paper is organized as follows: in next section we give the statements of our results in all the details. The proofs can be found in Section 4 and make use of some “abstract results” proved in [38] and [32] that, for the convenience of the reader, we recall in Section 3.

2 Main results

Before stating our results, we recall the definitions of solution and global solution of (1).

Definition 1. Assume (2)-(5). A function u in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ is a solution of (1) if it results

$$\int_0^T\int_\Omega\{-u\varphi_t+a(x,t,\nabla u)\nabla\varphi\}\,dxdt=\int_\Omega u_0\varphi(x,0)\,dx+\int_0^T\int_\Omega\left[\lambda\frac{u}{|x|^2}+f\right]\varphi\,dxdt \tag{19}$$

for every $\varphi\in W^{1,1}(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ satisfying $\varphi(T)=0$.

We point out that all the integrals in (19) are well defined. As a matter of fact, by (3) it follows that $a(x,t,\nabla u)\in(L^2(\Omega_T))^N$ and thanks to Hardy's inequality (7) it results

$$\int_0^T\int_\Omega\lambda\frac{u}{|x|^2}\varphi\,dxdt\le\lambda\left(\int_0^T\int_\Omega\frac{u^2}{|x|^2}\right)^{\frac12}\left(\int_0^T\int_\Omega\frac{\varphi^2}{|x|^2}\right)^{\frac12}\le$$
$$\frac{\lambda}{\Lambda_N}\|\nabla u\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}\,. \tag{20}$$

We recall that under the assumptions (2)-(5) and (8) there exists solutions of (1) (see [43]). Now, to extend the previous notion to that of global solution, we assume

$$f\in L^2_{loc}([0,+\infty);L^2(\Omega))\quad\text{and}\quad\mu\in L^2_{loc}([0,+\infty);L^2(\Omega)) \tag{21}$$

where μ is the function that appears in (3).

Definition 2. By a global solution of (1), or (equivalently) of

$$\begin{cases}u_t-\operatorname{div}(a(x,t,\nabla u))=\lambda\dfrac{u}{|x|^2}+f(x,t) & \text{in }\ \Omega\times(0,+\infty)\\ u(x,t)=0 & \text{on }\partial\Omega\times(0,+\infty),\\ u(x,0)=u_0(x) & \text{in }\ \Omega,\end{cases} \tag{22}$$

we mean a measurable function u that is a solution of (1) for every $T>0$ arbitrarily chosen.

We point out that (21) together with the previous structure assumptions guarantee that the integrals in (19) are well defined for every choice of $T>0$. Indeed, there exists only one global solution of (1). In detail, we have:

Theorem 1. *Assume* (2)-(5), (8) *and* (21). *Then there exists only one global solution u of* (1) *belonging to* $C_{loc}([0,+\infty);L^2(\Omega))\cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$. *In particular, for every* $t>0$ *it results*

$$\int_0^t\int_\Omega\{-u\varphi_t+a(x,t,\nabla u)\nabla\varphi\}\,dxdt+\int_\Omega[u(x,t)\varphi(x,t)-u_0\varphi(x,0)]\,dx=$$
$$\int_0^t\int_\Omega\left[\lambda\frac{u}{|x|^2}+f\right]\varphi\,dxdt\,, \tag{23}$$

for every $\varphi\in W^{1,1}_{loc}([0,+\infty);L^2(\Omega))\cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$.

As noticed in the introduction, if we change the initial data in (1), all the associated global solutions (to these different initial data) have the same asymptotic behavior. In detail, let us consider the following problem

$$\begin{cases} v_t - \operatorname{div}(a(x,t,\nabla v)) = \lambda \dfrac{v}{|x|^2} + f(x,t) & \text{in } \Omega \times (0,+\infty), \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega . \end{cases} \tag{24}$$

We have the following result:

Theorem 2. *Assume* (2)-(5), (8) *and* (21). *If* $v_0 \in L^2(\Omega)$, *then the global solutions* u *and* v *of, respectively,* (1) *and* (24) *belonging to* $C_{loc}([0,+\infty);L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H^1_0(\Omega))$ *satisfy*

$$\|u(t)-v(t)\|_{L^2(\Omega)} \le \frac{\|u_0 - v_0\|_{L^2(\Omega)}}{e^{\sigma t}} \quad \textit{for every } t > 0\,, \tag{25}$$

where

$$\sigma = \left(\alpha - \frac{\lambda}{\Lambda_N}\right) c_P \tag{26}$$

with c_P *Poincaré's constant*[3].
In particular, it results

$$\lim_{t\to+\infty} \|u(t)-v(t)\|_{L^2(\Omega)} = 0\,. \tag{28}$$

Remark 1. Notice that in the particular case $f \equiv 0$, choosing as initial datum $v_0 = 0$ we obtain that $v \equiv 0$ is global solution of (1). With such a choice in (25) it follows that

$$\|u(t)\|_{L^2(\Omega)} \le \frac{\|u_0\|_{L^2(\Omega)}}{e^{\sigma t}} \quad \text{for every } t > 0\,.$$

In the model case (6) the previous estimate can be found (among other interesting results) in [49] with $\sigma = \mu_1$ the first eigenvalue (see also [38]). We recall that decay estimates of the solutions in the same Lebesgue space where is the initial datum is not a peculiarity of problems with singular potentials since appear also for other parabolic problems (see [15], [38], [48] and the references therein).

An immediate consequence of Theorem 2 is that in the autonomous case

$$a(x,t,\xi) = a(x,\xi) \qquad f(x,t) = f(x) \tag{29}$$

[3] Poincaré's inequality:

$$c_P \int_\Omega u^2 dx \le \int_\Omega |\nabla u|^2 dx \qquad \text{for every } u \in H^1_0(\Omega)\,, \tag{27}$$

where c_P is constant depending only on N and on the bounded set Ω

all the global solutions of (1), whatever is the value of the initial datum u_0, tend (letting $t \to +\infty$) to the solution w of the associate elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + f(x) & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

In detail, we have:

Corollary 1 (Autonomous case). *Assume* (2)-(5)*,* (8)*,* (21) *and* (29)*. Let w be the unique solution of* (30) *in* $H_0^1(\Omega)$ *and u be the global solution of* (1) *belonging to* $C_{loc}([0,+\infty);L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$*. Then it results*

$$\|u(t) - w\|_{L^2(\Omega)} \le \frac{\|u_0 - w\|_{L^2(\Omega)}}{e^{\sigma t}} \quad \textit{for every } t > 0, \tag{31}$$

where σ *is as in* (26)*.*
In particular, it results

$$\lim_{t\to+\infty} \|u(t) - w\|_{L^2(\Omega)} = 0. \tag{32}$$

We show now that it is also possible to estimate the distance between the global solution u of (1) and the global solution v of the different parabolic problem

$$\begin{cases} v_t - \mathrm{div}(b(x,t,\nabla v)) = \lambda \dfrac{v}{|x|^2} + F(x,t) & \text{in } \Omega \times (0,+\infty), \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{33}$$

where $b(x,t,\xi) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfying

$$b(x,t,\xi)\xi \ge \alpha_0|\xi|^2, \qquad \alpha_0 > 0, \tag{34}$$

$$|b(x,t,\xi)| \le \beta_0[|\xi| + \mu_0(x,t)], \qquad \beta_0 > 0, \qquad \mu_0 \in L^2_{loc}([0,+\infty);L^2(\Omega)), \tag{35}$$

$$(b(x,t,\xi) - b(x,t,\xi')) \cdot (\xi - \xi') \ge \alpha_0|\xi - \xi'|^2. \tag{36}$$

$$v_0 \in L^2(\Omega) \qquad F \in L^2_{loc}([0,+\infty);L^2(\Omega)). \tag{37}$$

