Number of claims and ruin time for a refracted risk process

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Abstract In this paper, we consider a classical risk model refracted at given level. We give an explicit expression for the joint density of the ruin time and the cumulative number of claims counted up to ruin time. The proof is based on solving some integro-differential equations and employing the Lagrange's Expansion Theorem.

1 Introduction

Between 20.11.2017 and 8.12.2017 an international research institute MATRIX in Creswick, Australia, run research program *Mathematics of Risk* during which four five-hour workshops were given. In particular, Z. Palmowski presented a workshop entitled *Ruin probabilities: exact and asymptotic results*. This paper is closely related with the topics introduced during his lectures.

The joint density of the ruin time and the numbers of claims counted until ruin time has been already studied for a classical risk process over last years. Dickson [3] derived special expression for it using probabilistic arguments. Landriault et al. [11] analyzed this object for the Sparre Andersen risk model with the exponential claims. Later Frostig et al. [6] generalized it to the case of a renewal risk model with the

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phase-type claims and inter-arrival times. The main tool used there was the duality between the risk model and a workload of a single server queueing model. Zhao and Zhang [20] considered a delayed renewal risk model, where the claim size is Erlang(n) distributed and the inter-arrival time is assumed to be infinitely divisible.

Our goal is to derive expression for the joint density of the ruin time and the numbers of claims counted until ruin time for a refracted classical risk process (see Kyrianou and Loeffen [9] for a formal definition). It is also called a compound Poisson risk model under a threshold strategy. The latter process is a classical risk process whose dynamic is changed by subtracting off a fixed linear drift whenever the cumulative risk process is above a pre-specified level b. This subtracting of the linear drift corresponds to the dividend payments and the considered strategy is also known as a threshold strategy. Dividend strategies for insurance risk models were first proposed by De Finetti [2] to reflect more realistically the surplus cash flows in an insurance portfolio. More recently, many kind of risk related quantities under threshold dividend strategies have been studied by Lin and Pavlova [17], Zhu and Yang [22], Lu and Li [14], [15], [16], Badescu, Drekic and Landriault [1], Gao and Yin [7] (see references therein). The case when the drift of the refracted process is disappearing (everything above threshold b is paid as dividends) is called barrier strategy, see Lin et al. [18], Li and Garrido [12], Zhou [21] and in the references therein.

The paper is organized as follows. In Section 2 we define the model we deal with in this paper. In Section 3 we recall properties of the translation operator and the root of the Lundberg fundamental equation. In particular, we introduce the Lagrange's expansion theorem and some notation. In Section 4 we construct two integro-differential equations identifying the joint Laplace transform of joint density of the numbers of claims counted up to ruin time and the ruin time. Analytical solutions of these two integro-differential equations are given in Section 5. Applying the Lagrange's expansion theorem in Section 6 we give the expression for above mentioned density.

2 Model

The classical risk process is given by

$$U(t) = u + c_1 t - S(t), \tag{1}$$

where $U(0) = u$ denotes initial capital, c is the premium rate and $S(t) = \sum_{i=1}^{N_t} X_i$ represents the total amount of claims appeared up to time $t \geq 0$. That is, $\{X_i\}_{\{i \in \mathbb{N}\}}$ are non-negative i.i.d. random variables with pdf $f(x)$ and cdf $F(x)$ and $\{N_t\}_{\{t \geq 0\}}$ is an independent Poisson process with a parameter λ. To take into account dividend payments paid when regulated process (after deduction of dividends) is above fixed threshold level $b > 0$, we consider so-called refracted process given formally for $c_2 < c_1$ by:

$$dU_b(t) = \begin{cases} c_1 dt - dS(t), \, 0 \le U_b(t) \le b \\ c_2 dt - dS(t), \, U_b(t) > b \end{cases} \tag{2}$$

and $U_b(0) = u$. In this case $c_1 - c_2$ denotes intensity of dividend payments, see Figure 1.
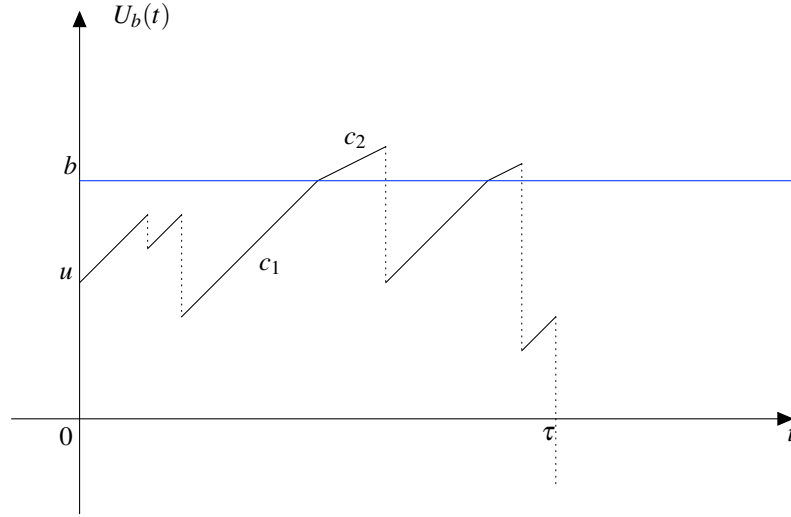

Fig. 1 : Graphical representation of the surplus process $U_b(t)$.

