A Bayesian sequential test for the drift of a fractional Brownian motion

Alexey Muravlev and Mikhail Zhitlukhin

Abstract We construct a sequential test for the sign of the drift of a fractional Brownian motion. We work in the Bayesian setting and assume the drift has a prior normal distribution. The problem reduces to an optimal stopping problem for a standard Brownian motion, obtained by a transformation of the observable process. The solution is described as the first exit time from some set, whose boundaries satisfy certain integral equations, which are solved numerically.

1 Introduction

Suppose one observes a fractional Brownian motion process (fBm) with linear drift and unknown drift coefficient. We are interested in sequentially testing the hypotheses that the drift coefficient is positive or negative. We consider a Bayesian setting where the drift coefficient has a prior normal distribution, and we use an optimality criteria of a test which consists of a linear penalty for the duration of observation and a penalty for a wrong decision proportional to the true value of the drift coefficient. The main result of this paper describes the structure of the exact optimal test in this problem, i.e. specifies a time to stop observation and a rule to choose between the two hypotheses.

The main novelty of our work compared to the large body of literature related to sequential tests (for an overview of the field, see e.g. [10, 20]) is that we work with fBm. To the best of our knowledge, this is the first non-asymptotic solution of a continuous-time sequential testing problem for this process. It is well-known

Alexey Muravlev
Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, Russia, e-mail: almurav@mi.ras.ru

Mikhail Zhitlukhin (corresponding author)
Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, Russia, e-mail: mikhailzh@mi.ras.ru

that a fBm is not a Markov process, neither a semimartingale except the particular case when it is a standard Brownian motion (standard Bm). As a consequence, many standard tools of stochastic calculus and stochastic control (Itô's formula, the HJB equation, etc.) cannot be directly applied in models based on fBm. Fortunately, in the problem we consider it turns out to be possible to change the original problem for fBm so that it becomes tractable. One of the key steps is a general transformation outlined in the note [13], which allows to reduce sequential testing problems for fBm to problems for diffusion processes.

In the literature, the result which is most closely related to ours is the sequential test proposed by H. Chernoff [3], which has exactly the same setting and uses the same optimality criterion, but considers only standard Bm. For a prior normal distribution of the drift coefficient, Chernoff and Breakwell [1, 4] found asymptotically optimal sequential tests when the variance of the drift goes to zero or infinity.

Let us mention two other recent results in the sequential analysis of fBm, related to estimation of its drift coefficient. Çetin, Novikov, and Shiryaev [2] considered a sequential estimation problem assuming a normal prior distribution of the drift with a quadratic or a δ-function penalty for a wrong estimate and a linear penalty for observation time. They proved that in their setting the optimal stopping time is non-random. Gapeev and Stoev [8] studied sequential testing and changepoint detection problems for Gaussian processes, including fBm. They showed how those problems can be reduced to optimal stopping problems and found asymptotics of optimal stopping boundaries. There are many more results related to fixed-sample (i.e. non-sequential) statistical analysis of fBm. See, for example, Part II of the recent monograph [19], which discusses statistical methods for fBm in details.

The remaining part of our paper is organized as follows. Section 2 formulates the problem. Section 3 describes a transformation of the original problem to an optimal stopping problem for a standard Bm and introduces auxiliary processes which are needed to construct the optimal sequential test. The main result of the paper – the theorem which describes the structure of the optimal sequential test – is presented in Section 4, together with a numerical solution.

2 Decision rules and their optimality

Recall that the fBm B_t^H, $t \geq 0$, with Hurst parameter $H \in (0,1)$ is a zero-mean Gaussian process with the covariance function

$$\operatorname{cov}(B_t^H, B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \qquad t, s \geq 0.$$

In the particular case $H = 1/2$ this process is a standard Brownian motion (standard Bm) and has independent increments; its increments are positively correlated in the case $H > 1/2$ and negatively correlated in the case $H < 1/2$.

