How to hear the corners of a drum

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Abstract We prove that the existence of corners in a class of planar domain, which includes all simply connected polygonal domains and all smoothly bounded domains, is a spectral invariant of the Laplacian with both Neumann and Robin boundary conditions. The main ingredient in the proof is a locality principle in the spirit of Kac's "principle of not feeling the boundary," but which holds uniformly up to the boundary. Albeit previously known for Dirichlet boundary condition, this appears to be new for Robin and Neumann boundary conditions, in the geometric generality presented here. For the case of curvilinear polygons, we describe how the same arguments using the locality principle are insufficient to obtain the analogous result. However, we describe how one may be able to harness powerful microlocal methods and combine these with the locality principles demonstrated here to show that corners are a spectral invariant; this is current work-in-progress [23].

1 Introduction

It is well known that "one cannot hear the shape of a drum" [8]. Mathematically, this means that there exist bounded planar domains which have the same eigenvalues for the Laplacian with Dirichlet boundary condition, in spite of the domains

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having different shapes. The standard example is shown in Figure 1. Two geometric characteristics of these domains are immediately apparent:

1. These domains both have corners.
2. Neither of these domains are convex.

This naturally leads to the following two open problems:

Problem 1. Can one hear the shape of a smoothly bounded drum?

Problem 2. Can one hear the shape of a convex drum?

The mathematical formulation of these problems are: if two smoothly bounded (respectively, convex) domains in the plane are isospectral for the Laplacian with Dirichlet boundary condition, then are they the same shape?

One could dare to conjecture that the answer to Problem 1 is yes, based on the isospectrality result of Zelditch [40]. He proved that if two analytically bounded domains both have a bilateral symmetry and are isospectral, then they are in fact the same shape. For certain classes of convex polygonal domains including triangles [5], [11]; parallelograms [15]; and trapezoids [12]; if two such domains are isospectral, then they are indeed the same shape. This could lead one to suppose that perhaps Problem 2 also has a positive answer.

Contemplating these questions lead the second author and Z. Lu to investigate whether smoothly bounded domains can be isospectral to domains with corners. In [16], they proved that for the Dirichlet boundary condition, "one can hear the corners of a drum" in the sense that a domain with corners cannot be isospectral to a smoothly bounded domain. Here we generalize that result to both Neumann and Robin boundary conditions.

The key technical tool in the proof is a locality principle for the Neumann and Robin boundary conditions in a general context which includes domains with only

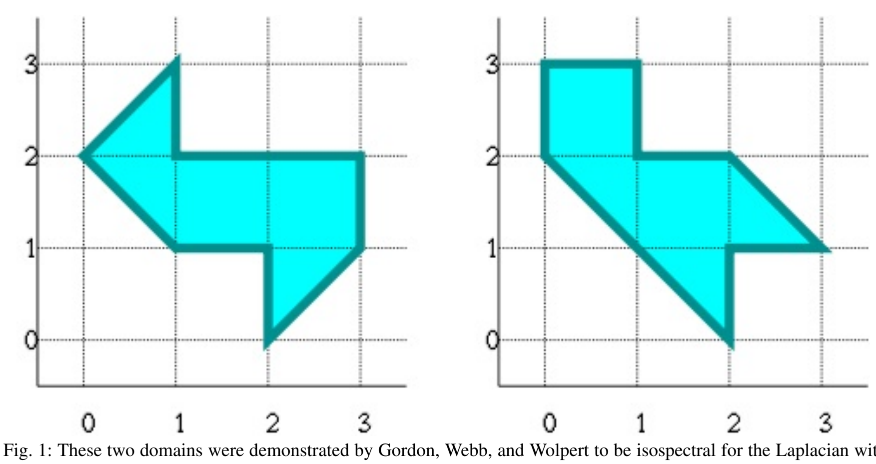

Fig. 1: These two domains were demonstrated by Gordon, Webb, and Wolpert to be isospectral for the Laplacian with Dirichlet boundary condition [9]. This image is from Wikipedia Commons.

piecewise smooth boundary. This locality principle may be of independent interest, because it not only generalizes Kac's "principle of not feeling the boundary" [13] but also unlike that principle, it holds uniformly up to the boundary. First, we explain Kac's locality principle. Let Ω be a bounded domain in $\mathbb{R}^2$, or more generally $\mathbb{R}^n$, because the argument works in the same way in all dimensions. Assume the Dirichlet boundary condition, and let the corresponding heat kernel for Ω be denoted by H, while the heat kernel for $\mathbb{R}^n$,

$$K(t,z,z') = (4\pi t)^{-n/2} e^{-d(z,z')^2/4t}. \tag{1}$$

Let

$$\delta = \min\{d(z,\partial\Omega), d(z',\partial\Omega)\}.$$

Then, there are constants $A, B > 0$ such that

$$|K(t,z,z') - H(t,z,z')| \leq A t^{-n/2} e^{-B\delta^2/t}.$$

This means that the heat kernel for Ω is $O(t^\infty)$[1] close to the Euclidean heat kernel, as long as we consider points z, z' which are at a positive distance from the boundary. Hence the heat kernel "does not feel the boundary."

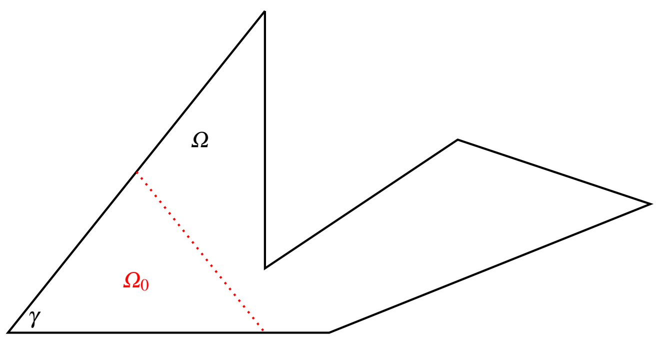

Fig. 2: Above, we have the polygonal domain Ω which contains the triangular domain, Ω_0. Letting $S = S_\gamma$ be a circular sector of opening angle γ and infinite radius, this is an example of an "exact geometric match," in the sense that Ω_0 is equal to a piece of S.

In a similar spirit, a more general locality principle is known to be true. The idea is that one has a collection of sets which are "exact geometric matches" to certain pieces of the domain, Ω. To describe the meaning of an "exact geometric match," consider a piece of the first quadrant near the origin in $\mathbb{R}^2$. A sufficiently small piece is an exact match for a piece of a rectangle near a corner. Similarly, for a surface with exact conical singularities, near a singularity of opening angle γ, a piece of an infinite cone with the same opening angle is an exact geometric match

[1] By $O(t^\infty)$, we mean $O(t^N)$ for any $N \in \mathbb{N}$.

to a piece of the surface near that singularity. For a planar example, see Figure 2. The locality principle states that if one takes the heat kernels for those "exact geometric matches," and restricts them to the corresponding pieces of the domain (or manifold), Ω, then those "model heat kernels" are equal to the heat kernel for Ω, restricted to the corresponding pieces of Ω, with error $O(t^\infty)$ as $t \downarrow 0$.

This locality principle is incredibly useful, because if one has exact geometric matches for which one can explicitly compute the heat kernel, then one can use these to compute the short time asymptotic expansion of the heat trace. Moreover, in addition to being able to compute the heat trace expansion, one can also use this locality principle to compute the zeta regularized determinant of the Laplacian as in [1].

Here, we shall give one application of the locality principle: "how to hear the corners of a drum."

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded, Lipschitz planar domain with piecewise smooth boundary. Moreover, assume that the (finitely many) points at which the boundary is not smooth are exact corners; that is, there exists a neighborhood of each corner in which the boundary of Ω is the union of two straight line segments. Assume that for at least one such corner, the interior angle is not equal to π.*

Then the Laplacian with either Dirichlet[2]*, Neumann, or Robin boundary condition is not isospectral to the Laplacian with the same boundary condition*[3] *on any smoothly bounded domain.*

To prove the result, we use a locality principle which is stated and proved in §2. We next introduce model heat kernels as well as the corresponding Green's functions for the "exact geometric matches" in §3. We proceed there to use the models together with our locality principle to compute the short time asymptotic expansion of the heat trace. Theorem 1 is then a consequence of comparing the heat trace expansions in the presence and lack of corners. In conclusion, we explain in how the locality principle *fails* to prove Theorem 1 for the case of curvilinear polygonal domains, in which the corners are not exact. An example of a non-exact corner of interior angle $\pi/2$ is the corner where the straight edge meets the curved edge in a half-circle. This motivates the discussion in §4 concerning the necessity and utility of microlocal analysis, in particular, the construction of the heat kernel for curvilinear polygonal domains and surfaces via the heat space and heat calculus in those settings. This construction, together with a generalization of Theorem 1 to all boundary conditions (including discontinuous, mixed boundary conditions), as well as to surfaces with both conical singularities and edges is currently in preparation and shall be presented in forthcoming work [23].

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[2] This result was proven in the Dirichlet case in [16].

[3] In particular, in the case of Robin boundary conditions, we assume the same Robin parameters for both domains.

2 The locality principle

We begin by setting notations and sign conventions and recalling fundamental concepts.

2.1 Geometric and analytic preliminaries

To state the locality principle, we make the notion of an "exact geometric match" precise. Let Ω be a domain, possibly infinite, contained in $\mathbb{R}^n$.

Definition 1. Assume that $\Omega_0 \subset \Omega \subset \mathbb{R}^n$, and $S \subset \mathbb{R}^n$. We say that S and Ω are *exact geometric matches on* Ω_0 if there exists a sub-domain $\Omega_c \subseteq \Omega$ which compactly contains Ω_0 and which is isometric to a sub-domain of S (which, abusing notation, we also call Ω_c). Recall that Ω_0 being compactly contained in Ω_c means that the distance from $\overline{\Omega_0}$ to $\overline{\Omega \setminus \Omega_c}$ is positive. A planar example is depicted in Figure 2.