Throughout this paper, we will assume that $c_2 > \lambda EX_1$, which means refracted process $U_b(t)$ tends to infinity almost surely. We can then consider the ruin time:

$$\tau = \inf\{t > 0, U_b(t) < 0\},$$

($\tau = \infty$ if ruin does not occur). Note that N_τ represents the number of claims counted until the ruin time. The main goal of this paper is identification of the density of (τ, N_τ). We start from analyzing its Laplace transform:

$$\phi(u) = E[r^{N_\tau} e^{-\delta \tau} \mathbb{I}(\tau < \infty)|U_b(0) = u] \tag{3}$$

$$= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} w(u,n,t) dt, \tag{4}$$

where

$$w(u,n,t) = P(N_\tau = n, \tau \in dt|U_b(0) = u)/dt$$

is the joint density of (τ, N_τ) when $U_b(0) = u$. In above definition we have $\delta > 0$ and $r \in (0,1]$. Later we will use the following notation

$$w_1(u,n,t) = w(u,n,t) \qquad \text{for } u \le b$$

and

$$w_2(u,n,t) = w(u,n,t) \qquad \text{for } u > b.$$

3 Preliminaries

In this section we introduce few facts used further in this paper. We start from recalling the translation operator T_s; see Dickson and Hipp [4]. For any integrable real-valued function f it is defined as

$$T_s f(x) = \int_x^\infty e^{-s(y-x)} f(y) dy, \ \ x \geq 0.$$

The operator T_s satisfies the following properties:

1. $T_s f(0) = \int_0^\infty e^{-sx} f(x) dx = \hat{f}(s)$ which is the Laplcae transform of f;
2. The operator T_s is commutative, i.e. $T_s T_r = T_r T_s$. Moreover, for $s \neq r$ and $x \geq 0$

$$T_s T_r f(x) = T_r T_s f(x) = \frac{T_s f(x) - T_r f(x)}{r - s}. \tag{5}$$

More properties of the translation operator T_s can be found in Li and Garrido [13] and Gerber and Shiu [8].

For any function g we will denote by $\hat{g}(s)$ its Laplace Transform, that is $\hat{g}(s) = \int_0^\infty e^{-sx} g(x) \, dx$. Next, for $i = 1, 2$ let ρ_i be the positive root of the Lundberg fundamental equation

$$c_i s - (\lambda + \delta) + \lambda r \hat{f}(s) = 0. \tag{6}$$

The positive roots always exists for $\delta > 0$; see Figure 2.

Lagrange's Expansion Theorem. In this paper we will also use the Lagrange's Expansion Theorem; see pages 251-326 of Lagrange [10]. Given two functions $\alpha(z)$ and $\beta(z)$ which are both analytic on and inside a contour D surrounding a point a, if r satisfies the inequality

$$|r\beta(z)| < |z - a|, \tag{7}$$

for every z on the perimeter of D, then $z - a - r\varphi(z)$, as a function of z, has exactly one zero η in the interior of D, and we further have:

$$\alpha(\eta) = \alpha(a) + \sum_{k=1}^\infty \frac{r^k}{k!} \frac{d^{k-1}}{dx^{k-1}} \left(\alpha'(x)\beta^k(x) \right)\big|_{x=a}. \tag{8}$$

Finally, we define also the impulse function

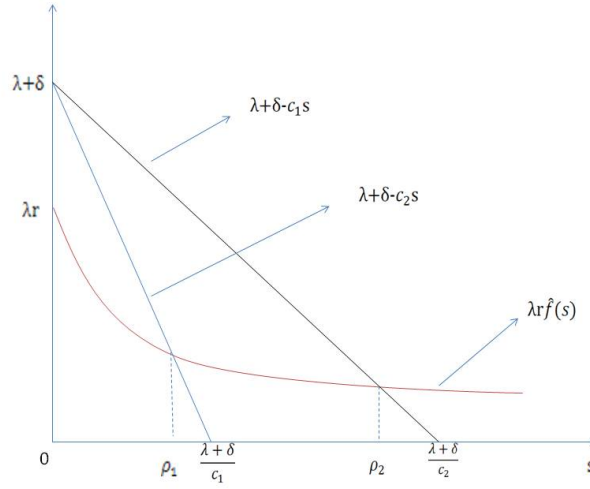$$\delta_x(t) = \begin{cases} 0, & t \neq x \\ \infty, & t = x \end{cases}$$

Fig. 2: Roots for Lundberg's fundamental equation.

with $\int_0^\infty \delta_x(t)dt = 1$. We denote $g^{k*}, k \geq 0$, with $g^{1*} = g$ and $g^{0*}(t) = \delta_0(t)$ the k-fold convolution of g with itself, where

$$(g*h)(t) = \int_0^t g(x)h(t-x)dx, \quad t \geq 0$$

for two functions g and h supported on $(0, \infty)$.

4 Integro-differential equations for the joint Laplace transform

In this section, we derive two integro-differential equations identifying $\phi(u)$ defined in (3). We will follow the idea given in Lin and Pavlova [17]. Denote

$$\phi(u) = \begin{cases} \phi_1(u), & u \leq b, \\ \phi_2(u), & u > b. \end{cases} \tag{9}$$

Theorem 1. *The joint Laplace transform ϕ satisfies the following integro-differential equations:*

$$\begin{cases} \phi_1'(u) = \frac{\lambda+\delta}{c_1}\phi_1(u) - \frac{\lambda r}{c_1}\int_0^u \phi_1(u-x)f(x)dx - \frac{\lambda r}{c_1}\bar{F}(u), & 0 \le u \le b \\ \phi_2'(u) = \frac{\lambda+\delta}{c_2}\phi_2(u) - \frac{\lambda r}{c_2}\left(\int_0^{u-b}\phi_2(u-x)f(x)dx + \int_{u-b}^u \phi_1(u-x)f(x)dx\right) - \frac{\lambda r}{c_2}\bar{F}(u), & u > b \end{cases}$$

$$\text{(10)}$$

with the boundary condition

$$\phi_1(b) = \phi_2(b) := \lim_{u \to b^+} \phi_2(u). \tag{11}$$

Remark 1. Note that from the integro-differential equations (10) follows that the joint Laplace transform with initial surplus above the barrier depends on the respective function with initial surplus below the barrier, but the reverse relationship does not hold true.