Suppose one observes the stochastic process

$$Z_t = \theta t + B_t^H, \qquad t \geq 0,$$

where $H \in (0,1)$ is known, and θ is a random variable independent of B^H and having a normal distribution with known mean $\mu \in \mathbb{R}$ and known variance $\sigma^2 > 0$.

It is assumed that neither the value of θ, nor the value of B_t^H can be observed directly, but the observer wants to determine whether the value of θ is positive or negative based on the information conveyed by the combined process Z_t. We will look for a sequential test for the hypothesis $\theta > 0$ versus the alternative $\theta \leq 0$. By a sequential test we call a pair $\delta = (\tau, d)$, which consists of a stopping time τ of the filtration $\mathscr{F}_t^Z$, generated by Z, and an $\mathscr{F}_\tau^Z$-measurable function d assuming values ± 1. The stopping time is the moment of time when observation is terminated and a decision about the hypotheses is made; the value of d shows which of them is accepted.

We will use the criterion of optimality of a decision rule consisting in minimizing the linear penalty for observation time and the penalty for a wrong decision proportional to the absolute value of θ. Namely, with each decision rule δ we associate the risk

$$R(\delta) = \mathrm{E}(\tau + |\theta| \, \mathrm{I}(d \neq \mathrm{sgn}(\theta))).$$

The problem consists in finding δ^* that minimizes $R(\delta)$ over all decision rules.

This problem was proposed by H. Chernoff in [3] for standard Bm, and we refer the reader to that paper for a rationale for this setting. The subsequent papers [1, 4, 5] include results about the asymptotics of the optimal test and other its properties, including a comparison with Wald's sequential probability ratio test. Our paper [21] contains a result which allows to find the exact (non-asymptotic) optimal test by a relatively simple numerical procedure.

3 Reduction to an optimal stopping problem

From the relation $|\theta| \, \mathrm{I}(d \neq \mathrm{sgn}(\theta)) = \theta^+ \mathrm{I}(d = -1) + \theta^- \mathrm{I}(d = 1)$, where $\theta^+ = \max(\theta, 0)$, $\theta^- = -\min(\theta, 0)$, and from that d is $\mathscr{F}_\tau^Z$-measurable, one can see that the optimal decision rule should be looked for among rules (τ, d) with $d = \min(\mathrm{E}(\theta^- \mid \mathscr{F}_\tau^Z), \mathrm{E}(\theta^+ \mid \mathscr{F}_\tau^Z))$. Hence, it will be enough to solve the optimal stopping problem which consists in finding a stopping time τ^* such that $R(\tau^*) = \inf_\tau R(\tau)$, where

$$R(\tau) = \mathrm{E}(\tau + \min(\mathrm{E}(\theta^- \mid \mathscr{F}_\tau^Z), \mathrm{E}(\theta^+ \mid \mathscr{F}_\tau^Z)))$$

(for brevity, we'll use the same notation R for the functional associated with a decision rule, and the functional associated with a stopping time).

We will transform the expression inside the expectation in $R(\tau)$ to the value of some process, constructed from a standard Bm. Introduce the process X_t, $t \geq 0$, by

$$X_t = C_H \int_0^t K_H(t,s)dZ_s$$

with the integration kernel $K_H(t,s) = (t-s)^{\frac{1}{2}-H} {}_2F_1(\frac{1}{2}-H,\ \frac{1}{2}-H,\ \frac{3}{2}-H,\ \frac{s-t}{t})$, where ${}_2F_1$ is the Gauss (ordinary) hypergeometric function, and the constant $C_H = \left(\frac{\Gamma(2-2H)}{2H\Gamma(\frac{1}{2}+H)(\Gamma(\frac{3}{2}-H))^3}\right)^{\frac{1}{2}}$; as usual, Γ denotes the gamma function.