Theorem 3. *Assume* (2)-(5)*,* (8)*,* (21) *and* (34)-(37)*. Then the global solutions u and v of, respectively,* (1) *and* (33) *belonging to* $C_{loc}([0,+\infty);L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$ *satisfy*

$$\|u(t) - v(t)\|^2_{L^2(\Omega)} \le \frac{\|u_0 - v_0\|^2_{L^2(\Omega)}}{e^{2\sigma_0 t}} + \int_0^t g(s)ds \quad \textit{for every } t \ge 0, \tag{38}$$

for every choice of

$$\sigma_0 < \sigma \tag{39}$$

where

$$g(s)=\frac{c_P}{\sigma-\sigma_0}\int_\Omega\left[|b(x,s,\nabla v(x,s))-a(x,s,\nabla v(x,s))|^2+\frac{1}{c_P}|f(x,s)-F(x,s)|^2\right]dx\,, \tag{40}$$

with σ and c_P are as in (26). *Moreover, if $g\in L^1((0,+\infty))$ then it results*

$$\|u(t)-v(t)\|^2_{L^2(\Omega)}\le\frac{\Lambda}{e^{\sigma_0 t}}+\int_{\frac{t}{2}}^{t}g(s)ds\quad \textit{for every}\ t>0\,, \tag{41}$$

where

$$\Lambda=\|u_0-v_0\|^2_{L^2(\Omega)}+\int_0^{+\infty}g(t)dt\,.$$

In particular, we have

$$\lim_{t\to+\infty}\|u(t)-v(t)\|_{L^2(\Omega)}=0\,. \tag{42}$$

Remark 2. The proof of Theorem 3 shows that the structure assumptions (34)-(36) on the operator b can be weakened. In particular, it is sufficient to assume that there exists a global solution v of (33) in $C_{loc}([0,+\infty);L^2(\Omega))\cap L^2_{loc}([0,+\infty);H^1_0(\Omega))$ satisfying

$$b(x,t,\nabla v)\in L^2_{loc}([0,+\infty);L^2(\Omega))\,.$$

Hence, also problems (33) which are not of parabolic type are allowed.

Moreover, with slight changes in the proof, it is also possible to choose a larger class of data f and F. In particular, an alternative option that can be done is $L^2_{loc}([0,+\infty);H^{-1}(\Omega))$.

Examples of operators satisfying all the assumptions of the previous Theorem (and hence for which (42) holds) are

$$\begin{cases} u_t-\operatorname{div}(a(x,t,\nabla u))=\lambda\dfrac{u}{|x|^2}+f(x,t) & \text{in }\ \Omega\times(0,+\infty),\\ u(x)=0 & \text{on }\partial\Omega\times(0,+\infty)\\ u(x,0)=u_0 & \text{in }\ \Omega\,. \end{cases}$$

and the model case

$$\begin{cases} v_t-\Delta v=\lambda\dfrac{u}{|x|^2}+F(x,t) & \text{in }\ \Omega\times(0,+\infty),\\ v(x)=0 & \text{on }\partial\Omega\times(0,+\infty)\\ v(x,0)=v_0 & \text{in }\ \Omega\,, \end{cases}$$

if we assume

$$[a(x,t,\nabla v)-\nabla v]\in L^2(\Omega\times(0,+\infty))\qquad [f(x,t)-F(x,t)]\in L^2(\Omega\times(0,+\infty)).$$

Remark 3. We point out that an admissible choice for the parameter λ in Theorem 3 is

$$\lambda=0\,,$$

i.e., the case of absence of the singular potential. In this particular but also interesting case, the previous result permits to estimate the difference of solutions of different evolution problems. Moreover, as noticed in Remark 2, only one of these two evolution problems is required to be of parabolic type.

A consequence of Theorem 3 is the possibility to estimate also the distance between global solutions of (1) and solutions of stationary problems, for example of elliptic type, without assuming to be in the autonomous case (29). These estimates (see Corollary 2 below) show that if the data and the operators of these different PDE problems (calculated on the solution of the stationary problem) are "sufficiently near", then the solutions of the evolution problems tend (for every choice of the initial data u_0) to the stationary solution.
In detail, let us consider the following stationary problem

$$\begin{cases} -\mathrm{div}(b(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + F(x) \text{ in } \Omega\,, \\ w(x) = 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega\,. \end{cases} \tag{43}$$

To stress the assumptions really needed, in what follows we do not assume any structure condition on b except that $b : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function. We have:

Corollary 2. *Assume* (2)-(5)*,* (8) *and* (21)*. Let F be in $L^2(\Omega)$ and $w \in H_0^1(\Omega)$ be such that*

$$b(x,\nabla w) \in (L^2(\Omega))^N\,. \tag{44}$$

If w is a solution of (43) *and $u \in C_{loc}([0,+\infty);L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$ is the global solution of* (1)*, then the following estimate holds true*

$$\|u(t)-w\|^2_{L^2(\Omega)} \le \frac{\|u_0-w\|^2_{L^2(\Omega)}}{e^{2\sigma_0 t}} + \int_0^t g(s)ds\,, \tag{45}$$

for every $t \ge 0$ and for every choice of σ_0 as in (39) *where*

$$g(s) = \frac{c_P}{\sigma - \sigma_0} \int_\Omega \left[|b(x,\nabla w(x)) - a(x,s,\nabla w(x))|^2 + \frac{1}{c_P}|f(x,s)-F(x)|^2 \right] dx\,. \tag{46}$$

Moreover, if $g \in L^1((0,+\infty))$, then it results

$$\|u(t)-w\|^2_{L^2(\Omega)} \le \frac{\Lambda}{e^{\sigma_0 t}} + \int_{\frac{t}{2}}^t g(s)ds \quad \textit{for every } t>0\,, \tag{47}$$

where

$$\Lambda = \|u_0-w\|^2_{L^2(\Omega)} + \int_0^{+\infty} g(t)dt\,.$$

In particular, it follows

$$\lim_{t\to+\infty} \|u(t)-w\|_{L^2(\Omega)} = 0\,. \tag{48}$$

Examples of operators satisfying all the assumptions of Corollary 2 (and hence for which (48) holds) are

$$\begin{cases} u_t - \operatorname{div}(\alpha(x,t)\nabla u) = \lambda \dfrac{u}{|x|^2} + f(x,t) & \text{in } \Omega \times (0,+\infty), \\ u(x) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases}$$

and

$$\begin{cases} -\Delta w = \lambda \dfrac{w}{|x|^2} + F(x) & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

with

$$[\alpha(x,t) - 1] \in L^2(\Omega \times (0,+\infty)) \qquad [f(x,t) - F(x)] \in L^2(\Omega \times (0,+\infty)) \tag{49}$$

or

$$\begin{cases} u_t - \operatorname{div}(\alpha(x,t)b(x,\nabla u)) = \lambda \dfrac{u}{|x|^2} + f(x,t) & \text{in } \Omega \times (0,+\infty), \\ u(x) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(b(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + F(x) & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

with $\alpha(x,t)$ and the data f and F satisfying (49).

3 Preliminary results

In this section we state two results that will be essential tools in proving the theorems presented above.

Theorem 4 (Theorem 2.8 in [38]). *Let u be in* $C((0,T);L^r(\Omega)) \cap L^\infty(0,T;L^{r_0}(\Omega))$ *where* $0 < r \le r_0 < \infty$. *Suppose also that* $|\Omega| < +\infty$ *if* $r \ne r_0$ *(no assumption are needed on* $|\Omega|$ *if* $r = r_0$*). If u satisfies*

$$\int_\Omega |u|^r(t_2) - \int_\Omega |u|^r(t_1) + c_1 \int_{t_1}^{t_2} \|u(t)\|^r_{L^r(\Omega)}\, dt \le 0 \quad \text{for every } 0 < t_1 < t_2 < T, \tag{50}$$

and there exists $u_0 \in L^{r_0}(\Omega)$ *such that*

$$\|u(t)\|_{L^{r_0}(\Omega)} \le c_2 \|u_0\|_{L^{r_0}(\Omega)} \quad \text{for almost every } t \in (0,T), \tag{51}$$

where c_i, $i = 1,2$ *are real positive numbers, then the following estimate holds true*

$$\|u(t)\|_{L^r(\Omega)} \le c_4 \frac{\|u_0\|_{L^{r_0}(\Omega)}}{e^{\sigma t}} \quad \text{for every } 0 < t < T, \tag{52}$$

where

$$c_4 = \begin{cases} c_2 |\Omega|^{\frac{1}{r} - \frac{1}{r_0}} & \text{if } r < r_0, \\ 1 & \text{if } r = r_0, \end{cases} \qquad \sigma = \frac{c_1}{r}.$$