Proof. Let first $0 \le u \le b$. Then conditioning on the occurrence of the first claim we will have two cases: the first claim occurs before the surplus has reached the barrier level b or it occurs after reaching this barrier. There are also two other cases at the moment of the arrival of the first claim: either the risk process starts all over again with new initial surplus or the first claim leads already to ruin. Hence:

$$
\begin{aligned}
\phi(u) &= \phi_1(u) \\
&= \int_0^{\frac{b-u}{c_1}} \lambda r e^{-\lambda t} e^{-\delta t} \left(\int_0^{u+c_1 t} \phi(u+c_1 t - x)f(x)dx + \bar{F}(u+c_1 t) \right) dt \\
&\quad + \int_{\frac{b-u}{c_1}}^{\infty} \lambda r e^{-\lambda t} e^{-\delta t} \left(\int_0^{b+c_2(t-\frac{b-u}{c_1})} \phi(b+c_2(t-\frac{b-u}{c_1}) - x)f(x)dx + \bar{F}(b+c_2(t-\frac{b-u}{c_1})) \right) dt \\
&= \lambda r \int_0^{\frac{b-u}{c_1}} e^{-(\lambda+\delta)t}\gamma(u+c_1 t)dt + \lambda r \int_{\frac{b-u}{c_1}}^{\infty} e^{-(\lambda+\delta)t}\gamma(b+c_2(t-\frac{b-u}{c_1}))dt, \tag{12}
\end{aligned}
$$

where $\gamma(t) = \int_0^t \phi(t-x)f(x)dx + \bar{F}(t)$.
Changing variables in (12) and rearranging leads to the following equation for $0 \le u \le b$:

$$\phi_1(u) = \frac{\lambda r}{c_1}e^{(\lambda+\delta)u/c_1}\int_u^b e^{-(\lambda+\delta)t/c_1}\gamma(t)dt + \frac{\lambda r}{c_2}e^{(\lambda+\delta)u/c_1}\int_b^{\infty} e^{-(\lambda+\delta)[t-(c_1-c_2)b/c_1]/c_2}\gamma(t)dt. \tag{13}$$

Differentiating both sides of (13) with respect to u yields first equation.

Similarly, for $u > b$ we have:

$$
\begin{aligned}
\phi(u) &= \phi_2(u) \\
&= \int_0^{\infty} \lambda r e^{-\lambda t} e^{-\delta t} \left(\int_0^{u+c_2 t} \phi(u+c_2 t - x)f(x)dx + \bar{F}(u+c_2 t) \right) dt \\
&= \lambda r \int_0^{\infty} e^{-(\lambda+\delta)t}\gamma(u+c_2 t)dt \\
&= \frac{\lambda r}{c_2}e^{(\lambda+\delta)u/c_2}\int_u^{\infty} e^{-(\lambda+\delta)t/c_2}\gamma(t)dt. \tag{14}
\end{aligned}
$$

Differentiating both sides of (14) with respect to u produces the second equation.

Note also that from equations (13) and (14) it follows that $\phi(u)$ is continuous at $u = b$ and hence (11) holds. This completes the proof.

5 The analytical expression for $\phi(u)$

In this section, we derive the analytical expression for $\phi_i(u)$ ($i = 1, 2$) using the translation operator introduced in Section 3.

Theorem 2. *The function $\phi_2(u)$ can be expressed analytically as follows:*

$$\phi_2(u) = \sum_{n=0}^{\infty} \left(\frac{\lambda r}{c_2}\right)^{n+1} (T_{\rho_2} f)^{n*} * h(u - b), \quad u > b, \tag{15}$$

where

$$h(u) := \int_u^{u+b} \phi_1(u + b - x) T_{\rho_2} f(x) dx + T_{\rho_2} \bar{F}(u + b). \tag{16}$$

Proof. We adopt the approach of Willmot and Dickson [19]. Consider the second equation in (10) for $u > b$. For a fixed $s > 0$, we multiply both sides of this equation by $e^{-s(u-b)}$ and integrate it with respect to u from b to ∞:

$$c_2 \int_b^{\infty} e^{-s(u-b)} \phi_2'(u) du$$

$$= (\lambda + \delta) T_s \phi_2(b) - \lambda r \int_b^{\infty} e^{-s(u-b)} \int_0^{u-b} \phi_2(u - x) f(x) dx du$$

$$\quad - \lambda r \int_b^{\infty} e^{-s(u-b)} \int_0^b \phi_1(y) f(u - y) dy du - \lambda r T_s \bar{F}(b)$$

$$= (\lambda + \delta) T_s \phi_2(b) - \lambda r \int_0^{\infty} e^{-sx} f(x) \int_{x+b}^{\infty} e^{-s(u-x-b)} \phi_2(u - x) du dx$$

$$\quad - \lambda r \int_0^b \phi_1(y) \int_b^{\infty} e^{-s(u-b)} f(u - y) du dy - \lambda r T_s \bar{F}(b)$$

$$= (\lambda + \delta) T_s \phi_2(b) - \lambda r \hat{f}(s) T_s \phi_2(b) - \lambda r \int_0^b \phi_1(y) T_s f(b - y) dy - \lambda r T_s \bar{F}(b).$$

Integrating by parts gives:

$$c_2 \int_b^{\infty} e^{-s(u-b)} \phi_2'(u) du = c_2 s T_s \phi_2(b) - c_2 \phi_2(b).$$

