A sixth order curvature flow of plane curves with boundary conditions

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Abstract We show that small energy curves under a particular sixth order curvature flow with generalised Neumann boundary conditions between parallel lines converge exponentially in the C^∞ topology in infinite time to straight line segments.

1 Introduction

Higher order geometric evolution problems have received increasing attention in the last few years. Particular geometric fourth order equations occur in physical problems and enjoy some interesting applications in mathematics. We mention in particular for curves the curve diffusion flow and L^2-gradient flow of the elastic energy, and for surfaces the surface diffusion and Willmore flows. Flows of higher even order than four have been less thoroughly investigated, but motivation for them and their elliptic counterparts comes for example from computer design, where higher order equations are desirable as they allow more flexibility in terms of prescribing boundary conditions [8]. Such equations have also found applications in medical imaging [10].

In this article we are interested in curves γ meeting two parallel lines with Neumann (together with other) boundary conditions evolving under the L^2 gradient flow for the energy

$$\int_\gamma k_s^2 ds.$$

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Here k_s denotes the first derivative of curvature with respect to the arc length parameter s. Particularly relevant to us is the corresponding consideration of the curve diffusion and elastic flow in this setting in [12]. Other relevant works on fourth order flow of curves with boundary conditions are [2, 3, 7]. Of course if one instead considers closed curves without boundary evolving by higher order equations, these have been more thoroughly studied; we mention in particular [1, 4, 5, 9, 11].

The remainder of this article is organised as follows. In Section 2 we describe the set-up of our problem, the normal variation of the energy and the boundary conditions. We define our corresponding gradient flow, discuss local existence and give the relevant evolution equations of various geometric quantities. We also state our main theorem in this part, Theorem 2.2. In Section 3 we state the relevant tools from analysis to be used including an interpolation inequality valid in our setting. Under the small energy condition (7) below, we show that the winding number of curves under our flow is constant and remains equal to zero. We show further that under this condition the length of the curve does not increase and the curvature and all curvature derivatives in L^2 are bounded under the flow. That these bounds are independent of time implies solutions exist for all time. In Section 4 we show under a smaller energy assumption that in fact the L^2 norm of the second derivative of curvature decays exponentially under the flow. As a corollary we obtain uniform pointwise exponential decay of curvature and all curvature derivatives to zero. A stability argument shows that the solution converges to a unique horizontal line segment. The exponential convergence of the flow speed allows us to describe the bounded region in which the solution remains under the flow.

2 The set-up

Let $\gamma_0 : [-1, 1] \to \mathbb{R}^2$ be a (suitably) smooth embedded (or immersed) regular curve. Denote by ds the arc length element and k the (scalar) curvature. We consider the energy functional

$$E[\gamma] = \frac{1}{2} \int_\gamma k_s^2 \, ds$$

where k_s is the derivative of curvature with respect to arc length. We are interested in the L^2 gradient flow for curves of small initial energy with Neumann boundary conditions.

Under the normal variation $\tilde{\gamma} = \gamma + \varepsilon F \nu$ a straightforward calculation yields

$$\frac{d}{d\varepsilon} E[\tilde{\gamma}] \bigg|_{\varepsilon=0} = -2 \int_\gamma \left(k_{s^4} + k^2 k_{ss} - \frac{1}{2} k k_s^2 \right) F \, ds$$
$$+ 2 \left[k_s F_{ss} + k_{ss} F_s + \left(k_{sss} + k^2 k_s \right) F \right]_{\partial \gamma}, \quad (1)$$

where $k_{s^4} = k_{ssss}$.

'Natural boundary conditions' for the corresponding L^2-gradient flow would ensure that the above boundary term is equal to zero. However, this term is rather complicated. In view of the first term in (1), we wish to take

$$F = k_{s^4} + k^2 k_{ss} - \frac{1}{2} k k_s^2 \tag{2}$$

and the corresponding gradient flow

$$\frac{\partial \gamma}{\partial t} = F\nu. \tag{3}$$

Differentiating the Neumann boundary condition (see also [12, Lemma 2.5] for example) implies

$$0 = -F_s(\pm 1, t) = -k_{s^5} - kk_s k_{ss} - k^2 k_{sss} + \frac{1}{2} k_s^3. \tag{4}$$

As in previous work, we will assume the 'no curvature flux condition' at the boundary,

$$k_s(\pm 1, t) = 0. \tag{5}$$

The boundary terms in (1) then disappear if we choose, for example,

$$k_{sss}(\pm 1, t) = 0. \tag{6}$$

This is in a way a natural choice because equation (4) then implies $k_{s^5}(\pm 1, t) = 0$. In fact by an induction argument we have

Lemma 2.1 *With Neumann boundary conditions and also* (5) *and* (6) *satisfied, a solution to the flow* (3) *satisfies* $k_{s^{2\ell+1}} = 0$ *on the boundary for* $\ell \in \mathbb{N}$.