Next, we recall the heat kernel in this context. The heat kernel, H, is the Schwartz kernel of the fundamental solution of the heat equation. It is therefore defined on $\Omega \times \Omega \times [0,\infty)$, and satisfies

$$H(t,z,z') = H(t,z',z), \quad (\partial_t + \Delta)H(t,z,z') = 0 \text{ for } t > 0,$$

$$H(0,z,z') = \delta(z-z'), \quad \text{in the distributional sense.}$$

Throughout we use the sign convention for the Laplacian, Δ, on $\mathbb{R}^n$, that

$$\Delta = -\sum_{j=1}^{n} \partial_j^2.$$

We consider two boundary conditions:

(N) the *Neumann boundary condition,* which requires the normal derivative of the function to vanish on the boundary;
(R) the *Robin boundary condition,* which requires the function, u, to satisfy the following equation on the boundary:

$$\alpha u + \beta \frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial u}{\partial \nu} \text{ is the outward pointing normal derivative.} \tag{2}$$

For $u_0 \in \mathscr{L}^2(\Omega)$, the heat equation with initial data given by u is then solved by

$$u(t,z) = \int_\Omega H(t,z,z')u_0(z')dz'.$$

Moreover, if Ω is a bounded domain, and $\{\phi_k\}_{k\geq 1}$ is an orthonormal basis for $\mathscr{L}^2(\Omega)$ consisting of eigenfunctions of the Laplacian satisfying the appropriate boundary condition, with corresponding eigenvalues $\{\lambda_k\}_{k\geq 1}$, then the heat kernel

$$H(t,z,z') = \sum_{k\geq 1} e^{-\lambda_k t}\phi_k(z)\phi_k(z').$$

2.2 Locality principle for Dirichlet boundary condition

In the general context of domains in $\mathbb{R}^n$ which have only piecewise smooth boundary, the key point is that the locality principle should hold *up to the boundary.* This differs from many previous presentations of a locality principle. For example, in [14, Theorem 1.1], it is proved that without any condition on the regularity of the boundary, for any choice of self-adjoint extension of the Laplacian on $\Omega \subset \mathbb{R}^n$, the heat kernel for this self adjoint extension of the Laplacian on Ω, denoted by H^Ω satisfies

$$|H^\Omega(t,z,z') - H^0(t,z,z')| \leq (C_a\rho(z,z')^{-n} + C_b)\cdot \frac{\exp\left(-\frac{(\rho(z)+\rho(z'))^2}{4t}\right)}{t^{2\lceil\frac{n+1}{2}\rceil - \frac{1}{2}}}.$$

Above, H^0 is the heat kernel for $\mathbb{R}^n$, $\rho(z) = \mathrm{dist}(z,\partial\Omega)$, $\rho(z,z') = \min(\rho(z),\rho(z'))$. The constants C_a and C_b can also be calculated explicitly according to [14]. Clearly, the estimate loses its utility as one approaches the boundary.

In the case of smoothly bounded domains, there is a result of Lück & Schick [17, Theorem 2.26], which implies the locality principle for both the Dirichlet and Neumann boundary conditions, and which holds all the way up to the boundary. We recall that result.[4]

Theorem 2 (Lück & Schick). *Let N be a Riemannian manifold possibly with boundary which is of bounded geometry. Let $V \subset N$ be a closed subset which carries the structure of a Riemannian manifold of the same dimension as N such that the inclusion of V into N is a smooth map respecting the Riemannian metrics. For fixed $p \geq 0$, let $\Delta[V]$ and $\Delta[N]$ be the Laplacians on p-forms on V and N, considered as unbounded operators with either absolute boundary conditions or with relative boundary conditions (see Definition 2.2 of [17]). Let $\Delta[V]^k e^{-t\Delta[V]}(x,y)$ and $\Delta[N]^k e^{-t\Delta[N]}(x,y)$ be the corresponding smooth integral kernels. Let k be a non-negative integer.*

Then there is a monotone decreasing function $C_k(K) : (0,\infty) \to (0,\infty)$ which depends only on the geometry of N (but not on V, x, y, t) and a constant C_2 depending only on the dimension of N such that for all $K > 0$ and $x,y \in V$ with $d_V(x) := d(x, N\setminus V) \geq K$, $d_V(y) \geq K$ and all $t > 0$:

[4] In the original statement of their result, Lück and Schick make the parenthetical remark "We make no assumptions about the boundaries of N and V and how they intersect." This could easily be misunderstood. If one carefully reads the proof, it is implicit that the boundaries are *smooth.* The arguments break down if the boundaries have singularities, such as corners. For this reason, we have omitted the parenthetical remark from the statement of the theorem.

$$\left|\Delta[V]^k e^{-t\Delta[V]}(x,y) - \Delta[N]^k e^{-t\Delta[N]}(x,y)\right| \leq C_k(K) e^{-\left(\frac{d_V(x)^2+d_V(y)^2+d(x,y)^2}{C_2 t}\right)}.$$

One may therefore compare the heat kernels for the Laplacian acting on functions, noting (see p. 362 of [30]) that relative boundary conditions are Dirichlet boundary conditions, and absolute boundary conditions are Neumann boundary conditions. We present this as a corollary to Lück and Schick's theorem.

Corollary 1. *Assume that S is an exact match for $\Omega_0 \subset \Omega$, for two smoothly bounded domains, Ω and Ω_0 in $\mathbb{R}^n$. Assume the same boundary condition, either Dirichlet or Neumann, for the Euclidean Laplacian on both domains. Then*

$$\left|H^\Omega(t,z,z') - H^S(t,z,z')\right| = O(t^\infty) \text{ as } t \downarrow 0, \quad \text{uniformly for } z,z' \in \Omega_0.$$

Proof. We use the theorem of Lück and Schick twice, once with $N = \Omega$ and once with $N = S$, with $V = \Omega_c$ in both cases. We set $k = 0$ and

$$K = \alpha = d(\Omega_0, S \setminus \Omega_c).$$

By the definition of an exact geometric match, $\alpha > 0$. In the $N = S$ case, the theorem reads

$$|H^S(t,z,z') - H^{\Omega_c}(t,z,z')| \leq C_0(\alpha) e^{-\frac{|\mathrm{dist}(z,S\setminus\Omega_c)|^2}{C_2 t} - \frac{|\mathrm{dist}(z',S\setminus\Omega_c)|^2}{C_2 t}} \leq C_0(\alpha) e^{-\frac{2\alpha^2}{C_2 t}}.$$

We conclude that

$$|H^S(t,z,z') - H^{\Omega_c}(t,z,z')| = O(t^\infty)$$

uniformly on Ω_0. The same statement holds with S replaced by Ω, and then the triangle inequality completes the proof. □

The assumption of smooth boundary is quite restrictive, and the proof in [17] relies heavily on this assumption. To the best of our knowledge, the first locality result which holds all the way up to the boundary and includes domains which have only piecewise smooth boundary, but may have corners, was demonstrated by van den Berg and Srisatkunarajah [29]. We note that this result is not stated in the precise form below in [29], but upon careful reading, it is straightforward to verify that this result is indeed proven in [29] and is used in several calculations therein.

Theorem 3 (van den Berg & Srisatkunarajah). *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Let H^Ω denote the heat kernel for the Laplacian on Ω with the Dirichlet boundary condition. Then, for $S = S_\gamma$, a sector of opening angle γ, and for any corner of Ω with opening angle γ, there is a neighborhood $\mathcal{N}_\gamma$ such that*

$$|H^\Omega(t,z,z') - H^{S_\gamma}(t,z,z')| = O(t^\infty), \quad \text{uniformly} \quad \forall (z,z') \in \mathcal{N}_\gamma \times \mathcal{N}_\gamma,$$

Above, H^{S_γ} denotes the heat kernel for S_γ with the Dirichlet boundary condition. Moreover, for any $\mathcal{N}_e \subset \Omega$ which is at a positive distance to all corners of Ω,

$$|H^{\Omega}(t,z,z') - H^{\mathbb{R}^2_+}(t,z,z')| = O(t^{\infty}), \quad \textit{uniformly} \quad \forall (z,z') \in \mathcal{N}_e \times \mathcal{N}_e.$$

Above, $H^{\mathbb{R}^2_+}$ *denotes the heat kernel for a half space with the Dirichlet boundary condition.*

The proof uses probabilistic methods. We are currently unaware of a generalization to domains with corners in higher dimensions. However, it is reasonable to expect that such a generalization holds. Since Theorem 1 has already been demonstrated for the Dirichlet boundary condition in [16], we are interested in the Neumann and Robin boundary conditions. For this reason, we shall give a proof of a locality principle for both Neumann and Robin boundary conditions which holds in all dimensions, for domains with piecewise smooth boundary (in fact, only piecewise $\mathscr{C}^3$ boundary is required), as long as we have a suitable estimate on the second fundamental form on the boundary. Moreover, our locality principle, similar to that of [29], allows one to compare the heat kernels all the way up to the boundary. For this reason, the locality principles demonstrated below may be of independent interest.

2.3 Locality principle for Neumann boundary condition

Here we prove a locality principle for the Neumann boundary condition for domains in $\mathbb{R}^n$ with piecewise $\mathscr{C}^2$ boundary satisfying some relatively general geometric assumptions. Since we consider both bounded and unbounded domains, we require a uniform version of an interior cone condition:

Definition 2. Let $\varepsilon > 0$ and $h > 0$. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the (ε, h)-cone condition if, for every $x \in \partial\Omega$, there exists a ball $B(x,\delta)$ centered at x of radius δ, and a direction ξ_x, such that for all $y \in B(x,\delta) \cap \Omega$, the cone with vertex y directed by ξ_x of opening angle ε and height h is contained in Ω.

Definition 3. Let $\varepsilon > 0$ and $h > 0$. We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the two-sided (ε, h)-cone condition if both Ω and $\mathbb{R}^n \setminus \Omega$ satisfy the (ε, h)-cone condition.

Theorem 4 (Locality Principle for Neumann Boundary Condition). *Let Ω, Ω_0, and S be domains in $\mathbb{R}^n$ such that S and Ω are exact geometric matches on Ω_0, as in Definition 1. Assume that both Ω and S satisfy the two-sided (ε, h)-cone condition for some $\varepsilon > 0$ and $h > 0$. Let H^{Ω} denote the heat kernel associated to the Laplacian on Ω, and let H^S denote the heat kernel on S, with the same boundary condition for ∂S as taken on $\partial\Omega$. Moreover, assume that there exists $\sigma \in \mathbb{R}$ such that the second fundamental form $\mathbb{I} \geq -\sigma$ holds on all the $\mathscr{C}^2$ pieces of $\partial\Omega$ and ∂S. Then*

$$\left| H^{\Omega}(t,z,z') - H^S(t,z,z') \right| = O(t^{\infty}) \text{ as } t \downarrow 0, \quad \text{uniformly for } z, z' \in \Omega_0.$$

Proof. We use a patchwork parametrix construction, as discussed in section 3.2 of [1]. This is a general technique to construct heat kernels whenever one has exact geometric matches for each part of a domain.

Let $\{\chi_j\}_{j=1}^2$ be a $\mathscr{C}^\infty$ partition of unity on Ω. Assume that $\tilde{\chi}_j \in \mathscr{C}^\infty(\Omega)$ is identically 1 on a small neighborhood of the support of χ_j and vanishes outside a slightly larger neighborhood. In particular, we choose χ_1 to be identically equal to one on Ω_0. Choose $\tilde{\chi}_1$ to be identically one on a strictly larger neighborhood and to have its support equal to Ω_c. We assume that the support of $\tilde{\chi}_2$ does not intersect Ω_0. We then define the patchwork heat kernel

$$G(t,z,z') := \sum_{j=1}^{2} \tilde{\chi}_j(z) H^S(t,z,z') \chi_j(z').$$

We claim that uniformly for all z, $z' \in \Omega_0$,

$$|H^\Omega(t,z,z') - G(t,z,z')| = O(t^\infty), \quad t \downarrow 0.$$

That is, we claim that the patchwork heat kernel is equal to the true heat kernel with an error that is $O(t^\infty)$ for small time. This claim immediately implies our result, since on Ω_0, $\chi_1 = 1$, and $\tilde{\chi}_1 = 1$, whereas χ_2 and $\tilde{\chi}_2$ both vanish, and thus $G(t,z,z') = H^S(t,z,z')$.