Hence

$$c_2 s T_s \phi_2(b) - c_2 \phi_2(b)$$

$$= (\lambda + \delta) T_s \phi_2(b) - \lambda r \hat{f}(s) T_s \phi_2(b) - \lambda r \int_0^b \phi_1(y) T_s f(b - y) dy - \lambda r T_s \bar{F}(b)$$

and simple rearranging leads to:

$$(c_2 s - (\lambda + \delta) + \lambda r \hat{f}(s)) T_s \phi_2(b) = c_2 \phi_2(b) - \lambda r \int_0^b \phi_1(y) T_s f(b - y) dy - \lambda r T_s \bar{F}(b).$$
(17)

Taking $s = \rho_2$ for the solution ρ_2 of the Lundberg Fundamental Equation (6) gives

$$c_2 \phi_2(b) = \lambda r \int_0^b \phi_1(y) T_{\rho_2} f(b - y) dy + \lambda r T_{\rho_2} \bar{F}(b).$$

Then equation (17) is equivalent to:

$$[c_2(s - \rho_2) + \lambda r \hat{f}(s) - \lambda r \hat{f}(\rho_2)] T_s \phi_2(b)$$
$$= \lambda r \int_0^b \phi_1(y) [T_{\rho_2} f(b - y) - T_s f(b - y)] dy + \lambda r [T_{\rho_2} \bar{F}(b) - T_s \bar{F}(b)].$$

Now dividing above equation by $s - \rho_2$ and using property 2 of the translation operator introduced in Section 2 produces:

$$c_2 T_s \phi_2(b) = \lambda r T_s T_{\rho_2} f(0) T_s \phi_2(b) + \lambda r \int_0^b \phi_1(y) T_s T_{\rho_2} f(b - y) dy + \lambda r T_s T_{\rho_2} \bar{F}(b).$$
(18)

Inverting the translation operators of (18) yields the following renewal equation for $\phi_2(u)$:

$$\phi_2(u) = \frac{\lambda r}{c_2} \left[\int_0^{u-b} \phi_2(u - x) T_{\rho_2} f(x) dx + \int_{u-b}^u \phi_1(u - x) T_{\rho_2} f(x) dx + T_{\rho_2} \bar{F}(u) \right].$$
(19)

Taking $y = u - b$ and $g(y) = \phi_2(y + b)$ we can rewrite (19) as follows:

$$g(y) = \frac{\lambda r}{c_2} \int_0^y g(y - x) T_{\rho_2} f(x) dx + \frac{\lambda r}{c_2} h(y), \qquad y > 0,$$

where

$$h(y) = h(u - b) = \int_{u-b}^u \phi_1(u - x) T_{\rho_2} f(x) dx + T_{\rho_2} \bar{F}(u), \quad u > b.$$

Hence

$$\phi_2(u) = g(y)$$
$$= \frac{\lambda r}{c_2} \int_0^y g(y - x) T_{\rho_2} f(x) dx + \frac{\lambda r}{c_2} h(y)$$
$$= \sum_{n=0}^\infty \left(\frac{\lambda r}{c_2} \right)^{n+1} (T_{\rho_2} f)^{n*} * h(y)$$
$$= \sum_{n=0}^\infty \left(\frac{\lambda r}{c_2} \right)^{n+1} (T_{\rho_2} f)^{n*} * h(u - b)$$

which completes the proof.

The expression for $\phi_1(u)$ could be also derived in terms of the translation operator.

Theorem 3. *The function $\phi_1(u)$ can be expressed analytically in the following form:*

$$\phi_1(u) = \phi_\infty(u) + \frac{\frac{\lambda r}{c_2}\left[\phi_\infty * T_{\rho_2}f(b) + T_{\rho_2}\bar{F}(b)\right] - \phi_\infty(b)}{v(b) - \frac{\lambda r}{c_2}v * T_{\rho_2}f(b)}v(u), \qquad (20)$$

where

$$\phi_\infty(u) := \sum_{n=0}^{\infty}\left(\frac{\lambda r}{c_1}\right)^{n+1}(T_{\rho_1}f)^{n*} * T_{\rho_1}\bar{F}(u) \qquad (21)$$

and

$$v(x) := \sum_{n=0}^{\infty}\left(\frac{\lambda r}{c_1}\right)^{n}(T_{\rho_1}f)^{n*} * p(x) \qquad (22)$$

with $p(x) = e^{\rho_1 x}$.

Proof. We will follow Landriault et al. [11]. Note that the first equation in (10) does not involve the barrier level b:

$$\phi_1'(u) = \frac{\lambda + \delta}{c_1}\phi_1(u) - \frac{\lambda r}{c_1}\int_0^u \phi_1(u-x)f(x)dx - \frac{\lambda r}{c_1}\bar{F}(u). \qquad (23)$$

The information about the barrier b is included in the boundary condition:

$$\phi_1(b) = \phi_2(b) := \lim_{u \to b^+}\phi_2(u).$$

Lin et al. [17] showed that the general solution of (23) is of the form

$$\phi_1(u) = \phi_\infty(u) + kv(u), \qquad (24)$$

where $\phi_\infty(u)$ is the joint Laplace transform of density of the ruin time and number of claims counted up to ruin time for the classical risk process (1) without any barrier applied. That is,

$$\phi_\infty(u) := \sum_{n=1}^{\infty}r^n\int_0^{\infty}e^{-\delta t}w_\infty(u,n,t)dt \qquad (25)$$

for

$$w_\infty(u,n,t) := P(N_\tau = n, \tau \in dt | U(0) = u)/dt. \qquad (26)$$

In above equation (24) the quantity k is a constant which we can specify by implementing (24) and (19):

$$k = \frac{\frac{\lambda r}{c_2}\left[\int_0^b \phi_\infty(b-x)T_{\rho_2}f(x)dx + T_{\rho_2}\bar{F}(b)\right] - \phi_\infty(b)}{v(b) - \frac{\lambda r}{c_2}\int_0^b v(b-x)T_{\rho_2}f(x)dx}. \qquad (27)$$

