Stochastic Maximum Principle on a Continuous-time Behavioral Portfolio Model *

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Abstract In this short note, we consider the optimization problem with probability distortion when the objective functional involves a running term which is given by an S-shaped function. A stochastic maximum principle is presented.

1 Introduction

There are several epoch-making achievements in the history of finance theory over the past 70 years. The first is the expected utility maximization proposed by von Neumann and Morgenstern [17]. It is premised on the tenets that decision makers are rational and consistently risk averse under uncertainty. Later on, a Nobel-prize-winning work, Markowitz's mean-variance model [12] came out. Along with these theories in continuous portfolio selection problems, many approaches, such as dynamic programming, stochastic maximum principle, martingale and convex duality have been developed, see Merton [13], Peng [14], Duffie and Epstein [4], Yong and Zhou [19], Karatzas et al. [9].

On the account of substantial phenomena violating the basic tenets of conventional financial theory, for instance, Allais paradox [1], Tversky and Kahneman [16] put forward cumulative prospect theory (CPT) and Benartzi and Thaler [3] proposed behavioral economics. Both of them integrate psychology with finance and economics. To study the continuous-time portfolio choice problem, we concentrate

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on CPT in this paper. Its key elements are: (1) benchmark (evaluated at terminal time T) serves as a base point to distinguish gains from losses (Without loss of generality, it is assumed to be 0 in this paper); (2) Utility functions are concave for gains and convex for losses, and steeper for losses than for gains; (3) Probability distortions (or weighting) are nonlinear transformation of the probability measures, which overweight small probabilities and underweight moderate and high probabilities.

There have been burgeoning research merge CPT into portfolio investment. Most of them are limited to the discrete-time setting, see for example Benartzi and Thaler [2], Shefrin and Statman [15], Levy and Levy [10]. The pioneering analytical research on continuous-time asset allocation featuring behavioral criteria is done by Jin and Zhou [7]. Since then, a few extensive works have been published, see He and Zhou ([5], [6]), Xu and Zhou [18], Jin and Zhou [8] and so on. Jin and Zhou developed a new theory to work out the optimal terminal value in a continuous-time CPT model. Nonetheless, their theory aims at a particular portfolio choice problem in a self-financing market.

This article is to deal with probability distortion for model with running utilities. In order to come closer to reality, bankruptcy is not allowed in our problem. The remainder is organized as follows. Next section will formulate a general continuous-time portfolio selection model under the CPT, featuring S-shaped utility functions and probability distortions. The stochastic maximum principle as well as a solvable example are finally presented.

2 Problem Formulation

Let $T>0$ be a fixed time horizon and $(\Omega,\mathscr{F},\mathbb{P},\{\mathscr{F}_t\}_{t\geq 0})$ a filtered complete probability space on which is defined a standard $\mathscr{F}_t$-adapted m-dimensional Brownian motion $W_t\equiv(W_t^1,\cdots,W_t^m)^\top$ with $W_0=0$. It is assumed that $\mathscr{F}_t=\sigma\{W_s:0\leq s\leq t\}$, augmented by all the null sets. Throughout this paper $A^\top$ denotes the transpose of a matrix A; $a^\pm$ denote the positive and negative parts of the real number a.

We define a positive state process

$$\begin{cases} dX_t &= b(t,u_t,X_t)dt+\sigma(t,u_t,X_t)dW_t \\ X_0 &= x_0>0, \end{cases} \tag{2.1}$$

and the agent's prospective functional

$$\begin{aligned} J(u_\cdot) = \mathbb{E}\int_0^T &\left(\zeta_+(u_t^+)\varpi_+'\big(1-F_{u_t^+}(u_t^+)\big)-\zeta_-(u_t^-)\varpi_-'\big(1-F_{u_t^-}(u_t^-)\big)\right)dt \\ &+\mathbb{E}\left(l(X_T)w'\big(1-F_{X_T}(X_T)\big)\right), \end{aligned} \tag{2.2}$$

where $u_\cdot$ is a control process taking values in a convex set $U \subseteq \mathbb{R}$. According to CPT, the following assumptions will be in force throughout this paper, where x denotes the state variable, u denotes the control variable.