Proposition 1 (Proposition 3.2 in [32]). *Assume $T \in (t_0, +\infty]$ and let $\phi(t)$ a continuous and non negative function defined in $[t_0, T)$ verifying*

$$\phi(t_2) - \phi(t_1) + M \int_{t_1}^{t_2} \phi(t)dt \le \int_{t_1}^{t_2} g(t)dt$$

for every $t_0 \le t_1 \le t_2 < T$ where M is a positive constant and g is a non negative function in $L^1_{loc}([t_0, T))$. Then for every $t \ge t_0$ we get

$$\phi(t) \le \phi(t_0) e^{-M(t-t_0)} + \int_{t_0}^{t} g(s)ds \qquad \forall t > t_0. \tag{53}$$

Moreover, if $T = +\infty$ and g belongs to $L^1((t_0, +\infty))$ there exists $t_1 \ge t_0$ (for example $t_1 = 2t_0$) such that

$$\phi(t) \le \Lambda e^{-\frac{M}{2}t} + \int_{\frac{t}{2}}^{t} g(s)ds \qquad \text{for every} \quad t \ge t_1, \tag{54}$$

where

$$\Lambda = \phi(t_0) + \int_{t_0}^{+\infty} g(s)ds.$$

In particular, we get that

$$\lim_{t \to +\infty} \phi(t) = 0.$$

4 Proofs of the results

4.1 Proof of Theorem 1

Let $T > 0$ arbitrarily fixed. The existence of a solution $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ of (1) can be found in [43]. We point out that since u_t belongs to $L^2(0, T; H^{-1}(\Omega))$ (thanks to the regularity of u and (20)) it follows that u belongs also to $C([0, T]; L^2(\Omega))$. Consequently, it results

$$\int_0^T \int_\Omega \{-u\varphi_t + a(x, t, \nabla u)\nabla \varphi\}\, dxdt + \int_\Omega [u(x, T)\varphi(x, T) - u_0 \varphi(x, 0)]\, dx = \int_0^T \int_\Omega \left[\lambda \frac{u}{|x|^2} + f\right] \varphi\, dxdt, \tag{55}$$

for every $\varphi \in W^{1,1}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Moreover, u is the unique solution of (1) belonging to $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. As a matter of fact, if there exists an other solution v of (1) in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, taking as test function $u-v$ in the equation satisfied by u and in that satisfied by v and subtracting the results[4] we deduce (using (4))

$$\frac{1}{2}\int_\Omega [u(x,T)-v(x,T)]^2 + \alpha \int_0^T\int_\Omega |\nabla(u-v)|^2 \leq \lambda \int_0^T\int_\Omega \frac{[u-v]^2}{|x|^2}\,. \tag{56}$$

By the previous estimate and Hardy's inequality (7) we obtain

$$\frac{1}{2}\int_\Omega [u(x,T)-v(x,T)]^2 + \left(\alpha - \frac{\lambda}{\Lambda_N}\right)\int_0^T\int_\Omega |\nabla(u-v)|^2 \leq 0\,.$$

from which the uniqueness follows since by assumption it results $\alpha - \frac{\lambda}{\Lambda_N} > 0$.

Hence, for every arbitrarily fixed $T > 0$ there exists a unique solution of (1) in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ that we denote $u^{(T)}$.

To conclude the proof, let us construct now the global solution of (1). For every $t \geq 0$ let us define $u(x,t) = u^{(T)}(x,t)$ where T is arbitrarily chosen satisfying $T > t$. We notice that by the uniqueness proved above this definition is well posed. Moreover, by construction this function satisfies the assertions of the theorem.