We express now the function ϕ_∞ in terms of a compound geometric distribution. Indeed, since ϕ_∞ also satisfies equation (23), taking Laplace transforms of its both sides for sufficiently large s gives:

$$(c_1 s - (\lambda + \delta) + \lambda r \hat{f}(s)) \hat{\phi}_\infty(s) = c_1 \phi_\infty(0) - \lambda r \hat{\bar{F}}(s), \ s \geq 0. \qquad (28)$$

To determine the constant term $c_1 \phi_\infty(0)$ in (28), we substitute the solution ρ_1 of the Lundberg Fundamental Equation (6) for s:

$$c_1 \phi_\infty(0) = \lambda r \hat{\bar{F}}(\rho_1) = \lambda r T_{\rho_1} \hat{\bar{F}}(0). \qquad (29)$$

Consequently, the equation (28) reduces to

$$[c_1(s - \rho_1) + \lambda r \hat{f}(s) - \lambda r \hat{f}(\rho_1)] \hat{\phi}_\infty(s) = \lambda r \hat{\bar{F}}(\rho_1) - \lambda r \hat{\bar{F}}(s).$$

Dividing above equation by $s - \rho_1$ and simple rearranging along with implementation of the formula (5) produces:

$$c_1 \hat{\phi}_\infty(s) = \lambda r \hat{\phi}_\infty(s) T_s T_{\rho_1} f(0) + \lambda r T_s T_{\rho_1} \bar{F}(0).$$

Inverting this Laplace transforms gives classical renewal equation:

$$\phi_\infty(u) = \frac{\lambda r}{c_1} \phi_\infty * T_{\rho_1} f(u) + \frac{\lambda r}{c_1} T_{\rho_1} \bar{F}(u) \qquad (30)$$

having the solution given as an Neumann infinite series (21).

To prove the last statement (22) note that the function $v(u)$ satisfies the following integro-differential equation:

$$c_1 v'(u) - (\lambda + \delta) v(u) + \lambda r \int_0^u v(u - x) f(x) dx = 0, \quad u \geq 0, \qquad (31)$$

with the initial condition $v(0) = 1$. To get the analytical expression of $v(u)$ we take the Laplace transforms of both sides of (31) for sufficiently large s ($s > \rho_1$). This yields:

$$c_1 s \hat{v}(s) - c_1 v(0) = (\lambda + \delta) \hat{v}(s) - \lambda r \hat{f}(s) \hat{v}(s).$$

Since $v(0) = 1$,

$$(s + \frac{\lambda r}{c_1} \hat{f}(s) - \frac{\lambda + \delta}{c_1}) \hat{v}(s) = 1. \qquad (32)$$

Recalling that ρ_1 is the root of (6), we can rewrite (32) as

$$(s - \rho_1 + \frac{\lambda r}{c_1} [\hat{f}(s) - \hat{f}(\rho_1)]) \hat{v}(s) = 1,$$

which, by dividing by $s - \rho_1$ and implementing (5), produces:

$$\hat{v}(s) = \frac{\lambda r}{c_1} \hat{v}(s) T_s T_{\rho_1} f(0) + \frac{1}{s - \rho_1}. \tag{33}$$

Inverting the Laplace transforms in (33) leads to the equation (22). Including all above identities in (24) completes the proof.

6 The joint density of (τ, N_τ)

In this section we give the joint density of the number of claims counted until ruin time and the ruin time using the Lagrange's Expansion theorem. We start with few facts that will be useful in the proof of the main result.

Recall that by $w_\infty(u, n, t)$ we denote the joint density of (τ, N_τ) for the classical risk process (1) (with infinite barrier $b = +\infty$); see (26). For $i = 1, 2$ we denote

$$g_i(x, 0, t) := \delta_{x/c_i}(t) e^{-\lambda x/c_i},$$
$$g_i(x, n, t) := x t^{n-1} e^{-\lambda t} \lambda^n f^{n*}(c_i t - x)/n!.$$

Following Dickson [3] we can state the following lemma.

Lemma 1. *We have*
$$w_\infty(u, 1, t) = \lambda e^{-\lambda t} \bar{F}(u + c_1 t).$$

For $n = 1, 2, 3, \ldots$ the following holds:

$$w_\infty(u, n+1, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \int_0^{u+c_1 t} f^{n*}(u + c_1 t - x) \lambda \bar{F}(x) dx$$
$$- c_1 \sum_{j=1}^n \int_0^t \frac{(\lambda s)^j}{j!} e^{-\lambda s} f^{j*}(u + c_1 s) w_\infty(0, n+1-j, t-s)) ds \tag{34}$$

where

$$w_\infty(0, n, t) = \frac{\lambda}{c_1} \int_0^{c_1 t} \bar{F}(x) g_1(x, n-1, t) dx, \quad n = 1, 2, \ldots. \tag{35}$$