To prove the claim, we follow the usual template. Observe that

$$E(t,z,z') := (\partial_t + \Delta) G(t,z,z) = \sum_{j=1}^{2} [\Delta, \tilde{\chi}_j(z)] H^S(t,z,z') \chi_j(z').$$

Each commutator $[\Delta, \tilde{\chi}_j(z)]$ is a first-order differential operator with support a positive distance from the support of χ_j. Thus $E(t,z,z')$ is a sum of model heat kernels and their first derivatives, cut off so that their spatial arguments are a positive distance from the diagonal. We claim each such term is $O(t^\infty)$. To obtain this estimate, we use [31, Theorem 1.1], which gives the estimate

$$|\nabla H^D(t,z,z')| \leq \frac{C_\alpha}{t^{(n+1)/2}} \exp\left(- \frac{|z-z'|}{C_\beta t} \right), \quad z, z' \in D,$$

for some constants $C_\alpha, C_\beta > 0$, for $D = \Omega$ and $D = S$. The setting there is not identical, so we note the places in the proof where minor modifications are required. First, the assumption that Ω is compact is used there to obtain estimates for *all* $t > 0$. In particular, the discreteness of the spectrum is used to obtain long time estimates by exploiting the first positive eigenvalue in (2.1) of [31]. Since we are only interested in $t \downarrow 0$, this long time estimate is not required. Next, compactness is used to be able to estimate the volume of balls, $|B(x, \sqrt{t})| \geq C_\varepsilon t^{\frac{n}{2}}$, for a uniform constant C_ε. However, we have this estimate due to the two-sided (ε, h)-cone condition which is satisfied for both Ω and S which are contained in $\mathbb{R}^n$. Moreover, we have verified [35] that the assumption of piecewise $\mathscr{C}^2$ boundary (rather than $\mathscr{C}^2$ boundary) is sufficient for the proof of [31, Theorem 1.1], as well as the references used therein: [32], [33], [34].[5]

Since the domains S and Ω satisfy the two-sided (ε,h)-cone condition, there are Gaussian upper bounds for the corresponding Neumann heat kernels. Specifically, as a result of [4, Theorems 6.1, 4.4], for any $T>0$, there exist $C_1, C_2 > 0$ such that

$$|H^S(t,z,z')| \leq C_1 t^{-\frac{n}{2}} e^{-\frac{|z-z'|^2}{C_2 t}}, \qquad |H^\Omega(t,z,z')| \leq C_1 t^{-\frac{n}{2}} e^{-\frac{|z-z'|^2}{C_2 t}} \tag{3}$$

on $(0,T]\times S\times S$ and $(0,T]\times\Omega\times\Omega$ respectively. The upshot is that each term in the sum defining $E(t,z,z')$ is uniformly $O(t^\infty)$ for all z and z' in Ω, and therefore

$$|E(t,z,z')| = O(t^\infty).$$

From here, the error may be iterated away using the usual Neumann series argument, as in [19] or section 4 of [27]. Letting $*$ denote the operation of convolution in time and composition in space, define

$$K := E - E*E + E*E*E - \dots.$$

It is an exercise in induction to see that $K(t,z,z')$ is well-defined and also $O(t^\infty)$ as t goes to zero, see for example the proof of parts a) and b) of Lemma 13 of [27]. Note that Ω is compact, which is key. Then the difference of the true heat kernel and the patchwork heat kernel is

$$H^\Omega(t,z,z') - G(t,z,z') = -(G*K)(t,z,z').$$

As in Lemma 14 of [27], this can also be bounded in straightforward fashion by $O(t^\infty)$, which completes the proof. □

The key ingredients in this patchwork construction are: (1) the model heat kernels satisfy off-diagonal decay estimates, and (2) the gradients of these model heat kernels satisfy similar estimates. The argument can therefore be replicated in any situation where all models satisfy those estimates. Here is one generalization:

Corollary 2. *Using the notation of Theorem 4, suppose that Ω is compact and that the heat kernels on both Ω and S satisfy off-diagonal bounds of the following form: if A and B are any two sets with $d(A,B)>0$, then uniformly for $z\in A$ and $z'\in B$, we have*

$$|H(t,z,z')| + |\nabla H(t,z,z')| = O(t^\infty) \text{ as } t\to 0. \tag{4}$$

Then the conclusion of Theorem 4 holds.

Proof. Apply the same method, with a partition of unity on Ω consisting of just two components, one cutoff function for Ω_0 where we use the model heat kernel H^S, and one cutoff function for the rest of Ω where we use H^Ω. The result follows.

[5] We have also verified in private communication with F. Y. Wang that the arguments in [31], [32], [33], [34] apply equally well under the curvature assumption $\mathbb{I} \geq -\sigma$ for piecewise $\mathscr{C}^2$ boundary.

Remark 1. The bounds (4) are satisfied, for example, by Neumann heat kernels on compact, convex domains with no smoothness assumptions on the boundary [31], as well as by both Dirichlet and Neumann heat kernels on sectors, half-spaces, and Euclidean space.

2.4 Locality for Robin boundary condition

In this section, we determine when locality results similar to those of Theorem 4 hold for the Robin problem. The answer is that in many cases they may be deduced from locality of the Neumann heat kernels. We consider a generalization of the classical Robin boundary condition (2)

$$\frac{\partial}{\partial n}u(x) + c(x)u(x) = 0, \qquad x \in \text{ the smooth pieces of } \partial D. \tag{5}$$

In the first version of the locality principle, to simplify the proof, we shall assume that $\Omega \subset S$, and that Ω is bounded. We note, however, that both of these assumptions can be removed in the corollary to the theorem. The statement of the theorem may appear somewhat technical, so we explain the geometric interpretations of the assumptions. Conditions (1) and (2) below are clear; they are required to apply our references [31] and [4]. Items (3), (4), and (5) mean that the (possibly unbounded) domain, $D = S$ (and as in the corollary, in which Ω may be unbounded, $D = \Omega$) has boundary which does not oscillate too wildly or "bunch up" and become space-filling. These assumptions are immediately satisfied when the domains are bounded or if the boundary consists of finitely many straight pieces (like a sector in $\mathbb{R}^2$, for example).

Theorem 5 (Locality Principle for Robin Boundary Condition). *Assume that Ω and S are exact geometric matches on Ω_0, as in Definition 1, with $\Omega_0 \subset \Omega \subset S \subset \mathbb{R}^n$. Assume that Ω is bounded. Let $K^S(t,x,y)$ and $K^\Omega(t,x,y)$ be the heat kernels for the Robin Laplacian with boundary condition* (5) *for $D = S$ and $D = \Omega$, respectively, for the same $c(x) \in \mathscr{L}^\infty(\partial S \cup \partial \Omega)$. Let $\alpha := dist(\Omega_0, S \setminus \Omega)$, and note that $\alpha > 0$ by our assumption of an exact geometric match. Define the auxiliary domain*

$$W := \{x \in \Omega : d(x, \Omega_0) \leq \alpha/2\}.$$

We make the following geometric assumptions:

1. *Both S and Ω satisfy the two-sided (ε, h)-cone condition;*
2. *Both S and Ω have piecewise $\mathscr{C}^3$ boundaries, and there exists a constant $\sigma \in \mathbb{R}$ such that the second fundamental form satisfies $\mathbb{I} \geq -\sigma$ on all the $\mathscr{C}^3$ pieces of both ∂S and $\partial \Omega$.*
3. *For any sufficiently small $r > 0$ and any $t > 0$, we have*

$$\sup_{x \in W} \int_0^t \int_{\partial S \setminus B(x,r)} \frac{1}{s^{\frac{n}{2}}} e^{-\frac{|x-z|^2}{s}} \sigma(dz) ds < \infty; \tag{6}$$

4. *For all $r > 0$ and all $x \in \mathbb{R}^n$, and both $D = S$ and $D = \Omega$, there is a constant C_D such that*
$$\mathscr{H}^{n-1}(\partial D \cap (B(x,r)) \leq C_D Vol_{n-1}(B_{n-1}(x,r)), \tag{7}$$
where $\mathscr{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure;
5. *If $G_n(x,y)$ is the free Green's function on $\mathbb{R}^n$, we have*
$$\sup_{x \in W} \int_{\partial\Omega} G_n(x,y)\sigma(dy) < \infty. \tag{8}$$

Then, uniformly on $\overline{\Omega}_0 \times \overline{\Omega}_0$, we have Robin locality:
$$|K^S(t,z,z') - K^\Omega(t,z,z')| = O(t^\infty), \qquad \forall z,z' \in \Omega_0, \quad t \to 0.$$

.

The assumptions that $\Omega \subset S$ and that Ω is bounded can both be removed:

Corollary 3. *Suppose we have an exact geometric match between Ω and S on the bounded domain Ω_0, and the Robin coefficient $c(x)$ agrees on a common open, bounded neighborhood Ω_c of Ω_0 in Ω and S. Then, as long as Theorem 5 holds for the pairs (Ω_0, Ω) and (Ω_0, S), the conclusion of Theorem 5 holds for the pair (Ω, S).*

Proof. Apply Theorem 5 to the pairs (Ω_0, Ω) and (Ω_0, S), using the same W, then use the triangle inequality. □

Before we prove Theorem 5, we discuss the geometric assumptions (6), (7), and (8), and give some sufficient conditions for them to hold. First, observe that regardless of what W is, (6) is immediately valid if S is a bounded domain whose boundary has finite $n-1$ dimensional Lebesgue measure. It is also valid if S is an infinite circular sector, by a direct computation, part of which is presented below.

Example 1. Let $S = S_\gamma \subset \mathbb{R}^2$ be a circular sector of opening angle γ and infinite radius. Assume that W and Ω are bounded domains such that $W \subset \Omega \subset S$, and assume for simplicity that W contains the corner of S; see figure 2 (the case where this does not happen is similar.) Then (6) holds. Indeed, let $r \in (0, \alpha/2)$ and $t > 0$, then
$$\begin{aligned}
&\sup_{x \in W} \int_0^t \int_{\partial S \setminus B(x,r)} \frac{1}{s} e^{-\frac{|x-z|^2}{s}} \sigma(dz) ds \\
\leq &\sup_{x \in W} \int_0^t \int_{\partial S \setminus \partial\Omega} \frac{1}{s} e^{-\frac{|x-z|^2}{s}} \sigma(dz) ds + \sup_{x \in W} \int_0^t \int_{(\partial S \cap \partial\Omega) \setminus B(x,r)} \frac{1}{s} e^{-\frac{|x-z|^2}{s}} \sigma(dz) ds \\
\leq &\, 2 \int_0^t \int_0^\infty \frac{1}{s} e^{-\frac{\tau^2}{s}} d\tau ds + \int_0^t \frac{1}{s} e^{-\frac{r^2}{s}} \int_{(\partial S \cap \partial\Omega)} \sigma(dz) ds < \infty.
\end{aligned}$$

Moreover, recalling the Green's function in two dimensions (9), we also have
$$\sup_{x \in W} \int_{\partial\Omega \cap B(x,r)} G_n(x,y)\sigma(dy) = \sup_{x \in W} \int_{\partial\Omega \cap B(x,r)} |\ln|x-z||\sigma(dz) \leq \int_0^\alpha |\ln \tau| d\tau < \infty.$$

As for (7), this is automatic if D is a bounded domain with piecewise $\mathscr{C}^1$ boundary. It is also true if D is a circular sector (in fact here $C_D = 2$).