We make the following assumptions throughout this article.

(H.1) $b(\cdot,\cdot,\cdot) : [0,T] \times U \times \mathbb{R}^+ \to \mathbb{R}$, $\sigma(\cdot,\cdot,\cdot) : [0,T] \times U \times \mathbb{R}^+ \to \mathbb{R}$, *are continuously differentiable with respect to* (u,x) *with Lipschitz continuous first derivatives. We further assume* $b(t,u,0) = \sigma(t,u,0) = 0$.

(H.2) $\zeta_\pm(\cdot), l(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *are differentiable, strictly increasing, strictly concave, with* $\zeta_\pm(0) = l(0) = 0$ *and* $\zeta'_\pm(0+) = l'(0+) = \infty$.

(H.3) $\varpi_\pm(\cdot), w(\cdot) : [0,1] \to [0,1]$, *are differentiable and strictly increasing, with* $\varpi_\pm(0) = w(0) = 0$, $\varpi_\pm(1) = w(1) = 1$. *Moreover, the first derivatives of* $\varpi_\pm(\cdot), w(\cdot)$ *are all bounded.*

Let

$$\mathscr{U} = \Big\{ u : [0,T] \times \Omega \to U \mid u_t \text{ is } \mathscr{F}_t\text{-adapted and } \mathbb{E}\int_0^T |u_t|^4 dt < \infty \Big\}.$$

Definition 1. A control process $u_\cdot \in \mathscr{U}$ is said to be admissible, and $(u_\cdot, X_\cdot)$ is called an admissible pair, if

1. $X_\cdot$ is the unique solution of equation (2.1) under $u_\cdot$;
2. For any $t \in [0,T]$, the distribution functions of $u_\cdot^\pm$ are continuous except at 0;
3. $\mathbb{E}\int_0^T \big|\zeta_\pm(u_t^\pm)\varpi'_\pm\big(1 - F_{u_t^\pm}(u_t^\pm)\big)\big|^8 dt < \infty$.
4. $\mathbb{E}\int_0^T \Big(\big|\frac{d}{du}\ln\zeta_\pm(u_t^\pm)\big|^8 + \big|\zeta''_\pm(u_t^\pm)\big|^4\Big) dt < \infty$.

The set of all admissible controls is denoted by $\mathscr{U}_{ad}$.

Meanwhile, the following technical assumption for the terminal state are in force throughout this paper.

Assumption 1 *The terminal state* X_T *corresponding to the control process* $u_\cdot \in \mathscr{U}_{ad}$ *is supposed to has continuous distribution function. Besides,*

$$\mathbb{E}\big|l(X_T)w'\big(1 - F_{X_T}(X_T)\big)\big|^8 + \mathbb{E}\Big|\frac{d}{dx}\ln l(X_T)\Big|^8 + \mathbb{E}\big|l''(X_T)\big|^4 < \infty. \tag{2.3}$$

Problem. Our optimal control problem is to find $\bar{u}_\cdot \in \mathscr{U}_{ad}$ such that

$$J(\bar{u}_\cdot) = \max_{u_\cdot \in \mathscr{U}_{ad}} J(u_\cdot). \tag{2.4}$$

3 A necessary Condition for Optimality

The current section presents our main result of the article. Let $(\bar{u}_\cdot, \bar{X}_\cdot)$ be an optimal pair of the problem (2.4). We proceed to presenting the condition it must satisfy. To this end, we formulate the adjoint equation

$$\begin{cases} dp_t &= -\big(b_x(t,\bar{u}_t,\bar{X}_t)p_t + \sigma_x(t,\bar{u}_t,\bar{X}_t)q_t\big)dt + q_t dW_t, \\ p_T &= l'(\bar{X}_T)w'(1-F_{\bar{X}_T}(\bar{X}_T)). \end{cases} \tag{3.1}$$

Here is the necessary condition we obtained for the optimality of the control.