Proof. Using Lagrange's Expansion Theorem presented in Section 2 with $\alpha(z) = e^{-zx}$, $\beta(z) = -\frac{\lambda}{c_i} \hat{f}(s)$, $a = (\lambda + \delta)/c_i$ and $D = \{z \mid |z - a| \leq a\}$ $(i = 1, 2)$ and the Lundberg fundamental equation (6) we can conclude the following identity:

$$e^{-\rho_i x} = e^{-(\lambda+\delta)x/c_i} + \sum_{n=1}^\infty \frac{r^n}{n!} \frac{d^{n-1}}{ds^{n-1}} \left(-x e^{-sx} \left(-\frac{\lambda}{c_i} \hat{f}(s) \right)^n \right) \Big|_{s=(\lambda+\delta)/c_i}$$

$$= e^{-(\lambda+\delta)x/c_i} + \sum_{n=1}^\infty \frac{r^n}{n!} \frac{d^{n-1}}{ds^{n-1}} \left((-1)^{n+1} \lambda^n x/c_i^n \int_0^\infty e^{-s(x+y)} f^{n*}(y) dy \right) \Big|_{s=(\lambda+\delta)/c_i}$$

$$= e^{-(\lambda+\delta)x/c_i} + \sum_{n=1}^\infty \frac{\lambda^n r^n}{n! c_i^n} \int_0^\infty x(x+y)^{n-1} e^{-(\lambda+\delta)(x+y)/c_i} f^{n*}(y) dy.$$

Substituting $t := (x+y)/c_i$ and rearranging leads to:

$$e^{-\rho_i x} = e^{-(\lambda+\delta)x/c_i} + \sum_{n=1}^{\infty} r^n \frac{\lambda^n}{n!} \int_{x/c_i}^{\infty} x t^{n-1} e^{-\lambda t} e^{-\delta t} f^{n*}(c_i t - x) dt$$

$$= \sum_{n=0}^{\infty} r^n \int_{x/c_i}^{\infty} e^{-\delta t} g_i(x,n,t) dt. \tag{36}$$

Therefore,

$$T_{\rho_i} f(x) = \int_x^{\infty} e^{-\rho_i(u-x)} f(u) du$$

$$= \int_x^{\infty} \sum_{n=0}^{\infty} r^n \int_{(u-x)/c_i}^{\infty} e^{-\delta t} g_i(u-x,n,t) dt f(u) du$$

$$= \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_x^{c_i t + x} f(u) g_i(u-x,n,t) du dt. \tag{37}$$

Since $\phi_\infty(u)$ defined in (25) is the joint Laplace transform under the classical compound Poisson risk model without a barrier we can use Dickson [3] to complete the proof.

Moreover, the following result holds true.

Lemma 2. *The function $v(u)$ given in (22) equals*

$$v(u) = \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \varpi(u,n,t) dt, \tag{38}$$

where

$$\varpi(u,0,t) := g_1(-u,0,t),$$

$$\varpi(u,n,t) := \sum_{m=1}^{n} \left(\frac{\lambda}{c_1}\right)^m \int_0^{c_1 t} \int_0^u g_{c_1}(y,n-m,t) b_m(u-x,y+x) dx dy + g_{c_1}(-u,n,t), \quad n \geq 1$$

$$b_n(u,y) := \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \int_0^u (u-x)^{n-1} f^{(n-j)*}(y+u-x) f^{j*}(x) dx.$$

Proof. Our goal is to express $v(u)$ as the Laplace transform:

$$v(u) = \int_0^{\infty} e^{-\rho_1 t} \xi(u,t) dt. \tag{39}$$

We start from definition (22):

$$v(u) = \sum_{n=0}^{\infty} \left(\frac{\lambda r}{c_1}\right)^n (T_{\rho_1} f)^{n*} * p(u)$$

$$= \sum_{n=1}^{\infty} \left(\frac{\lambda r}{c_1} \right)^n \int_0^u (T_{\rho_1} f)^{n*}(u-x) e^{\rho_1 x} dx + e^{\rho_1 u}. \tag{40}$$

Using Dickson and Willmot [5] we can obtain the following representation:

$$(T_{\rho_i} f)^{n*}(u) = \int_0^{\infty} e^{-\rho_i y} b_n(u,y) dy \tag{41}$$

for

$$b_n(u,y) := \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \int_0^u (u-x)^{n-1} f^{(n-j)*}(y+u-x) f^{j*}(x) dx.$$

By (40)

$$v(u) = \sum_{n=1}^{\infty} \left(\frac{\lambda r}{c_1} \right)^n \int_0^u \int_0^{\infty} e^{-\rho_1 y} b_n(u-x,y) dy e^{\rho_1 x} dx + e^{\rho_1 u}$$

$$= \sum_{n=1}^{\infty} \left(\frac{\lambda r}{c_1} \right)^n \int_0^{\infty} e^{-\rho_1 t} \int_0^u b_n(u-x,t+x) dx dt$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\lambda r}{c_1} \right)^n \int_{-u}^0 e^{-\rho_1 t} \int_{-t}^u b_n(u-x,t+x) dx dt + \int_0^{\infty} e^{-\rho_1 t} \delta_{-u}(t) dt.$$

Comparing the coefficients of $e^{-\rho_1 t}$ in (39) gives:

$$\xi(u,t) = \sum_{n=1}^{\infty} \left(\frac{\lambda r}{c_1} \right)^n \int_0^u b_n(u-x,t+x) dx + \delta_{-u}(t); \tag{42}$$

see also [11]. Using (36) and (42) in (39) we end up with:

$$v(u) = \int_0^{\infty} e^{-\rho_1 y} \xi(u,y) dy + e^{\rho_1 u}$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} r^n \int_{y/c_1}^{\infty} e^{-\delta t} g_{c_1}(y,n,t) dt \, \xi(u,y) dy$$

$$= \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_0^{c_1 t} g_{c_1}(y,n,t) \xi(u,y) dy dt$$

$$= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \left(\sum_{m=1}^n (\frac{\lambda}{c_1})^m \int_0^{c_1 t} \int_0^u g_{c_1}(y,n-m,t) b_m(u-x,y+x) dx dy + g_{c_1}(-u,n,t) \right) dt$$

$$+ \int_0^{\infty} e^{-\delta t} g_{c_1}(-u,0,t) dt$$

which completes the proof.