The condition (8) is also easy to satisfy:

Proposition 1. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ which has piecewise $\mathscr{C}^3$ boundary. Let $W \subset \Omega$ be a compact set, then* (8) *holds.*

Proof. Recall that

$$G_n(x,y) = \begin{cases} |\ln|x-y||, & \text{if } n = 2; \\ |x-y|^{2-n}, & \text{if } n \geq 3. \end{cases} \tag{9}$$

Since W is compact, it is enough to prove that

$$x \mapsto \int_{\partial\Omega} G_n(x,y)\sigma(dy) \tag{10}$$

is a continuous function on W.

Fix $x \in W$. Let $\varepsilon > 0$ and $\{x_j\}_{j=1}^{\infty} \subset W$ be a sequence such that $x_j \to x$. Since $\partial\Omega$ is piecewise $\mathscr{C}^3$, and $G_n(x,y)$ is in $\mathscr{L}^1_{loc}$, we can choose $\delta > 0$ such that

$$\int_{\partial\Omega \cap B(x,2\delta)} G_n(x,y)\sigma(dy) < \varepsilon, \qquad \int_{\partial\Omega \cap B(x,2\delta)} G_n(x_j,y)\sigma(dy) < \varepsilon, \tag{11}$$

for sufficiently large $j \in \mathbb{N}$, such that for these j we also have $|x - x_j| < \delta$. To see this, we note that $G_n(x,y) = G_n(|x-y|) = G_n(r)$, where $r = |x-y|$, and similarly, $G_n(x_j,y) = G_n(r_j)$ with $r_j = |x_j - y|$. Thus, choosing the radius, 2δ, sufficiently small, since G_n is locally $\mathscr{L}^1(\partial\Omega)$ integrable, and $\partial\Omega$ is piecewise $\mathscr{C}^3$, we can make the above integrals as small as we like.

Now, we note that $G_n(x_j,y) \to G_n(x,y)$ as $j \to \infty$, for $y \in \partial\Omega \setminus B(x,2\delta)$. Moreover, since Ω and thus $\partial\Omega$ are both compact, $G_n(x_j,y) < C = C(\delta)$ for $y \in \partial\Omega \setminus B(x,2\delta)$. The Dominated Convergence Theorem therefore implies

$$\left| \int_{\partial\Omega \setminus B(x,2\delta)} (G_n(x,y) - G_n(x_j,y))\sigma(dy) \right| < \varepsilon$$

for sufficiently large $j \in \mathbb{N}$. This, together with (11), implies that the function (10) is continuous on W.

In summary, we have

Corollary 4. *The locality principle, Theorem 5, holds in the case where Ω is a bounded domain in $\mathbb{R}^n$ with piecewise $\mathscr{C}^3$ boundary, and S is any domain with piecewise $\mathscr{C}^3$ boundary such that Ω and S are an exact geometric match on the bounded subdomain Ω_0 as in Definition 1. Moreover, we assume that:*

1. *Both S and Ω satisfy the two-sided (ε, h)-cone condition;*
2. *There exists a constant $\sigma \in \mathbb{R}$ such that the second fundamental form satisfies $\mathbb{I} \geq -\sigma$ on both ∂S and $\partial\Omega$;*
3. *S satisfies* (6) *and* (7)*;*

4. *The Robin coefficient* $c(x) \in \mathscr{L}^\infty(\partial S \cup \partial\Omega)$ *agrees on a common open bounded neighborhood* Ω_c *in* Ω *and* S.

Remark 2. In particular, all assumptions are satisfied if Ω is a bounded polygonal domain in $\mathbb{R}^2$, and S is a circular sector in $\mathbb{R}^2$.

The proof of Theorem 5 is accomplished by proving several estimates, in the form of lemmas and propositions below. Since the domains S and Ω satisfy the two-sided (ε, h)-cone condition, there are Gaussian upper bounds for the corresponding Neumann heat kernels as given in (3). With this in mind, define

$$F_1(t) := \sup_{(s,x,z)\in(0,t]\times W\times(\overline{S\cap\Omega})} \left| H^S(s,x,z) - H^\Omega(s,x,z) \right|,$$
$$F_2(t) := \sup_{(s,x,z)\in(0,t]\times W\times S\setminus\Omega} \left| H^S(s,x,z) \right|,$$
$$F_3(t) := \sup_{(s,x,z)\in(0,t]\times W\times\partial\Omega\setminus\partial S} \left| H^\Omega(s,x,z) \right|.$$

It now follows from (3) and Theorem 4 that

$$F(t) := \max(F_1(t), F_2(t), F_3(t)) = O(t^\infty), \qquad t \to 0. \tag{12}$$

The reason we require the Neumann heat kernels is because, as in [25, 39][6], the Robin heat kernels, $K^S(t,x,y)$ and $K^\Omega(t,x,y)$, can be expressed in terms of $H^S(t,x,y)$ and $H^\Omega(t,x,y)$ in the following way. Define

$$k_0^D(t,x,y) = H^D(t,x,y), \quad D = S \text{ and } D = \Omega,$$

and

$$k_m^D(t,x,y) = \int_0^t \int_{\partial D} H^D(s,x,z)c(z)k_{m-1}^D(t-s,z,y)\sigma(dz)ds \tag{13}$$

for $m \in \mathbb{N}$. Then

$$K^S(t,x,y) = \sum_{m=1}^\infty k_m^S(t,x,y), \qquad K^\Omega(t,x,y) = \sum_{m=1}^\infty k_m^\Omega(t,x,y).$$

Let us define the function

$$\begin{aligned} A(t,x) := \int_0^t \int_{\partial S} \left| H^S(s,x,z)c(z) \right| \sigma(dz)ds + \int_0^t \int_{\partial\Omega} \left| H^\Omega(s,x,z)c(z) \right| \sigma(dz)ds \\ =: A_1(t,x) + A_2(t,x). \end{aligned} \tag{14}$$

on $(0,1] \times W$. The following lemma, in particular, shows that $A(t,x)$ is a well defined function.

---

[6] We note that the result is stated for compact domains. However, the construction is purely formal and works as long as the series converges. Under our assumptions, we shall prove that it does.

Lemma 1. *The function $A(t,x)$ is uniformly bounded on $(0,1]\times W$.*

Proof. For $n=1$ the lemma follows from (3). Hence, we assume here $n\geq 2$. For any $x\in\overline{W}$, $A_j(t,x)$, $j=1,2$, is an increasing function with respect to the variable $t\in(0,1]$. Therefore, it is sufficient to prove that $A_j(x):=A_j(1,x)$ is bounded on W, for $j=1,2$.

Let us choose $0<\rho<\min(\alpha/2,1)$. Without loss of generality, setting $C_1=C_2=1$ in (3), we obtain

$$\begin{aligned} &A_1(x)+A_2(x)\\ &\leq \int_0^1\int_{\partial S\setminus B(x,\rho)} s^{-\frac{n}{2}}e^{-\frac{|x-z|^2}{s}}|c(z)|\sigma(dz)ds+\int_0^1\int_{\partial S\cap B(x,\rho)} s^{-\frac{n}{2}}e^{-\frac{|x-z|^2}{s}}|c(z)|\sigma(dz)ds\\ &+\int_0^1\int_{\partial\Omega\setminus B(x,\rho)} s^{-\frac{n}{2}}e^{-\frac{|x-z|^2}{s}}|c(z)|\sigma(dz)ds+\int_0^1\int_{\partial\Omega\cap B(x,\rho)} s^{-\frac{n}{2}}e^{-\frac{|x-z|^2}{s}}|c(z)|\sigma(dz)ds\\ &=: J_1(x)+J_2(x)+J_3(x)+J_4(x). \end{aligned}$$

The boundedness of $J_1(x)$ on W follows from (6) and the assumption that $c(z)\in\mathscr{L}^\infty$. For $J_3(x)$ we estimate using only that $\partial\Omega$ is bounded and thus, since it is piecewise $\mathscr{C}^3$, has finite measure,

$$J_3(x)\leq\|c\|_\infty\int_0^1\frac{1}{s^{\frac{n}{2}}}e^{-\frac{\rho^2}{s}}\int_{\partial\Omega\setminus B(x,\rho)}\sigma(dz)ds<\infty.$$

Since $\rho<\alpha/2$, $\partial S\cap B(x,\rho)=\partial\Omega\cap B(x,\rho)$ for $x\in W$, and hence by Fubini's theorem and a change of variables

$$\begin{aligned} J_2(x)=J_4(x)&=\int_0^1\int_{\partial\Omega\cap B(x,\rho)} s^{-\frac{n}{2}}e^{-\frac{|x-z|^2}{s}}|c(z)|\sigma(dz)ds\\ &\leq\|c\|_\infty\int_{\partial\Omega\cap B(x,\rho)}\frac{1}{|x-z|^{n-2}}\int_{|x-z|^2}^{+\infty}\tau^{\frac{n}{2}-2}e^{-\tau}d\tau\sigma(dz). \end{aligned}$$

For $n>2$, the second integral is uniformly bounded, and hence, (8) implies that $J_2(x)$ and $J_4(x)$ are bounded on W. If on the other hand $n=2$, then

$$J_2(x)=J_4(x)=\int_{\partial\Omega\cap B(x,\rho)}|c(z)|\int_{|x-z|^2}^{+\infty}\tau^{-1}e^{-\tau}d\tau\sigma(dz).$$

Since $\rho<1$, $\rho^2<\rho<1$, so we can write

$$\begin{aligned} J_2(x)=J_4(x)&\leq\int_{\partial\Omega\cap B(x,\rho)}|c(z)|\int_{|x-z|^2}^{1}\tau^{-1}d\tau\sigma(dz)+\int_{\partial\Omega\cap B(x,\rho)}|c(z)|\int_1^{+\infty}e^{-\tau}d\tau\sigma(dz)\\ &\leq\|c\|_\infty\int_{\partial\Omega\cap B(x,\rho)}\left|\ln|x-z|^2\right|\sigma(dz)+\|c\|_\infty\int_{\partial\Omega\cap B(x,\rho)}\sigma(dz), \end{aligned}$$

which is finite by (8), the boundedness of $\partial\Omega$ and the piecewise $\mathscr{C}^3$ smoothness of the boundary. □

Corollary 5. *In the notation of Lemma 1, we have*

$$\lim_{T\to 0}\sup_{(t,x)\in(0,T]\times W} A(t,x)=0.$$

Proof. Consider the functions $A_j(t,x)$. They are monotone increasing in t for each x, and they are continuous in x for each t by continuity of solutions to the heat equation. We claim that as $t\to 0$, $A(t,x)$ approaches zero pointwise. To see this write the time integral from 0 to t in each $A_j(t,x)$, $j=1,2$, as a time integral over $[0,1]$ by multiplying the integrand by the characteristic function $\chi_{[0,t]}$. For example,

$$A_1(t,x)=\int_0^1\int_{\partial S}\chi_{[0,t]}|H^S(s,x,z)c(z)|\,\sigma(dz)ds.$$

The integrands are bounded by $|H^S(s,x,z)c(z)|$, which is integrable by Lemma 1. For each x, they converge to zero as $t\to 0$. So by the Dominated Convergence Theorem applied to each $A_j(t,x)$, we see that $A(t,x)\to 0$ as $t\to 0$ for each x.