Let us now state precisely the flow problem.

Let $\eta_\pm(\mathbb{R})$ denote two parallel vertical lines in $\mathbb{R}^2$, with distance between them $|e|$. We consider a family of plane curves $\gamma : [-1, 1] \times [0, T] \to \mathbb{R}^2$ satisfying the evolution equation (3) with normal speed given by (2), boundary conditions

$$\gamma(\pm 1, t) \in \eta_\pm(\mathbb{R})$$

$$\langle \nu, \nu_{\eta_\pm} \rangle (\pm 1, t) = 0$$
$$k_s(\pm 1, t) = k_{sss}(\pm 1, t) = 0$$

and initial condition

$$\gamma(\cdot, 0) = \gamma_0(\cdot)$$

for initial smooth regular curve γ_0.

Theorem 2.2 *There exists a universal constant* $C > 0$ *such that the following holds. For the flow problem described above, if the initial curve* γ_0 *satisfies* $\omega = 0$ *and*

$$\delta = \left(\frac{\sqrt{5129} - 67}{80} \right) \pi^3 - \|k_s\|_2^2 L_0^3 > 0, \tag{7}$$

where L_0 is the length of γ_0, then the solution exists for all time $T = \infty$ and converges exponentially to a horizontal line segment γ_∞ with $dist(\gamma_\infty, \gamma_0) < C/\delta$.

In the above statement and throughout the article we use ω to denote the *winding number*, defined here as

$$\omega := \frac{1}{2\pi} \int_\gamma k\, ds.$$

Remarks:

- The condition (7) is not optimal. By a standard argument it can be weakened for example to the requirement of Lemma 3.4, namely

$$\frac{\pi^3}{7} - \|k_s\|_2^2 L_0^3 > 0.$$

 Details of this argument will appear in a future article. It is an open question whether the requirement can be further weakened.
- The exponential decay facilitates an explicit estimate on the distance of γ_∞ to γ_0.

Local existence of a smooth regular curve solution $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$ to the flow problem $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$ is standard. If γ_0 also satisfies compatibility conditions, then the solution is smooth on $[0, T)$. In this article we focus on the case of smooth initial γ_0. However, γ_0 may be much less smooth; equation (3) is smoothing. We do not pursue this here.

Similarly as in [12] and elsewhere we may derive the following:

Lemma 2.3 *Under the flow* (3) *we have the following evolution equations*

(i) $\frac{\partial}{\partial t} ds = -kF\, ds;$

(ii) $\frac{\partial}{\partial t} k = F_{ss} + k^2 F;$

(iii) $\frac{\partial}{\partial t} k_s = F_{sss} + k^2 F_s + 3kk_s F;$

(iv)

$$\partial_t k_{s^l} = k_{s^{l+6}} + \sum_{q+r+u=l} (c_{qru}^1 k_{s^{q+4}} k_{s^r} k_{s^u} + c_{qru}^2 k_{s^{q+3}} k_{s^{r+1}} k_{s^u}$$

$$+ c_{qru}^3 k_{s^{q+2}} k_{s^{r+2}} k_{s^u} + c_{qru}^4 k_{s^{q+2}} k_{s^{r+1}} k_{s^{u+1}})$$

$$+ \sum_{a+b+c+d+e=l} c_{abcde} k_{s^a} k_{s^b} k_{s^c} k_{s^d} k_{s^e}$$

for constants $c_{qru}^1, c_{qru}^2, c_{qru}^3, c_{qru}^4, c_{abcde} \in \mathbb{R}$ with $a, b, c, d, e, q, r, u \geq 0$. In particular,

(v)

$$\frac{\partial}{\partial t} k_{ss} = k_{s^8} + 10 k_s k_{s^3} k_{ss} + \frac{21}{2} k_s^2 k_{s^4} + 12 k k_{s^5} k_s + 14 k k_{s^4} k_{ss} + 5 k k_{s^3}^2 + 2 k^2 k_{s^6}$$
$$+ \frac{11}{2} k^2 k_s^2 k_{ss} + 8 k^3 k_{s^3} k_s + 5 k^3 k_{ss}^2 + k^4 k_{s^4} - 4 k k_s^4.$$

3 Controlling the geometry of the flow

We begin with the following standard result for functions of one variable.