As follows from [9] (see also earlier results [14, 12]), $B_t = C_H \int_0^t K_H(t,s)dB_s^H$ is a standard Bm, and by straightforward computation we obtain the representation

$$dX_t = B_t + \theta L_H t^{\frac{1}{2}-H} dt,$$

and the filtrations of the processes Z_t and X_t coincide. The constant L_H in the above formula is defined by $L_H = (2H(\frac{3}{2}-H)B(\frac{1}{2}+H,\ 2-2H))^{-\frac{1}{2}}$, where B is the beta function.

Now one can find that the conditional distribution $\mathrm{Law}(\theta \mid \mathscr{F}_\tau^X)$ is normal and transform the expression for the risk to

$$R(\tau) = \mathrm{E}\Big(\tau - \frac{1}{2}Y_\tau\Big) + const,$$

where *const* denotes some constant (depending on μ, σ, H), the value of which is not essential for what follows, and Y_t, $t \geq 0$, is the process satisfying the SDE

$$dY_t = t^{\frac{1}{2}-H}\big((\sigma L_H)^{-2} + t^{2-2H}/(2-2H)\big)d\widetilde{B}_t, \qquad Y_0 = \mu L_H,$$

where $\widetilde{B}$ is another standard Bm, the innovation Brownian motion (see e.g. Chapter 7.4 in [11]). For brevity, denote $\gamma = (2-2H)^{-1}$. Then under the following monotone change of time

$$t(r) = \left(\frac{(2-2H)r}{(\sigma L_H)^2(1-r)}\right)^\gamma, \qquad r \in [0,1),$$

where t runs through the half-interval $[0,\infty)$ when r runs through $[0,1)$, the process

$$W_r = (\sigma L_H)^{-1} Y_{t(r)} - \mu\sigma^{-1}$$

is a standard Bm in $r \in [0,1)$, and the filtrations $\mathscr{F}_r^W$ and $\mathscr{F}_{t(r)}^X$ coincide. Denote

$$M_{\sigma,H} = \frac{2}{\sigma}\left(\frac{2-2H}{\sigma^2 L_H^2}\right)^\gamma.$$

Then the optimal stopping problem for X in t-time is equivalent to the following optimal stopping problem for W in r-time:

$$V = \inf_{\rho<1} \mathrm{E}\left(M_{\sigma,H}\left(\frac{\rho}{1-\rho}\right)^\gamma - \left|W_\rho + \frac{\mu}{\sigma}\right|\right). \tag{1}$$

Namely, if ρ^* is the optimal stopping time for V, then the optimal decision rule $\delta^* = (\tau^*, d^*)$ is given by

$$\tau^* = t(\rho^*), \qquad d^* = \mathrm{I}(a_{\tau^*} > 0) - \mathrm{I}(a_{\tau^*} \le 0). \tag{2}$$

4 The main results

In this section we formulate a theorem about the solution of problem (1), which gives an optimal sequential test through transformation (2). Throughout we will assume that σ and H are fixed and will denote the function

$$f(t) = M_{\sigma,H}\left(\frac{t}{1-t}\right)^{\gamma}.$$

It is well-known that under general conditions the solution of an optimal stopping problem for a Markov process can be represented as the first time when the process enters a certain set (a stopping set). Namely, let us first rewrite our problem in the Markov setting by allowing the process W_t to start from any point $(t,x) \in [0,1) \times \mathbb{R}$:

$$V(t,x) = \inf_{\rho<1-t} \mathrm{E}(f(t+\rho) - |W_\rho + x|) - f(t). \tag{3}$$

For example, for the quantity V from (1) we have $V = V(0, \frac{\mu}{\sigma})$. We subtract $f(t)$ in the definition of $V(t,x)$ to make the function $V(t,x)$ bounded. For $t = 1$ we define $V(1,x) = -|x|$.

The following theorem describes the structure of the optimal stopping time in problem (3). In its statement, we set

$$t_0 = t_0(H) := \max\left(0, \frac{1-2H}{4(1-H)}\right).$$

Obviously, $t_0 > 0$ for $H < \frac{1}{2}$ and $t_0 = 0$ for $H \ge \frac{1}{2}$.