Now we have a monotone family of continuous functions converging pointwise to a continuous function (zero) on the compact set W. By Dini's theorem, this convergence is in fact uniform, which is precisely what we want. □

To use this, fix a small number A to be chosen later. Then Corollary 5 allows us to find $T>0$ such that

$$A(t,x)<A,\qquad (t,x)\in(0,T]\times W. \tag{15}$$

Next we prove the following two auxiliary propositions.

Proposition 2. *The following inequality holds with $D=S$ and $D=\Omega$:*

$$\int_0^t\int_{\partial D}\left|k_m^D(s,x,z)c(z)\right|\sigma(dz)ds\le 2^{m+1}A^{m+1} \tag{16}$$

on $(0,T]\times W$, for any $m\in\mathbb{N}$. Moreover, an identical inequality holds when $k_m^D(s,x,z)$ is replaced by $k_m^D(s,z,x)$.

Proof. By induction. For $m=0$, recalling the definition of $A(t,x)$, (14),

$$\int_0^t\int_{\partial D}|k_0^D(s,x,z)c(z)|\sigma(dz)ds=\int_0^t\int_{\partial D}|H^D(s,x,z)c(z)|\sigma(dz)ds\le A(t,x)<A.$$

We have thus verified the base case. Now, we assume that (16) holds for $k\le m$. Consider $k=m+1$:

$$\int_0^t\int_{\partial D}\left|k_{m+1}^D(s,x,z)c(z)\right|\sigma(dz)ds$$
$$=\int_0^t\int_{\partial D}\int_0^s\int_{\partial D}\left|H^D(\tau,z,\zeta)k_m^D(s-\tau,\zeta,x)c(\zeta)c(z)\right|\sigma(d\zeta)d\tau\sigma(dz)ds.$$

Changing variables:

$$\begin{aligned}
&\int_0^t \int_{\partial D} \int_0^s \int_{\partial D} \left| H^D(\tau,z,\zeta) k_m^D(s-\tau,\zeta,x) c(\zeta) c(z) \right| \sigma(d\zeta) d\tau \sigma(dz) ds \\
&\leq \int_0^t \int_{\partial D} \int_0^s \int_{\partial D} \left| H^D(s-\tau,z,\zeta) k_m^D(\tau,\zeta,x) c(\zeta) c(z) \right| \sigma(d\zeta) d\tau \sigma(dz) ds \\
&\leq \int_0^t \int_{\partial D} \int_0^t \int_{\partial D} \left| H^D(|s-\tau|,z,\zeta) k_m^D(\tau,\zeta,x) c(\zeta) c(z) \right| \sigma(d\zeta) d\tau \sigma(dz) ds \\
&\leq \int_0^t \int_{\partial D} \left(\int_0^t \int_{\partial D} \left| H^D(|s-\tau|,z,\zeta) c(z) \right| \sigma(dz) ds \right) \left| k_m^D(\tau,\zeta,x) c(\zeta) \right| \sigma(d\zeta) d\tau.
\end{aligned} \tag{17}$$

For the integrand, we compute

$$\begin{aligned}
&\int_0^t \int_{\partial D} \left| H^D(|s-\tau|,z,\zeta) c(z) \right| \sigma(dz) ds \\
&= \int_0^\tau \int_{\partial D} \left| H^D(|s-\tau|,z,\zeta) c(z) \right| \sigma(dz) ds + \int_\tau^t \int_{\partial D} \left| H^D(|s-\tau|,z,\zeta) c(z) \right| \sigma(dz) ds \\
&= \int_0^\tau \int_{\partial D} \left| H^D(\tau-s,z,\zeta) c(z) \right| \sigma(dz) ds + \int_0^{t-\tau} \int_{\partial D} \left| H^D(s,z,\zeta) c(z) \right| \sigma(dz) ds < 2A.
\end{aligned}$$

Therefore, from the induction hypothesis and (17), we obtain

$$\begin{aligned}
&\int_0^t \int_{\partial D} \int_0^s \int_{\partial D} \left| H^D(\tau,z,\zeta) k_m^D(s-\tau,\zeta,x) c(\zeta) c(z) \right| \sigma(d\zeta) d\tau \sigma(dz) ds \\
&\leq 2A \int_0^t \int_{\partial D} \left| k_m^D(\tau,\zeta,x) c(\zeta) \right| \sigma(d\zeta) d\tau \leq 2A \cdot 2^{m+1} A^{m+1} = 2^{m+2} A^{m+2},
\end{aligned}$$

as desired.

The estimates with x and z reversed are proved similarly. Note in particular that the base case works because $k_0^D = H_0^D$ is a Neumann heat kernel and is thus symmetric in its spatial arguments. □

We need one more lemma concerning pointwise bounds for k_m^D, which uses the geometric assumption (7).

Lemma 2. *Let $D = S$ or Ω. There exists $T_0 > 0$ such that for all m, all $t < T_0$, all $x \in D$, and all $y \in D$,*

$$|k_m^D(t,x,y)| \leq \frac{C_1}{2^m} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{C_2 t}}.$$

Proof. The proof proceeds by induction. The base case is $m = 0$, which is (3).

Now assume we have the result for $k = m$. Using the iterative formula (13), we have

$$|k_{m+1}^D(t,x,y)| \leq ||c||_\infty \int_0^t \int_{\partial D} |H^D(s,x,z) k_m^D(t-s,z,y)| \sigma(dz) ds. \tag{18}$$

Using (3) and the inductive hypothesis, we see that the integrand is bounded by

$$C_1 C_1 2^{-m} s^{-n/2} (t-s)^{-n/2} e^{-\frac{1}{C_2}\left(\frac{|x-z|^2}{s}+\frac{|z-y|^2}{t-s}\right)}.$$

First assume that D is a half-space. We do the estimate in the case $n=2$, because the general case is analogous. Hence, we use the coordinates $x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$, and estimate using $\{z_2=0\} \subset \mathbb{R}^2$ for ∂D. Dropping the constant factors, and saving the integral with respect to time for later, we therefore estimate

$$\int_{\mathbb{R}} s^{-1}(t-s)^{-1} e^{-\frac{|x-z|^2}{C_2 s}-\frac{|y-z|^2}{C_2(t-s)}} dz_1.$$

Without loss of generality, we shall assume that $x=(0,0)$. Then we are estimating

$$\int_{\mathbb{R}} s^{-1}(t-s)^{-1} e^{\frac{-z_1^2(t-s)-s|y-z|^2}{C_2 s(t-s)}} dz_1.$$

Since $z \in \partial D$, we have $z_2=0$. For the sake of simplicity, set $y_2=0$; the case where y_2 is nonzero is similar. Given this assumption, we set

$$z := z_1, \quad y := y_1,$$

and estimate

$$\int_{\mathbb{R}} s^{-1}(t-s)^{-1} e^{\frac{-z^2(t-s)-s(y-z)^2}{C_2 s(t-s)}} dz.$$

We do the standard trick of completing the square in the exponent. This gives

$$\int_{\mathbb{R}} s^{-1}(t-s)^{-1} \exp\left[-\left(\frac{\sqrt{t}z-\frac{sy}{\sqrt{t}}}{\sqrt{C_2}\sqrt{s}\sqrt{t-s}}\right)^2-\frac{y^2}{C_2(t-s)}+\frac{sy^2}{C_2 t(t-s)}\right] dz.$$

We therefore compute the integral over $\mathbb{R}$ in the standard way, obtaining

$$s^{-1/2}(t-s)^{-1/2}\sqrt{\frac{C_2\pi}{t}} e^{-\frac{y^2}{C_2(t-s)}+\frac{sy^2}{C_2 t(t-s)}} = s^{-1/2}(t-s)^{-1/2}\sqrt{\frac{C_2\pi}{t}} e^{\frac{-ty^2+sy^2}{C_2 t(t-s)}}$$

$$= s^{-1/2}(t-s)^{-1/2}\sqrt{\frac{C_2\pi}{t}} e^{-\frac{y^2}{C_2 t}}.$$

Finally, we compute the integral with respect to s,

$$\int_0^t \frac{1}{\sqrt{s}} \frac{1}{\sqrt{t-s}} ds = \pi.$$

Hence, the total expression is bounded from above by

$$\pi\sqrt{\frac{C_2\pi}{t}} e^{-\frac{y^2}{C_2 t}}.$$

Since we had assumed that $x=0$, we see that this is indeed

$$\pi\sqrt{\frac{C_2\pi}{t}}e^{-\frac{|x-y|^2}{C_2t}}.$$

Recalling the constant factors, we have

$$|k_{m+1}^D(t,x,y)| \leq C_1C_1||c||_\infty 2^{-m}\pi\sqrt{\frac{C_2\pi}{t}}e^{-\frac{|x-y|^2}{C_2t}}.$$

Now we note that the power of t is $t^{-(n-1)/2}$ for dimension $n=2$. Hence, we re-write the above estimate as

$$|k_{m+1}^D(t,x,y)| \leq C_1C_1||c||_\infty 2^{-m}\pi\sqrt{t}t^{-1}\sqrt{C_2\pi}e^{-\frac{|x-y|^2}{C_2t}}.$$

We then may choose for example

$$t \leq T_0 = \frac{1}{4(C_1+1)^2(||c||_\infty+1)^2\pi^3(C_2+1)}$$

$$\implies \sqrt{t} \leq \frac{1}{2(C_1+1)(||c||_\infty+1)\pi^{\frac{3}{2}}\sqrt{C_2+1}}.$$

This ensures that

$$|k_{m+1}^D(t,x,y)| \leq C_1 2^{-(m+1)}t^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{C_2t}}, \quad n=2.$$

We note that in general, for $\mathbb{R}^n$, by estimating analogously, noting that the integral will be over $\mathbb{R}^{n-1}$, we obtain

$$|k_{m+1}^D(t,x,y)| \leq ||c||_\infty 2^{-m}\pi(C_2\pi)^{n/2}t^{-(n-1)/2}e^{-\frac{|x-y|^2}{C_2t}}.$$

So, in the general-n case, we let

$$T_0 = \frac{1}{4(C_1+1)^2(||c||_\infty+1)^2\pi^{2+n}(C_2+1)^n}.$$

Then, for all $t \leq T_0$, we have

$$|k_{m+1}^D(t,x,y)| \leq C_1 2^{-m-1}t^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{C_2t}}.$$

Now consider the case where D is a general domain, not necessarily a half-space. As before, we have

$$|k_{m+1}^D(t,x,y)| \leq \frac{||c||_\infty C_1^2}{2^m}\int_0^t\int_{\partial D} s^{-n/2}(t-s)^{-n/2}e^{-\frac{1}{C_2}\left(\frac{|x-z|^2}{s}+\frac{|z-y|^2}{t-s}\right)}\sigma(dz)ds. \quad (19)$$

We claim that the right-hand side of (19) is less than or equal to C_D, the constant from (7), times the corresponding integral in the case where D is a half-plane through x and y. Assuming this claim, we get the same bound as for a half-plane, but with an extra C_D, and adjusting T_0 to absorb C_D as well, by putting an extra $(C_D+1)^2$ in the denominator, completes the proof.