Lemma 3.1 (Poincaré-Sobolev-Wirtinger (PSW) inequalities) *Suppose* $f : [0, L] \to \mathbb{R}$, $L > 0$ *is absolutely continuous.*

- *If* $\int_0^L f \, ds = 0$ *then*

$$\int_0^L f^2 ds \le \frac{L^2}{\pi^2} \int_0^L f_s^2 ds \text{ and } \|f\|_\infty^2 \le \frac{2L}{\pi} \int_0^L f_s^2 ds.$$

- *Alternatively, if* $f(0) = f(L) = 0$ *then*

$$\int_0^L f^2 ds \le \frac{L^2}{\pi^2} \int_0^L f_s^2 ds \text{ and } \|f\|_\infty^2 \le \frac{L}{\pi} \int_0^L f_s^2 ds.$$

To state the interpolation inequality we will use, we first need to set up some notation. For normal tensor fields S and T we denote by $S \star T$ any linear combination of S and T. In our setting, S and T will be simply curvature k or its arc length derivatives. Denote by $P_n^m(k)$ any linear combination of terms of type $\partial_s^{i_1} k \star \partial_s^{i_2} k \star \ldots \star \partial_s^{i_n} k$ where $m = i_1 + \ldots + i_n$ is the total number of derivatives.

The following interpolation inequality for closed curves appears in [1], for our setting with boundary we refer to [3].

Proposition 3.2 *Let* $\gamma : I \to \mathbb{R}^2$ *be a smooth curve. Then for any term* $P_n^m(k)$ *with* $n \ge 2$ *that contains derivatives of k of order at most* $\ell - 1$,

$$\int_I |P_n^m(k)| \, ds \le c L^{1-m-n} \|k\|_2^{n-m} \|k\|_{\ell,2}^p$$

where $p = \frac{1}{\ell} \left(m + \frac{1}{2} n - 1 \right)$ *and* $c = c(\ell, m, n)$. *Moreover, if* $m + \frac{1}{2} < 2\ell + 1$ *then* $p < 2$ *and for any* $\varepsilon > 0$,

$$\int_I |P_n^m(k)| \, ds \le \varepsilon \int_I \left| \partial_s^\ell k \right|^2 ds + c \varepsilon^{\frac{-p}{2-p}} \left(\int_I |k|^2 ds \right)^{\frac{n-p}{2-p}} + c \left(\int_I |k|^2 ds \right)^{m+n-1}.$$

Our first result concerns the winding number of the evolving curve γ. In view of the Neumann boundary condition, in our setting the winding number must be a multiple of $\frac{1}{2}$.

Lemma 3.3 *Under the flow* (3), $\omega(t) = \omega(0)$.

Proof. We compute using Lemma 2.3 (i)

$$\frac{d}{dt}\int k\,ds = \int F_{ss}\,ds + \int k^2F\,ds - \int k^2F\,ds = 0,$$

so ω is constant under the flow. $\square$

Remarks:

- It follows immediately that the average curvature $\overline{k}$ satisfies

$$\overline{k} := \frac{1}{L}\int_\gamma k\,ds \equiv 0$$

  under the flow (3). This is important for applying the inequalities of Lemma 3.1.
- Unlike the situation in [12], here small energy does not automatically imply that the winding number is close to zero. Indeed, one may add loops (or half-loops) of circles that contribute an arbitrarily small amount of the energy $L^3\|k_s\|_2^2$. Note that such loops must all be similarly oriented, as a change in contribution from positive to negative winding will necessitate a quantum of energy (for example a figure-8 style configuration with $\omega = 0$ can not have small energy despite comprising essentially only mollified arcs of circles).

Next we give an estimate on the length of the evolving curve in the case of small initial energy. Of course, this result does not require the energy as small as (7).

Lemma 3.4 *Under the flow* (3) *with* $\omega(0) = 0$,

$$\frac{d}{dt}L[\gamma(t)] \le 0.$$

Proof. We compute using integration by parts

$$\frac{d}{dt}L[\gamma(t)] = -\int kF\,ds = -\int k_{ss}^2\,ds + \frac{7}{2}\int k^2k_s^2\,ds \le -\left[1 - \frac{7L^3}{\pi^3}\|k_s\|_2^2\right]\int k_{ss}^2\,ds$$

where we have used Lemma 3.1. The result follows by the small energy assumption. $\square$

Thus under the small energy assumption we have the length of the evolving curve bounded above and below:

$$|e| \le L[\gamma] \le L_0.$$

We are now ready to show that the L^2-norm of curvature remains bounded, independent of time.