Theorem 1. *1) There exists a function $A(t)$ defined on $(t_0, 1]$, which is continuous, strictly decreasing, and strictly positive for $t < 1$ with $A(1) = 0$, such that for any $t > t_0$ and $x \in \mathbb{R}$ the optimal stopping time in the problem* (3) *is given by*

$$\rho^*(t,x) = \inf\{s \ge 0 : |W_s + x| \ge A(t+s)\}.$$

Moreover, for any $t \in (t_0, 1]$ the function $A(t)$ satisfies the inequality

$$A(t) \le \frac{(1-t)^{\gamma}}{2M_{\sigma,H}t^{\gamma-1}}. \tag{4}$$

2) The function $A(t)$ is the unique function which is continuous, non-negative, satisfies (4)*, and solves the integral equation*

$$G(t,A(t)) = \int_t^1 F(t,A(t),s,A(s))ds, \qquad t \in (t_0,1), \tag{5}$$

with the functions $G(t,x) = \mathrm{E}\,|\zeta\sqrt{1-t}+x|-x$ *and* $F(t,x,s,y) = f'(s)\,\mathrm{P}(|\zeta\sqrt{s-t}+x| \le y)$ *for a standard normal random variable* ζ.

Regarding the statement of this theorem, first note that for $H < \frac{1}{2}$, the stopping boundary $A(t)$ is described only for $t > t_0 > 0$. The main reason is that the method of proof we use to show that $A(t)$ is continuous and satisfies the integral equation requires it to be of bounded variation (at least, locally). In particular, this is a sufficient condition for applicability of the Itô formula with local time on curves [16], which is used in the proof. In the case $H \ge \frac{1}{2}$ and for $t \ge t_0$ in the case $H < \frac{1}{2}$ by a direct probabilistic argument we can prove that $A(t)$ is monotone and therefore has bounded variation; this argument however doesn't work for $t < t_0$ in the case $H < \frac{1}{2}$, and, as a formal numerical solution shows, the boundary $A(t)$ seems to be indeed not monotone in that case. Of course, the assumption of bounded variation can be relaxed while the Itô formula can still be applicable (see e.g. [16, 6, 7]), however verification of weaker sufficient conditions is problematic. Although the general scheme to obtain integral equations of type (5) and prove uniqueness of their solutions has been discovered quite a while ago (the first full result was obtained by Peskir for the optimal stopping problem for American options, see [17]), and has been used many times in the literature for various optimal stopping problems (a large number of examples can be found in [18]), we are unaware of any of its applications in the case when stopping boundaries are not monotone and/or cannot be transformed to monotone ones by space and/or time change. Nevertheless, a formal numerical solution of the integral equation produces stopping boundaries which "look smooth", but we admit that a rigor proof of this fact in the above-mentioned cases remains an open question.

Note also, that in the case $H \ge \frac{1}{2}$ the space-time transformation we apply to pass from the optimal stopping problem for the process $a(t)$ to the problem for W_r is essential from this point of view, because the boundaries in the problem for $a(t)$ are not monotone. Moreover, they are not monotone even in the case $H = \frac{1}{2}$, when $a(t)$ is obtained by simply shifting X_t in time and space, see [3, 21].

The second remark we would like to make is that in the case $H > \frac{1}{2}$ we do not know whether $A(0)$ is finite. In the case $H = \frac{1}{2}$ the finiteness of $A(0)$ follows from inequality (4), which is proved by a direct argument based on comparison with a simpler optimal stopping problem (one can see that it extends to $t = 0$). It seems that a deeper analysis may be required for the case $H > \frac{1}{2}$, which is beyond this paper.

Figure 1 shows the stopping boundary $A(t)$ for different values H computed by solving equation (5) numerically. A description of the numerical method, based on "backward induction", can be found, for example, in [15].