To prove this claim, we use the so-called layer cake representation: rewrite the right-hand side of (19), without the outside constants, as

$$\int_0^t s^{-n/2}(t-s)^{-n/2}\int_{\partial D}\int_0^\infty \chi_{\{f(s,t,x,y,z)<a\}}e^{-a}\,da\sigma(dz)ds, \tag{20}$$

where naturally

$$f(s,t,x,y,z) := \frac{1}{C_2}\left(\frac{|x-z|^2}{s}+\frac{|z-y|^2}{t-s}\right).$$

The representation (20) may seem odd at first but reverts to (19) upon integration in a. Switching the order of integration in (20) (valid by Fubini-Tonelli , since everything is positive) and evaluating the z-integral, this becomes

$$\int_0^t s^{-n/2}(t-s)^{-n/2}\int_0^\infty \mathscr{H}^{n-1}(\partial D\cap\{z: f(s,t,x,y,z)<a\})e^{-a}\,dads. \tag{21}$$

Let us more closely examine the set $\{z: f(s,t,x,y,z)<a\}$. It is the set where

$$\left(1-\frac{s}{t}\right)|x-z|^2+\frac{s}{t}|z-y|^2<\frac{1}{t}C_2as(t-s).$$

It is straightforward to compute that this set is in fact a ball centered at the point $P(s,t,x,y) := (1-\frac{s}{t})x+\frac{s}{t}y$, with radius squared equal to

$$R^2(s,t,x,y) := \max\left\{0,\ \frac{1}{t}C_2as(t-s)-\frac{s}{t}\left(1-\frac{s}{t}\right)|y-x|^2\right\}.$$

Therefore (21) equals

$$\int_0^t s^{-n/2}(t-s)^{-n/2}\int_0^\infty \mathscr{H}^{n-1}(\partial D\cap B_n(P,R))e^{-a}\,dads. \tag{22}$$

By the assumption (7), this is bounded by

$$C_D\int_0^t s^{-n/2}(t-s)^{-n/2}\int_0^\infty \mathrm{Vol}_{n-1}(B_{n-1}(P,R))e^{-a}\,dads. \tag{23}$$

However, in the event that D is a half-space with x and $y\in\partial D$ (so also $P\in\partial D$), we have $\partial D\cap B_n(P,R)=B_{n-1}(P,R)$, so (22) equals

$$\int_0^t s^{-n/2}(t-s)^{-n/2}\int_0^\infty \mathrm{Vol}_{n-1}(B_{n-1}(P,R))e^{-a}\,dads. \tag{24}$$

Therefore, the integral (22) for general D is bounded by C_D times the integral (22) for a half-space. Since (22) is equal to the right-hand side of (19) without the preceding constants, the claim is proven. This completes the proof of Lemma 2. □

Remark 3. The key is that the integral is half an order better in t than the true heat kernel, which is a critical feature of the difference between Robin and Neumann heat kernels. It allows us to utilize the extra $\sqrt{t}$ to obtain the additional factor of 2^{-m} which is required for the induction step in the next proposition.

Now, we establish the main estimate to prove Theorem 5. Let

$$G(t) = \max\left\{F(t),\, 2C_1 t^{-(n/2)} e^{-\frac{(\alpha/2)^2}{C_2 t}}\right\}.$$

We note that of course we still have $G(t) = O(t^\infty)$ as $t \downarrow 0$.

Proposition 3. *There exists $T > 0$ such that the estimate*

$$|k_m^S(t,x,y) - k_m^\Omega(t,x,y)| \leq G(t)\cdot 7\cdot 2^{-m} \tag{25}$$

holds for all $(t,x,y) \in (0,T] \times W \times \overline{\Omega}_0$.

Proof. We choose T small enough so that $T < T_0$ in Proposition 3 and so that (15) holds with $A = 1/4$.

Now proceed by induction. The base case is instantaneous by definition of k_0 and of $F(t)$, using our locality principle for the Neumann case. So assume that (25) holds for $k = m$; we will prove it for $k = m+1$. Using some algebraic manipulations,

$$\begin{aligned} I := |k_{m+1}^S(t,x,y) - k_{m+1}^\Omega(t,x,y)| &\leq I_1 + I_2 + I_3 \\ := \int_0^t \int_{\partial S\cap\partial\Omega} \Big| H^S(s,x,z)k_m^S(t-s,z,y) &- H^\Omega(s,x,z)k_m^\Omega(t-s,z,y)\Big| |c(z)|\sigma(dz)ds \\ &+ \int_0^t \int_{\partial S\setminus\partial\Omega} |H^S(s,x,z)k_m^S(t-s,z,y)c(z)|\sigma(dz)ds \\ &+ \int_0^t \int_{\partial\Omega\setminus\partial S} |H^\Omega(s,x,z)k_m^\Omega(t-s,z,y)c(z)|\sigma(dz)ds \end{aligned}$$

We estimate these terms separately, beginning with I_1.

$$\begin{aligned} I_1 \leq &\int_0^t \int_{\partial S\cap\partial\Omega} \Big| H^S(s,x,z) - H^\Omega(s,x,z)\Big| |k_m^S(t-s,z,y)||c(z)|\sigma(dz)ds \\ &+ \int_0^t \int_{W\cap\partial S\cap\partial\Omega} \Big|k_m^S(t-s,z,y) - k_m^\Omega(t-s,z,y)\Big| \Big|H^\Omega(s,x,z)\Big| |c(z)|\sigma(dz)ds \\ &+ \int_0^t \int_{(\Omega\setminus W)\cap\partial S\cap\partial\Omega} \Big|k_m^S(t-s,z,y) - k_m^\Omega(t-s,z,y)\Big| \Big|H^\Omega(s,x,z)\Big| |c(z)|\sigma(dz)ds. \end{aligned}$$

The first term in the first integral is bounded by $F(t)$, since $x \in W$ and $z \in \overline{\Omega}$, so we may pull it out. We estimate the other term with Proposition 2 and get a bound of $F(t)\cdot 2^{m+1}A^{m+1} = F(t)\cdot 2^{-(m+1)}$ for the first integral.

For the second integral, we pull out the supremum of the first term using the inductive hypothesis. We estimate the other term using the definition of A and we get a bound of $G(t)\cdot 7\cdot 2^{-m-2}$.

For the third integral, we use Lemma 2 to pull out the first term, ignoring the difference and just estimating both k terms separately. Since $|z-y|\geq \alpha/2$ on this region, the supremum is less than $2^{-m}G(t)$ by Lemma 2. We estimate the other term using the definition of A and we get $1/4$, giving a bound of $2^{-m-2}G(t)$. Overall, we have

$$I_1 \leq G(t)(2^{-m-1}+7\cdot 2^{-m-2}+2^{-m-2}).$$

Next we estimate the terms I_2, I_3. In each, we pull out the supremum of $H^S(s,x,z)$ over the relevant region, and observe that it is bounded above by $F(t)$. For the term remaining in the integral we use Proposition 2. Since $F(t)\leq G(t)$, we obtain a bound of $G(t)\cdot 2^{-m-1}$ for each of these two terms. Putting it all together, we see

$$I \leq G(t)(3\cdot 2^{-m-1}+7\cdot 2^{-m-2}+2^{-m-2}) = G(t)\cdot 2^{-m-1}\left(3+\frac{7}{2}+\frac{1}{2}\right) = G(t)\cdot 7\cdot 2^{-m-1},$$

as desired. □

Proof. Finally, we prove Theorem 5. By Proposition 3,

$$|K^S(t,x,y)-K^\Omega(t,x,y)| \leq \sum_{m=0}^{\infty} |k_m^S(t,x,y)-k_m^\Omega(t,x,y)| \leq \sum_{m=0}^{\infty} 7G(t)2^{-m} = 14\cdot G(t),$$

which is $O(t^\infty)$ as $t\to 0$. □

3 Hearing the corners of a drum

As a consequence of the work in the previous section, the locality principle holds for both the Neumann and Robin boundary conditions when Ω is a bounded domain as described in Theorem 1, and S is either a whole space, a half-space, a circular sector, or a smoothly bounded domain which is an exact geometric match for some piece of Ω. Therefore, to compute the heat trace expansion for $\Omega \subset \mathbb{R}^2$ satisfying the hypotheses of Theorem 1, it suffices to chop the domain into pieces and, depending on the piece, replace the true heat kernel with one of the following:

- the heat kernel for an infinite circular sector with the same opening angle and boundary conditions near a corner of Ω,
- the heat kernel for a smoothly bounded domain which is an exact match to Ω away from all the corners. Note that such a domain can be produced by rounding off each corner.

Henceforth we consider the Neumann boundary condition or the classical Robin boundary condition as in (2), so that in (5), $c(x)$ is a constant, specifically $c(x) = \alpha/\beta$. We shall compute the heat kernel for each model by first computing the Green's function.