Proposition 3.5 *Under the flow* (3) *with* $\omega(0) = 0$, *there exists a universal* $C > 0$ *such that*

$$\|k\|_2^2 \le \|k\|_2^2\Big|_{t=0} + C.$$

Proof. Using integration by parts, Lemma 2.3 and the interpolation inequality Proposition 3.2

$$\frac{d}{dt}\int k^2 ds = -2\int k_{s^3}^2 ds + 5\int k_{ss}^2 k^2 ds + 5\int k_{ss}k_s^2 k\, ds + \int k_{ss}k^5 ds - \frac{1}{2}\int k_s^2 k^4 ds$$

$$\leq (-2+3\varepsilon)\int k_{s^3}^2 ds + C\|k\|_2^{14} \leq -\frac{\pi^6}{L_0^6}\int k^2 ds + \frac{C\pi^7}{|e|},$$

where we have also used Lemma 3.1 and the length bounds. The result follows. $\square$

Moreover, we may show similarly using the evolution equation for k_{s^ℓ} that all derivatives of curvature are bounded in L^2 independent of time.

Proposition 3.6 *Under the flow* (3) *with* $\omega(0) = 0$, *there exists a universal $C > 0$ such that, for all $\ell \in \mathbb{N}$,*

$$\|k_{s^\ell}\|_2^2 \leq \left.|k_{s^\ell}\|_2^2\right|_{t=0} + C.$$

Pointwise bounds on all derivatives of curvature follow from Lemma 3.1. It follows that the solution of the flow remains smooth up to and including the final time, from which we may (if $T < \infty$) apply again local existence. This shows that the flow exists for all time, that is, $T = \infty$.

4 Exponential convergence

Using Lemma 2.3 (i) and (v) and integrating by parts to reduce the order of the derivatives we obtain

Lemma 4.1 *Under the flow* (3),

$$\frac{d}{dt}\int k_{ss}^2 ds = -2\int k_{s^5}^2 ds + 4\int k^2 k_{s^4}^2 ds - \int k_s^2 k_{s^3}^2 ds - 8\int kk_{ss}k_{s^3}^2 ds - 2\int k^4 k_{s^3}^2 ds$$

$$+ \frac{1}{3}\int k_{ss}^4 ds - \frac{1}{2}\int k^2 k_s^2 k_{ss}^2 ds + 5\int k^3 k_{ss}^3 ds + \frac{8}{5}\int k_s^6 ds.$$

Further integration by parts, use of Lemma 3.1 and throwing away some negative terms gives

Corollary 4.2 *Under the flow* (3) *with* $\omega(0) = 0$,

$$\frac{d}{dt}\int k_{ss}^2 ds \leq \left[-2 + \frac{67L^3}{2\pi^3}\|k_s\|_2^2 + \frac{20L^6}{\pi^6}\|k_s\|_2^4\right]\|k_{s^5}\|_2^2.$$

Under the small energy condition (7), the coefficient of $\|k_{s^5}\|_2^2$ of Corollary 4.2 is bounded above by $-\delta$. Using also Lemma 3.1 we obtain

Corollary 4.3 *There exists a $\delta > 0$ such that, under the flow,*

$$\frac{d}{dt} \|k_{ss}\|_2^2 \leq -\delta \|k_{ss}\|_2^2.$$

It follows that $\|k_{ss}\|_2^2$ decays exponentially to zero.

Proof. **Completion of the proof of Theorem 2.2:** Exponential decay of $\|k_{ss}\|_2^2$ implies exponential decay of $\|k\|_2^2$, $\|k_s\|_2^2$, $\|k\|_\infty$, $\|k_s\|_\infty$ via Lemma 3.1. Exponential decay of $\|k_{s^\ell}\|_2$ and $\|k_{s^\ell}\|_\infty$ then follows by a standard induction argument involving integration by parts and the curvature bounds of Propositions 3.5 and 3.6. That $\|k_s\|_2^2 \to 0$ implies subsequential convergence to straight line segments (horizontal, in view of boundary conditions). A stability argument (see [12] for the details of a similar argument) gives that in fact the limiting straight line is unique; all eigenvalues of the linearised operator

$$\mathscr{L}u = u_{x^6}$$

are negative apart from the first zero eigenvalue, which corresponds precisely to vertical translations. By Hale-Raugel's convergence theorem [6] uniqueness of the limit follows. Although we don't know the precise height of the limiting straight line segment, we can estimate a-priori its distance from the initial curve, since

$$|\gamma(x,t) - \gamma(x,0)| = \left| \int_0^t \frac{\partial \gamma}{\partial t}(x,\tau)\, d\tau \right| \leq \int_0^t |F|\, d\tau \leq \frac{C}{\delta}\left(1 - e^{-\delta t}\right).$$

$\square$

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