Using above lemmas we will prove the main result of this paper.

Theorem 4. *For $0 \leq u \leq b$ and $m > 1$ the joint density of the number of claims until ruin N_τ and the time to ruin τ is given by*

$$w_1(u,1,t) = \frac{\lambda}{c_2} e^{-\lambda t} \bar{F}(c_2 t + b + \frac{c_2}{c_1}(u-b))$$

$$w_1(u,m,t) = e^{-\frac{\lambda b}{c_1}} \left[\sum_{n=1}^{m} \vartheta(u,m,n,t-\frac{b}{c_1}) - \sum_{n=1}^{m-1} \int_0^{t-\frac{b}{c_1}} \varsigma(b,m-n,t-\frac{b}{c_1}-z)w_1(u,n,z)dz \right],$$

(43)

where for $n \geq 1$

$$\varsigma(b,0,t) := \varpi(b,0,t) = g_{c_1}(-b,0,t),$$

$$\varsigma(b,n,t) := \varpi(b,n,t) - \sum_{m=0}^{n-1} \frac{\lambda}{c_2} \int_0^b \int_0^t \varpi(b-x,n-1-m,t-z) \int_x^{c_2 z+x} f(y)g_2(y-x,m,z)dydzdx,$$

$$\gamma(b,1,t) := \frac{\lambda}{c_2} \int_b^{c_2 t+b} \bar{F}(y)g_2(y-b,0,t)dy,$$

$$\gamma(b,n,t) := \sum_{m=0}^{n-2} \frac{\lambda}{c_2} \int_0^b \int_0^t w_\infty(b-x,n-m-1,t-z) \int_x^{c_2 z+x} f(y)g_2(y-x,m,z)dydzdx,$$

$$\vartheta(u,m,n,t) := \int_0^t \varsigma(b,m-n,t-z)w_\infty(u,n,z) + (\gamma(b,n,t-z) - w_\infty(b,n,t-z))\varpi(u,m-n,z)dz.$$

Proof. In order to get the joint density $w(u,n,t)$, we have to take inverse Laplace transform with respect to δ rather than ρ_1 and ρ_2. To do this we must find firstly the relationship between transforms with respect to ρ_1, ρ_2 and δ by applying the Lagrange's Expansion theorem. For convenience, we will denote:

$$\chi(b) := v(b) - \frac{\lambda r}{c_2} v * T_{\rho_2} f(b).$$

(44)

Then we can rewrite (20) as follows:

$$\chi(b)\phi_1(u) = \chi(b)\phi_\infty(u) + \frac{\lambda r}{c_2} \left[\phi_\infty * T_{\rho_2} f(b) + T_{\rho_2} \bar{F}(b) \right] v(u) - \phi_\infty(b)v(u).$$

(45)

Putting (37) and (38) into (44) we will derive:

$$\chi(b) = \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \varpi(b,n,t)dt - \frac{\lambda r}{c_2} \int_0^b v(b-x)T_{\rho_2} f(x)dx$$

$$= \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \varpi(b,n,t)dt - \frac{\lambda r}{c_2} \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} \int_0^b \int_0^{\infty} e^{-\delta t} \varpi(b-x,n-m,t)dt \int_0^{\infty} e^{-\delta z}$$

$$\int_x^{c_2 z+x} f(y)g_2(y-x,m,z)dydzdx$$

$$= \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \varpi(b,n,t)dt - \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \{ \sum_{m=0}^{n-1} \frac{\lambda}{c_2} \int_0^b \int_0^t \varpi(b-x,n-1-m,t-z)$$

$$\int_x^{c_2 z+x} f(y)g_2(y-x,m,z)dydzdx\}dt$$

$$= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \{ \varpi(b,n,t) - \sum_{m=0}^{n-1} \frac{\lambda}{c_2} \int_0^b \int_0^t \varpi(b-x,n-1-m,t-z) \int_x^{c_2 z+x} f(y)$$

$$g_2(y-x,m,z)dydzdx\}dt + \int_0^{\infty} e^{-\delta t} \varpi(b,0,t)dt$$

$$= \sum_{n=0}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \varsigma(b,n,t)dt. \tag{46}$$

Similarly, using Lemma 1, we can check that:

$$\frac{\lambda r}{c_2} \left[\phi_{\infty} * T_{\rho_2} f(b) + T_{\rho_2} \bar{F}(b) \right] = \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \gamma(b,n,t)dt. \tag{47}$$

Using (38), (46) and (47) in (45) we obtain:

$$\sum_{m=1}^{\infty} r^m \int_0^{\infty} e^{-\delta t} \sum_{n=1}^{m} \int_0^t \varsigma(b,m-n,t-z)(w_1(u,n,z) - w_{\infty}(u,n,z))\,dzdt$$

$$= \sum_{m=1}^{\infty} r^m \int_0^{\infty} e^{-\delta t} \sum_{n=1}^{m} \int_0^t (\gamma(b,n,t-z) - w_{\infty}(b,n,t-z))\varpi(u,m-n,z)dzdt$$

or equivalently that

$$\sum_{n=1}^{m} \int_0^t \varsigma(b,m-n,t-z)w_1(u,n,z)dz$$

$$= \sum_{n=1}^{m} \int_0^t \varsigma(b,m-n,t-z)w_{\infty}(u,n,z)$$

$$+ (\gamma(b,n,t-z) - w_{\infty}(b,n,t-z))\varpi(u,m-n,z)dz.$$

Now, if $m=1$ then

$$\int_0^t \varsigma(b,0,t-z)w_1(u,1,z)dz = \vartheta(u,1,1,t).$$

In this case

$$\int_0^t \delta_{-b/c_1}(t-z)e^{\frac{\lambda b}{c_1}}w_1(u,1,z)dz = e^{\frac{\lambda b}{c_1}}w_1(u,1,t+\frac{b}{c_1})$$

$$= \frac{\lambda}{c_2}e^{-\lambda t}\bar{F}(c_2 t+b+\frac{c_2}{c_1}u)$$

and

$$w_1(u,1,t) = \frac{\lambda}{c_2} e^{-\lambda(t+\frac{b}{c_1})} \bar{F}(c_2 t + b + \frac{c_2}{c_1}(u-b)).$$