The Green's function for a circular sector of infinite radius and opening angle γ, with the Neumann boundary condition, is, in polar coordinates,

$$G_N(s,r,\phi,r_0,\phi_0) = \frac{1}{\pi^2}\int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \times \Big\{\cosh(\pi - |\phi_0 - \phi|)\mu + \frac{\sinh\pi\mu}{\sinh\gamma\mu}\cosh(\phi+\phi_0-\gamma)\mu + \frac{\sinh(\pi-\gamma)\mu}{\sinh\gamma\mu}\cosh(\phi-\phi_0)\mu\Big\}d\mu. \quad (26)$$

For the Robin boundary condition,

$$G_R(s,r,\phi,r_0,\phi_0) = \frac{1}{\pi^2}\int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \times \Big\{\cosh(\pi - |\phi_0 - \phi|)\mu + \frac{\sinh\pi\mu}{\sinh\gamma\mu}\cosh(\phi+\phi_0-\gamma)\mu + \frac{\sinh(\pi-\gamma)\mu}{\sinh\gamma\mu}\cosh(\phi-\phi_0)\mu - \frac{\sinh\pi\mu}{\sinh\gamma\mu}\left(e^{(\phi+\phi_0-\gamma)\mu}\frac{\alpha}{\alpha+\beta\mu} + e^{-(\phi+\phi_0-\gamma)\mu}\frac{\alpha}{\alpha-\beta\mu}\right)\Big\}d\mu. \quad (27)$$

Above, $K_{i\mu}$ is the modified Bessel function of the second kind, and s is the spectral parameter of the resolvent, $(\Delta+s)^{-1}$. The derivation of these formulas stems from Fedosov's study of Kontorovich-Lebedev transforms [7] and shall be presented in our forthcoming work [23]. Using functional calculus techniques, as we do in [23], one may rigorously justify the statement that

$$H(t,r,\phi,r_0,\phi_0) = \mathscr{L}^{-1}\left(G(s,r,\phi,r_0,\phi_0)\right)(t),$$

where H denotes the heat kernel and $\mathscr{L}^{-1}$ denotes the inverse Laplace transform taken with respect to s. This allows us to pass from the Green's functions to the heat kernels on a sector, and we may then compute the short time asymptotic expansions of the heat traces using our locality principles.

3.1 Heat trace calculations

Let Ω be a domain with corners as described in Theorem 1. Assume that Ω has n corners. Let $\mathscr{N}_i$ be a neighborhood of the i^{th} corner consisting of a circular sector of radius R, with R sufficiently small so that each $\mathscr{N}_i$ can be taken to equal to Ω_0 in the definition of exact geometric match corresponding to S_i, where S_i is the infinite circular sector of interior angle equal to the angle θ_i. Then let U be a smoothly bounded domain such that U can be taken equal to S in the definition of exact geometric match, with $\Omega_0 = \Omega \setminus \{\mathscr{N}_i\}_{i=1}^n$. By our locality principles, the heat trace

$$\int_{\Omega} H^{\Omega}(t,z,z)dz = \int_{\Omega\setminus\{\mathcal{N}_i\}_{i=1}^n} H^U(t,z,z)dz + \sum_{i=1}^n \int_{\mathcal{N}_i} H^{S_i}(t,z,z)dz + O(t^\infty). \quad (28)$$

The calculation of the asymptotics of the integral of $H^U(t,z,z)$ is well-known and may be extracted from [21] and [39]. More interesting is the calculation of the heat trace near the corners. Let us define:

$$A := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\cosh(\pi - |\phi_0 - \phi|)\mu d\mu,$$

$$B := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\frac{\sinh \pi\mu}{\sinh \gamma\mu}\cosh(\phi + \phi_0 - \gamma)\mu d\mu$$

$$C := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\frac{\sinh(\pi-\gamma)\mu}{\sinh \gamma\mu}\cosh(\phi - \phi_0)\mu d\mu,$$

and

$$E := \int_0^\infty K_{i\mu}(rs)K_{i\mu}(r_0 s)\frac{\sinh \pi\mu}{\sinh \gamma\mu}\left(e^{(\phi+\phi_0-\gamma)\mu}\frac{\alpha}{\alpha+\beta\mu} + e^{-(\phi+\phi_0-\gamma)\mu}\frac{\alpha}{\alpha-\beta\mu}\right)d\mu.$$

With this terminology, the Neumann Green's function for an infinite sector is given by

$$\frac{1}{\pi^2}(A+B+C),$$

while the Robin Green's function for an infinite sector is

$$\frac{1}{\pi^2}(A+B+C-E).$$

We shall compute the heat trace contributions from each of these terms. In each calculation, we will take the inverse Laplace transform, restrict to the angular diagonal $\phi = \phi_0$ (which commutes with $\mathscr{L}^{-1}$), integrate in ϕ (same), restrict to $r = r_0$, and integrate in r.

3.1.1 Heat trace contribution from the A term

Setting $\phi = \phi_0$, we have by [10, 6.794.1]

$$\int_0^\infty K_{ix}(r\sqrt{s})K_{ix}(r_0\sqrt{s})\cosh(\pi x)dx = \frac{\pi}{2}K_0(\sqrt{(r-r_0)^2 s}).$$

Then, by [6, 5.16.35], we have

$$\mathscr{L}^{-1}[A] = \mathscr{L}^{-1}\left[\frac{\pi}{2}K_0(\sqrt{(r-r_0)^2 s})\right] = \frac{\pi}{2}\frac{1}{2}\frac{1}{t}e^{-\frac{(r-r_0)^2}{4t}}.$$

Hence for $\phi = \phi_0$,

$$\frac{1}{\pi^2}\mathscr{L}^{-1}(A) = \frac{e^{-\frac{(r-r_0)^2}{4t}}}{4\pi t}. \tag{29}$$

Setting $r = r_0$ gives $(4\pi t)^{-1}$, and integrating over $\mathscr{N}_i$, the contribution from this term to the heat trace is the usual area term:

$$\frac{A(\mathscr{N}_i)}{4\pi t}.$$

3.1.2 Heat trace contribution from the B term

Now we investigate the contribution from B. The first simplification is to restrict to $\phi = \phi_0$, then compute

$$\int_0^\gamma B|_{\phi=\phi_0} d\phi = \int_0^\gamma K_{ix}(rs)K_{ix}(r_0 s)\frac{\sinh \pi x}{\sinh \gamma x}\cosh(2\phi-\gamma)x d\phi.$$

The only dependence on the angle is in the cosh term, which may be explicitly integrated, and we obtain

$$\int_0^\gamma B|_{\phi=\phi_0} d\phi = \int_0^\infty K_{ix}(r\sqrt{s})K_{ix}(r_0\sqrt{s})\frac{\sinh \pi x}{x}dx = \frac{\pi^2}{2}I_0(r_0\sqrt{s})K_0(r\sqrt{s}),$$

where in the last equality we have used [10, 6.794.10]. Now take the inverse Laplace transform:

$$\mathscr{L}^{-1}\left[\int_0^\gamma B|_{\phi=\phi_0} d\phi\right] = \mathscr{L}^{-1}\left[\frac{\pi^2}{2}I_0(r_0\sqrt{s})K_0(r\sqrt{s})\right] = \frac{\pi^2}{2}\frac{1}{2t}e^{-\frac{r^2+r_0^2}{4t}}I_0(\frac{rr_0}{2t}). \tag{30}$$

Thus, we see that

$$\frac{1}{\pi^2}\mathscr{L}^{-1}\left[\int_0^\gamma B|_{\phi=\phi_0} d\phi\right] = \frac{1}{4t}e^{-\frac{r^2+r_0^2}{4t}}I_0(\frac{rr_0}{2t}). \tag{31}$$

To compute the trace, we make a change of variables, by setting

$$u = \frac{r^2}{2t}, \quad du = \frac{r}{t}dr.$$

Therefore,

$$\frac{1}{4t}\int_0^R e^{-r^2/2t}I_0\left(\frac{r^2}{2t}\right)rdr = \frac{1}{4}\int_0^{\frac{R^2}{2t}} e^{-u}I_0(u)du.$$

By [36, p. 79 (3)] with $\nu = 1$,

$$uI_1'(u) + I_1(u) = uI_0(u). \tag{32}$$

By [36, p. 79 (4)] with $\nu = 0$,

$$uI_0'(u) = uI_1(u). \tag{33}$$

We use these to compute

$$\begin{aligned}\frac{d}{du}\left(e^{-u}u(I_0(u)+I_1(u))\right) &= e^{-u}\left(-uI_0(u)-uI_1(u)+I_0(u)+I_1(u)+uI_0'(u)+uI_1'(u)\right)\\ &= e^{-u}\left(-uI_1(u)+I_0(u)+uI_0'(u)\right) \quad \text{(by (32))}\\ &= e^{-u}I_0(u) \quad \text{(by (33))}.\end{aligned}$$

Next, define

$$g(u) := e^{-u}u(I_0(u)+I_1(u)), \tag{34}$$

and note that we have computed

$$g'(u) = e^{-u}I_0(u).$$

We therefore have

$$\int_0^{R^2/2t} e^{-u}I_0(u)du = \left(g(R^2/2t) - g(0)\right).$$

Since $I_0(0) = 1$ and $I_1(0) = 0$ [36], it follows that $g(0) = 0$, and we therefore compute that

$$\int_0^{R^2/2t} e^{-u}I_0(u)du = g(R^2/2t) = e^{-R^2/2t}\frac{R^2}{2t}(I_0(R^2/2t)+I_1(R^2/2t)).$$

For large arguments, the Bessel functions admit the following asymptotic expansions (see [36])

$$I_j(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{1}{2x}\left(j^2-\frac{1}{4}\right)+\sum_{k=2}^{\infty}c_{j,k}x^{-k}\right), \quad x \gg 0, \quad j = 0,1.$$

Consequently, for $x = R^2/2t$,

$$\begin{aligned}g(R^2/2t) = \frac{R^2}{2t}e^{-R^2/2t}(I_0(R^2/2t)+I_1(R^2/2t)) &= \frac{R^2}{2t}\left(\frac{2}{\sqrt{2\pi(R^2/2t)}}\right) - O\left(\frac{1}{(R^2/2t)^{3/2}}\right)\\ &= \frac{R}{\sqrt{\pi t}} + O(\sqrt{t}), \quad t \downarrow 0.\end{aligned}$$

Recalling the factor of $\frac{1}{4}$, we see that the trace of B contributes

$$\frac{R}{4\sqrt{\pi t}} + O(\sqrt{t}), \quad t \downarrow 0. \tag{35}$$

Observe that this is precisely the usual perimeter term:

$$\frac{\ell(\mathcal{N}_i \cap \partial\Omega)}{8\sqrt{\pi t}} + O(\sqrt{t}).$$

3.1.3 Heat trace contribution from the C term

Next, we compute the trace of the C term. This is done following [29]. The cosh term drops out when $\phi = \phi_0$. Integrating with respect to the angle gives a factor of γ. We define

$$R(t) = -\mathcal{L}^{-1}\left(\frac{\gamma}{\pi^2}\int_0^\infty \frac{\sinh(\pi-\gamma)x}{\sinh(\gamma x)}\int_R^\infty K_{ix}^2(r\sqrt{s})rdr\right).$$

It is shown in [29] that

$$R(t) = O(e^{-c/t}),$$

and in fact an estimate is also obtained there for the constant $c > 0$. Hence, it suffices to compute

$$\mathcal{L}^{-1}\left(\frac{\gamma}{\pi^2}\int_0^\infty dx \frac{\sinh(\pi-\gamma)x}{\sinh\gamma x}\int_0^\infty K_{ix}^2(rs)rdr\right).$$