□

4.2 Proof of Theorem 2

Let u and v be as in the statement of Theorem 2. Taking as test function $u-v$ in (1) and in (24) and subtracting the equations obtained in this way, we deduce (using assumption (4)) that for every $0 \leq t_1 < t_2$ it results

$$\frac{1}{2}\int_\Omega [u(x,t_2)-v(x,t_2)]^2\,dx - \frac{1}{2}\int_\Omega [u(x,t_1)-v(x,t_1)]^2 + \alpha \int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 \leq \lambda \int_{t_1}^{t_2}\int_\Omega \frac{[u-v]^2}{|x|^2}\,. \tag{57}$$

Using again Hardy's inequality (7), from (57) we deduce

$$\int_\Omega [u(x,t_2)-v(x,t_2)]^2\,dx - \int_\Omega [u(x,t_1)-v(x,t_1)]^2 + c_0 \int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 \leq 0\,, \tag{58}$$

where we have defined

[4] the use here and below of these test functions can be made rigorous by means of Steklov averaging process

$$c_0 = 2\left(\alpha - \frac{\lambda}{\Lambda_N}\right). \tag{59}$$

Thanks to Poincaré's inequality (27) by the previous estimate we get for every $0 \le t_1 < t_2$

$$\int_\Omega [u(x,t_2) - v(x,t_2)]^2\,dx - \int_\Omega [u(x,t_1) - v(x,t_1)]^2 + c_1 \int_{t_1}^{t_2}\int_\Omega |u-v|^2 \le 0\,, \tag{60}$$

where $c_1 = c_P c_0$. Notice that by (60) (choosing $t_2 = t$ and $t_1 = 0$) it follows also that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \le \|u_0 - v_0\|_{L^2(\Omega)}\,.$$

Now the assert follows applying Theorem 4 with $r = r_0 = 2$. □

4.3 Proof of Corollary 1

The assertion (31) follows by Theorem 2 once noticed that, thanks to the assumption (29), the solution $w \in H_0^1(\Omega)$ of (30) is also the global solution $w(x,t) \equiv w(x) \in C_{loc}([0,+\infty];L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$ of the following parabolic problem

$$\begin{cases} w_t - \operatorname{div}(a(x,\nabla w)) = \lambda \dfrac{w}{|x|^2} + f(x) & \text{in } \ \Omega \times (0,+\infty), \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ w(x,0) = w(x) & \text{in } \ \Omega\,. \end{cases}$$

□

4.4 Proof of Theorem 3

Let u and v be the global solutions in $C_{loc}([0,+\infty];L^2(\Omega)) \cap L^2_{loc}([0,+\infty);H_0^1(\Omega))$ of, respectively, (1) and (33). Taking u-v in both the problems (1) and (33) and subtracting the results we obtain for every $0 \le t_1 < t_2$

$$\begin{aligned} &\frac{1}{2}\int_\Omega [u(x,t_2) - v(x,t_2)]^2\,dx - \frac{1}{2}\int_\Omega [u(x,t_1) - v(x,t_1)]^2 + \\ &\int_{t_1}^{t_2}\int_\Omega [a(x,t,\nabla u) - b(x,t,\nabla v)]\nabla(u-v) \le \\ &\quad \lambda \int_{t_1}^{t_2}\int_\Omega \frac{(u-v)^2}{|x|^2} + \int_{t_1}^{t_2}\int_\Omega (f-F)(u-v)\,, \end{aligned}$$

which is equivalent to the following estimate

$$\frac{1}{2}\int_\Omega [u(x,t_2)-v(x,t_2)]^2\,dx - \frac{1}{2}\int_\Omega [u(x,t_1)-v(x,t_1)]^2 +$$
$$\int_{t_1}^{t_2}\int_\Omega [a(x,t,\nabla u)-a(x,t,\nabla v)]\nabla(u-v) \le \lambda \int_{t_1}^{t_2}\int_\Omega \frac{(u-v)^2}{|x|^2} +$$
$$\int_{t_1}^{t_2}\int_\Omega (f-F)(u-v) + \int_{t_1}^{t_2}\int_\Omega [b(x,t,\nabla v)-a(x,t,\nabla v)]\nabla(u-v)\,. \quad (61)$$