Similarly, if $m = 2$ then

$$\int_0^t \delta_{-b/c_1}(t-z) e^{\frac{\lambda b}{c_1}} w_1(u,2,z) dz = \sum_{n=1}^2 \vartheta(u,2,n,t) - \int_0^t \varsigma(b,1,t-z) w_1(u,1,z) dz$$

and

$$w_1(u,2,t) = e^{-\frac{\lambda b}{c_1}} \left[\sum_{n=1}^2 \vartheta(u,2,n,t-\frac{b}{c_1}) - \int_0^{t-\frac{b}{c_1}} \varsigma(b,1,t-\frac{b}{c_1}-z) w_1(u,1,z) dz \right].$$

Similarly we can prove the assertion for any $m > 1$.

Theorem 5. *For $u > b$ and $m > 1$ the joint density of the number of claims until ruin N_τ and the time to ruin τ is given by*

$$w_2(u,m,t) = (\frac{\lambda}{c_2})^m \sum_{k=0}^{m-1} \sum_{n=0}^{m-k-1} \int_0^{u-b} \int_0^t \int_0^{c_2 z} g_2(y,k,z) b_{m-k-n-1}(u-b-x,y) \varepsilon(x,n,t-z) dy dz dx,$$

$$(48)$$

where

$$\varepsilon(u,0,t) := \int_{u+b}^{c_2 t + u + b} \bar{F}(y) g_2(y - u - b, 0, t) dy,$$

$$\varepsilon(u,m,t) := \sum_{n=1}^m \int_u^{u+b} \int_0^t w_1(u+b-x,n,t-z) \int_x^{c_2 z + x} f(y) g_2(y-x,m-n,z) dy dz dx$$

$$+ \int_{u+b}^{c_2 t + u + b} \bar{F}(y) g_2(y - u - b, m, z) dy, \ \ n \geq 1.$$

Proof. To obtain an expression for $w_2(u,m,t)$ we first consider $h(x)$ defined in (16). Using (37) we can derive:

$$h(u) = \int_u^{u+b} \sum_{m=1}^\infty r^m \int_0^\infty e^{-\delta t} w_1(u+b-x,m,t) dt$$

$$\times \sum_{n=0}^\infty r^n \int_0^\infty e^{-\delta z} \int_x^{c_2 z + x} f(y) g_2(y-x,n,z) dy dz dx$$

$$+ \sum_{n=0}^\infty r^n \int_0^\infty e^{-\delta t} \int_{u+b}^{c_2 t + u + b} \bar{F}(y) g_2(y - u - b, n, t) dy dt$$

$$= \sum_{n=1}^\infty r^n \int_0^\infty e^{-\delta t} \left[\sum_{m=1}^n \int_u^{u+b} \int_0^t w_1(u+b-x,m,t-z) \right.$$

$$\times \int_x^{c_2 z + x} f(y) g_2(y-x,n-m,z) dy dz dx$$

$$+ \int_{u+b}^{c_2 t+u+b} \bar{F}(y) g_2(y-u-b,n,z) dy \Bigg] dt \tag{49}$$

$$+ \int_0^\infty e^{-\delta t} \int_{u+b}^{c_2 t+u+b} \bar{F}(y) g_2(y-u-b,0,t) dy dt$$

$$= \sum_{n=0}^\infty r^n \int_0^\infty e^{-\delta t} \varepsilon(u,n,t) dt. \tag{50}$$

Moreover, substituting (41), (49) and (36) into (15) gives:

$$\phi_2(u) = \sum_{m=0}^\infty \left(\frac{\lambda r}{c_2}\right)^{m+1} \int_0^{u-b} (T_{\rho_2} f)^{m*}(u-b-x) h(x) dx$$

$$= \sum_{m=0}^\infty \left(\frac{\lambda r}{c_2}\right)^{m+1} \int_0^{u-b} \int_0^\infty e^{-\rho_2 y} b_m(u-b-x,y) dy \sum_{n=0}^\infty r^n \int_0^\infty e^{-\delta t} \varepsilon(x,n,t) dt dx$$

$$= \sum_{m=0}^\infty \left(\frac{\lambda r}{c_2}\right)^{m+1} \int_0^{u-b} \int_0^\infty \sum_{k=0}^\infty r^k \int_{y/c_2}^\infty e^{-\delta z} g_2(y,k,z) dz b_m(u-b-x,y) dy$$

$$\times \sum_{n=0}^\infty r^n \int_0^\infty e^{-\delta t} \varepsilon(x,n,t) dt dx$$

$$= \sum_{m=1}^\infty r^m \int_0^\infty e^{-\delta t} \left\{ \left(\frac{\lambda}{c_2}\right)^m \sum_{k=0}^{m-1} \sum_{n=0}^{m-k-1} \int_0^{u-b} \int_0^t \int_0^{c_2 z} g_2(y,k,z) b_{m-k-n-1}(u-b-x,y) \right.$$

$$\left. \times \varepsilon(x,n,t-z) dy dz dx \right\} dt. \tag{51}$$

Comparing equations (51) and (4) completes the proof.

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