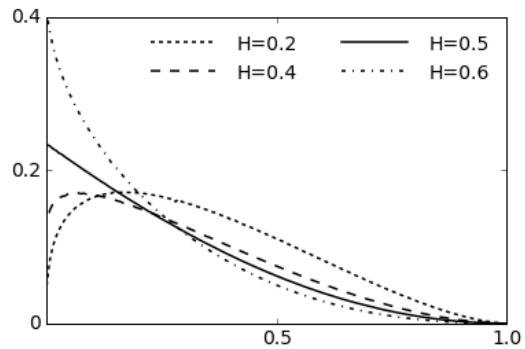

Fig. 1 The stopping boundary $A(t)$ for different values of H and $\sigma = 1$.

References

1. J. Breakwell, H. Chernoff (1964). Sequential tests for the mean of a normal distribution II (large t). *The Annals of Mathematical Statistics* 35, 162-173.
2. U. Çetin, A. Novikov, A.N. Shiryaev (2013). Bayesian sequential estimation of a drift of fractional Brownian motion. *Sequential Analysis* 32, 288–296.
3. H. Chernoff (1961). Sequential tests for the mean of a normal distribution. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* 1, 79–91.
4. H. Chernoff (1965). Sequential test for the mean of a normal distribution III (small t). *The Annals of Mathematical Statistics* 36, 28–54.
5. H. Chernoff (1965). Sequential Tests for the Mean of a Normal Distribution IV (Discrete Case). *The Annals of Mathematical Statistics* 36, 55–68.
6. N. Eisenbaum (2006). Local timespace stochastic calculus for Lévy processes. *Stochastic processes and their applications* 116, 757–778.
7. H. Föllmer, P. Protter, A.N. Shiryayev (1995). Quadratic covariation and an extension of Itô's formula. *Bernoulli* 1, 149–169.
8. P.V. Gapeev, Y.I. Stoev (2017). On the Laplace transforms of the first exit times in one-dimensional non-affine jumpdiffusion models. *Statistics & Probability Letters* 121, 152–162.
9. C. Jost (2006). Transformation formulas for fractional Brownian motion. *Stochastic Processes and their Applications* 116, 1341–1357.
10. T.L. Lai (1997). On optimal stopping problems in sequential hypothesis testing. *Statistica Sinica* 7, 33–51.
11. R.S. Liptser, A.N. Shiryaev (2001). *Statistics of Random Processes I: General Theory*, 2nd ed. Springer.
12. G.M. Molchan, Y.I. Golosov (1969). Gaussian stationary processes with asymptotical power spectrum. *Doklady Akademii Nauk SSSR* 184, 546–549.
13. A.A. Muravlev (2013). Methods of sequential hypothesis testing for the drift of a fractional Brownian motion. *Russian Mathematical Surveys* 68, 577–579.
14. I. Norros, E. Valkeila, J. Virtamo (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli* 5, 571–587.
15. J.L. Pedersen, G. Peskir (2002). On nonlinear integral equations arising in problems of optimal stopping. *Functional analysis VII: proceedings*. Dubrovnik, Croatia, 17–26 September 2001.
16. G. Peskir (2005). A change-of-variable formula with local time on curves. *Journal of Theoretical Probability* 18, 499–535.
17. G. Peskir (2005). On the American option problem. *Mathematical Finance* 15, 169–181.
18. G. Peskir, A. Shiryaev (2006). *Optimal stopping and free-boundary problems*. Birkhäuser Basel.

19. K. Tanaka (2017). *Time series analysis: nonstationary and noninvertible distribution theory*. John Wiley & Sons.
20. A. Tartakovsky, I. Nikiforov, M. Basseville (2014). *Sequential analysis: Hypothesis testing and changepoint detection*. CRC Press.
21. M.V. Zhitlukhin, A.A. Muravlev (2013). On Chernoff's hypotheses testing problem for thqe drift of a Brownian motion. *Theory of Probability and its Applications* 57, 708–717.