By assumption (4), Hardy's inequality (7) and (61) we deduce

$$\frac{1}{2}\int_\Omega [u(x,t_2)-v(x,t_2)]^2\,dx - \frac{1}{2}\int_\Omega [u(x,t_1)-v(x,t_1)]^2 +$$
$$\left(\alpha - \frac{\lambda}{\Lambda_N}\right)\int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 \le \quad (62)$$
$$\int_{t_1}^{t_2}\int_\Omega (f-F)(u-v) + \int_{t_1}^{t_2}\int_\Omega [b(x,t,\nabla v)-a(x,t,\nabla v)]\nabla(u-v)\,.$$

We estimate the last two integrals in (62). Let $\theta \in (0,1)$ a constant that we will choose below. It results (using Young's and Poincaré's inequalities)

$$\int_{t_1}^{t_2}\int_\Omega (f-F)(u-v) \le \frac{\theta}{2}C_0 c_P \int_{t_1}^{t_2}\int_\Omega (u-v)^2 + \frac{1}{2\theta C_0 c_P}\int_{t_1}^{t_2}\int_\Omega |f-F|^2 \le$$
$$\frac{\theta}{2}C_0 \int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 + \frac{1}{2\theta C_0 c_P}\int_{t_1}^{t_2}\int_\Omega |f-F|^2$$

where c_P is Poincaré's constant defined in (27) and $C_0 = \left(\alpha - \frac{\lambda}{\Lambda_N}\right)$. Moreover, we have

$$\int_{t_1}^{t_2}\int_\Omega [b(x,t,\nabla v)-a(x,t,\nabla v)]\nabla(u-v) \le \frac{\theta}{2}C_0 \int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 +$$
$$\frac{1}{2\theta C_0}\int_{t_1}^{t_2}\int_\Omega |b(x,t,\nabla v)-a(x,t,\nabla v)|^2$$

By the previous estimates we deduce that

$$\int_\Omega [u(x,t_2)-v(x,t_2)]^2\,dx - \int_\Omega [u(x,t_1)-v(x,t_1)]^2 +$$
$$2(1-\theta)C_0 \int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 \le$$
$$\frac{1}{\theta C_0 c_P}\int_{t_1}^{t_2}\int_\Omega |f-F|^2 + + \frac{1}{\theta C_0}\int_{t_1}^{t_2}\int_\Omega |b(x,t,\nabla v)-a(x,t,\nabla v)|^2\,.$$

which implies (again by Poincaré's inequality)

$$\int_{\Omega}[u(x,t_2)-v(x,t_2)]^2\,dx-\int_{\Omega}[u(x,t_1)-v(x,t_1)]^2+ \\ M\int_{t_1}^{t_2}\int_{\Omega}|u-v|^2\le\int_{t_1}^{t_2}g(s)\,ds \tag{63}$$

where $M=2c_P(1-\theta)C_0=2(1-\theta)\sigma$ (where σ is as in (26)) and

$$g(s)=\frac{1}{\theta C_0}\int_{\Omega}\left[\frac{1}{c_P}|f(x,s)-F(x,s)|^2+|b(x,s,\nabla v)-a(x,s,\nabla v)|^2\right]dx. \tag{64}$$

Denoting $\sigma_0=(1-\theta)\sigma$ (i.e., $\theta=1-\frac{\sigma_0}{\sigma}$) and applying Proposition 1 with $\phi(t)=\int_{\Omega}[u(x,t)-v(x,t)]^2\,dx$ and $t_0=0$, the assertions follow.
□

4.5 *Proof of Corollary 2*

The asserts follow observing that $w(x,t)=w(x)$ is also a global solution in

$$C_{loc}([0,+\infty];L^2(\Omega))\cap L^2_{loc}([0,+\infty);H^1_0(\Omega))$$

of the following evolution problem

$$\begin{cases} w_t-\mathrm{div}(b(x,\nabla w))=\lambda\dfrac{w}{|x|^2}+F(x) & \text{in } \Omega\times(0,+\infty), \\ w(x,t)=0 & \text{on } \partial\Omega\times(0,+\infty), \\ w(x,0)=w(x) & \text{in } \Omega. \end{cases}$$

□

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