Theorem 2. *If $\bar{u}.$ is the optimal control with the state trajectory $\bar{X}.$, then there exists a pair $(p.,q.)$ of adapted processes which satisfies (3.1) such that a.e. $t\in[0,T]$,*

$$p_t b_u(t,\bar{u}_t,\bar{X}_t) + \sigma_u(t,\bar{u}_t,\bar{X}_t)q_t = \begin{cases} -\zeta'_+\big(\bar{u}_t^+\big)\varpi'_+\big(1-F_{\bar{u}_t^+}(\bar{u}_t^+)\big) & \text{if } \bar{u}_t > 0, \\ -\zeta'_-\big(\bar{u}_t^-\big)\varpi'_-\big(1-F_{\bar{u}_t^-}(\bar{u}_t^-)\big) & \text{if } \bar{u}_t < 0, \end{cases} \quad a.s.. \tag{3.2}$$

Recall the state equation (2.1) and the adjoint equation (3.1). Given an optimal control $\bar{u}.$, there exists a unique solution $\bar{X}(\bar{u}.)$ to the state equation. As p_T is known, the unique solution $(p(\bar{u}.),q(\bar{u}.))$ for the backward SDE (3.1) is obtained. Plugging $\bar{X}(\bar{u}.)$ and $(p(\bar{u}.),q(\bar{u}.))$ into (3.2), the optimal control $\bar{u}.$ is narrowed to one of the solution of so obtained algebraic equation.

In what follows, we present a solvable example and compare the result with the one without probability distortions. The process $u_t^{\pm}$ in the objective functional are replaced by $u_t^{\pm}X_t$, signifying the proportion of wealth process. We study a case with compounded cost function.

Example 1. Let $u_t, X_t > 0$, $b(t,u,x) = -ux$, and $\sigma(t,u,x) = x$. We take utility function $\zeta_+(x) = \frac{x^\alpha}{\alpha}(0<\alpha<1)$, and distortion function (see, Lopes [11])

$$\varpi_+(p) = \nu p^{\gamma+1} + (1-\nu)[1-(1-p)^{\beta+1}], \qquad \gamma,\beta \geq 0, 0 \leq \nu \leq 1.$$

Then,

$$dX_t = -u_t X_t dt + X_t dW_t, \qquad X_0 = x_0,$$

and

$$J(u.) = \mathbb{E}\int_0^T \Big(\frac{1}{\alpha}(u_t X_t)^\alpha \varpi'_+\big(1-F_{u_t X_t}(u_t X_t)\big) + X_t\Big)dt.$$

By Theorem 2, its optimal solution $(\bar{u}.,\bar{X}.)$ should satisfy

$$p_t = \big(\bar{u}_t\bar{X}_t\big)^{\alpha-1}\varpi'_+\big(1-F_{\bar{u}_t\bar{X}_t}(\bar{u}_t\bar{X}_t)\big),\ a.e. t\in[0,T], a.s., \tag{3.3}$$

and

$$dp_t = \big(\bar{u}_t p_t - q_t - \big(\bar{u}_t\bar{X}_t\big)^{\alpha-1}\varpi'_+\big(1-F_{\bar{u}_t\bar{X}_t}(\bar{u}_t\bar{X}_t)\big)\bar{u}_t - 1\big)dt + q_t dW_t, \quad p_T = 0.$$

It yields $p_t = T-t,\ q_t = 0,\ \forall t\in[0,T]$. Plugging back to equality (3.3), we obtain that $\bar{u}_t\bar{X}_t = \Big(\frac{T-t}{(1-\nu)(\beta+1)}\Big)^{1/(\alpha-1)}$, a.e. $t\in[0,T]$, a.s.. Solving the state equation, we arrive at

$$\bar{u}_t = (T-t)^{1/(\alpha-1)}/V_t\big(x_0((1-\nu)(\beta+1))^{1/(\alpha-1)} + \int_0^t \frac{(T-s)^{1/(\alpha-1)}}{V_s}ds, \quad a.s.,$$

where $V_t = \exp\{B_t - \frac{t}{2}\}$.

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