Here we use [10, 6.521.3]. As in that notation we have $a = s = b$, we must compute instead the limit of the expression as $b \to a$,

$$\lim_{b\to a}\frac{\pi(ab)^{-\nu}(a^\nu+b^\nu)}{2\sin(\nu\pi)(a+b)}\frac{f(a)-f(b)}{a-b}, \quad f(t) = t^\nu.$$

Then, since

$$f'(t) = \nu t^{\nu-1}$$

we have

$$\lim_{b\to a}\frac{\pi(ab)^{-\nu}(a^\nu+b^\nu)}{2\sin(\nu\pi)(a+b)}\frac{f(a)-f(b)}{a-b} = \frac{\pi a^{-2\nu}(2a^\nu)}{4\sin(\nu\pi)a}\nu a^{\nu-1} = \frac{\pi\nu}{2\sin(\pi\nu)a^2}.$$

Inserting our parameters, we have that

$$\int_0^\infty K_{ix}^2(r\sqrt{s})rdr = \frac{\pi x}{2\sinh(\pi x)s}.$$

So we must compute

$$\mathcal{L}^{-1}\left\{\frac{\gamma}{\pi^2}\int_0^\infty \frac{\sinh(\pi-\gamma)x}{\sinh\gamma x}\frac{\pi x}{2s\sinh(\pi x)}dx\right\}.$$

This calculation has been done in [29, p. 122] using [10]; we have independently verified these calculations as well. The result is given in [29, (2.10)]:

$$\frac{\pi^2-\gamma^2}{24\pi\gamma}.$$

Thus, we see that C contributes to the trace the usual "corner contribution":

$$\frac{\pi^2-\gamma^2}{24\pi\gamma}+O(t^\infty). \tag{36}$$

3.2 Trace of the E term

Let

$$\theta := \phi+\phi_0-\gamma.$$

Then,

$$\left(e^{(\phi+\phi_0-\gamma)x}\frac{\alpha}{\alpha+\beta x}+e^{-(\phi+\phi_0-\gamma)x}\frac{\alpha}{\alpha-\beta x}\right)=\frac{2\alpha^2\cosh(\theta x)-2\alpha\beta x\sinh(\theta x)}{\alpha^2-\beta^2x^2}. \tag{37}$$

It is straightforward to verify that because cosh is an even function, setting $\phi=\phi_0$,

$$\int_0^\gamma \sinh((2\phi-\gamma)x)d\phi=0.$$

Hence, when we are computing the trace, this term will not contribute anything. The term with cosh gives

$$\int_0^\gamma \frac{2\alpha^2\cosh(\theta x)}{\alpha^2-\beta^2x^2}d\phi=\frac{2\alpha^2\sinh(\gamma x)}{x(\alpha^2-\beta^2x^2)}.$$

Hence, after restricting to the angular diagonal and integrating with respect to ϕ, we obtain

$$\int_0^\gamma E d\phi = 2\alpha^2\int_0^\infty K_{ix}(r\sqrt{s})K_{ix}(r_0\sqrt{s})\frac{\sinh \pi x}{x(\alpha^2-\beta^2x^2)}dx.$$

Observe that after these calculations, there is no longer any dependence on the angle, γ. Hence, the contribution of the E term to the heat trace is *independent* of the angle, γ. We may therefore compute this contribution by taking $\gamma=\pi$. In that case, we have an explicit formula for $\mathscr{L}^{-1}(E_\pi)$, (see [2, (3.19)], and more classically [3, §14.2]),

$$\mathscr{L}^{-1}(E_\pi):=\widetilde{E_\pi}=-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-x')^2}{4t}}\frac{\alpha}{\beta}e^{\frac{\alpha(y+y')}{\beta}}e^{\frac{\alpha^2 t}{\beta^2}}\operatorname{erfc}\left(\frac{y+y'}{\sqrt{4t}}+\frac{\alpha}{\beta}\sqrt{t}\right).$$

3.2.1 Trace of the $\widetilde{E}_\pi$ term

The restriction of $\widetilde{E}_\pi$ to the diagonal yields

$$-\frac{1}{\sqrt{4\pi t}}\frac{\alpha}{\beta}e^{\frac{2\alpha y}{\beta}}e^{\frac{\alpha^2 t}{\beta^2}}\operatorname{erfc}\left(\frac{y}{\sqrt{t}}+\frac{\alpha}{\beta}\sqrt{t}\right).$$

We have to integrate this over the semicircle $x^2+y^2 \leq R^2$. Doing the x-integration first yields

$$\int_0^R -\frac{\sqrt{R^2-y^2}}{\sqrt{\pi t}}\frac{\alpha}{\beta}e^{\frac{2\alpha y}{\beta}}e^{\frac{\alpha^2 t}{\beta^2}}\operatorname{erfc}\left(\frac{y}{\sqrt{t}}+\frac{\alpha}{\beta}\sqrt{t}\right)dy.$$

We make a substitution by setting

$$u=\frac{y}{\sqrt{t}},$$

so we obtain

$$-\frac{\alpha}{\sqrt{\pi}\beta}e^{\frac{\alpha^2 t}{\beta^2}}\int_0^{R/\sqrt{t}} e^{2\alpha u\sqrt{t}/\beta}\sqrt{R^2-t^2u^2}\operatorname{erfc}(u+\frac{\alpha}{\beta}\sqrt{t})du.$$

We shall use integration by parts, noting that

$$\frac{d}{dz}\left(z\operatorname{erfc}(z)-\frac{e^{-z^2}}{\sqrt{\pi}}\right)=\operatorname{erfc}(z).$$

So,

$$\int_0^{R/\sqrt{t}} e^{2\alpha u\sqrt{t}/\beta}\sqrt{R^2-t^2u^2}\operatorname{erfc}(u+\frac{\alpha}{\beta}\sqrt{t})du$$

$$=e^{2\alpha\sqrt{t}u/\beta}\sqrt{R^2-t^2u^2}\left[\left(u+\frac{\alpha\sqrt{t}}{\beta}\right)\operatorname{erfc}(u+\alpha\sqrt{t}/\beta)-\frac{e^{-(u+\alpha\sqrt{t}/\beta)^2}}{\sqrt{\pi}}\right]_{u=0}^{R/\sqrt{t}}$$

$$-\int_0^{R/\sqrt{t}}\left(\frac{2\alpha\sqrt{t}}{\beta}-\frac{t(tu)}{R^2-t^2u^2}\right)e^{2\alpha\sqrt{t}u/\beta}\sqrt{R^2-t^2u^2}\operatorname{erfc}(u+\alpha\sqrt{t}/\beta)du.$$

It is a straightforward exercise to prove that

$$\int_0^{R/\sqrt{t}} e^{2\alpha\sqrt{t}u/\beta}\operatorname{erfc}(u+\alpha\sqrt{t}/\beta)du$$

is uniformly bounded as $t\downarrow 0$. Hence the second term is $O(\sqrt{t})$ as $t\downarrow 0$, for we can pull out a factor of $\sqrt{t}$ and keep every other term in the integrand bounded above by a constant. However, as $t\downarrow 0$, the first term converges to $R\pi^{-1/2}$. Hence, the E term gives a contribution of

$$-\frac{\alpha R}{\pi\beta} = -\frac{\alpha}{2\pi\beta}\ell(\mathscr{N}_i \cap \partial\Omega).$$

This is the usual perimeter term in the Robin setting [39].

3.3 Heat trace expansions and proof of Theorem 1

It is now straightforward to use (28) to compute the heat trace asymptotics for Ω by combining our explicit computations of the integrals over $\mathscr{N}_i$ with the known asymptotics [21, 39] for the integrals over $\Omega \setminus \{\mathscr{N}_i\}$. The trace of the C term, which gives the corner contribution, is the same as it is in the Dirichlet setting. Compared to a domain without a corner (in which turning the corner contributes to the curvature), each corner of angle θ_k contributes the following to the t^0 term in the heat asymptotics:

$$\frac{\pi^2-\theta_k^2}{24\pi\theta_k} - \frac{\pi-\theta_k}{12\pi} = \frac{\pi^2+\theta_k^2}{24\pi\theta_k} - \frac{1}{12}.$$

We therefore obtain:

(N) for the Neumann boundary condition,

$$\mathrm{tr} e^{-t\Delta} \sim \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{\chi(\Omega)}{6} - \frac{n}{12} + \sum_{k=1}^{n} \frac{\pi^2+\theta_k^2}{24\pi\theta_k} + O(\sqrt{t}),$$

(R) for the Robin boundary condition,

$$\mathrm{tr} e^{-t\Delta} \sim \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{\chi(\Omega)}{6} - \frac{|\partial\Omega|\alpha}{2\pi\beta} - \frac{n}{12} + \sum_{k=1}^{n} \frac{\pi^2+\theta_k^2}{24\pi\theta_k} + O(\sqrt{t}).$$

Now let $\tilde{\Omega}$ be a smoothly bounded domain in the plane. The heat trace expansions have been computed by [21] for the Neumann boundary condition and [39] for the Robin condition. These are, respectively,

(N) for the Neumann boundary condition,

$$\mathrm{tr} e^{-t\Delta} \sim \frac{|\tilde{\Omega}|}{4\pi t} + \frac{|\partial\tilde{\Omega}|}{8\sqrt{\pi t}} + \frac{\chi(\tilde{\Omega})}{6} + O(\sqrt{t}),$$

(R) for the Robin boundary condition,

$$\mathrm{tr} e^{-t\Delta} \sim \frac{|\tilde{\Omega}|}{4\pi t} + \frac{|\partial\tilde{\Omega}|}{8\sqrt{\pi t}} + \frac{\chi(\tilde{\Omega})}{6} - \frac{|\partial\tilde{\Omega}|\alpha}{2\pi\beta} + O(\sqrt{t}).$$

Proof (Theorem 1). If two domains are isospectral, then they have the same heat trace. Hence, for each power of t in such an expansion, the coefficient must be identical for both domains. Now, let us assume that Ω satisfies the assumptions in Theorem 1, so it has at least one corner of interior angle not equal to π. Let the interior angles at the corners be $\{\theta_k\}_{k=1}^n$. Let us assume that $\tilde{\Omega}$ is a smoothly bounded domain, and that the we have taken the same boundary condition for the Laplacian for both Ω and $\tilde{\Omega}$. Assume for the sake of contradiction that Ω and $\tilde{\Omega}$ are isospectral. Therefore, their heat trace coefficients coincide. Hence, they have the same area and perimeter. Since the same boundary condition is taken for both domains, and thus the same values of α and β in the Robin case, we should have

$$\frac{\chi(\Omega)}{6} - \frac{n}{12} + \sum_{k=1}^{n} \frac{\pi^2 + \theta_k^2}{24\pi\theta_k} = \frac{\chi(\tilde{\Omega})}{6}. \tag{38}$$

We have assumed that Ω is simply connected, but we make no such assumption on $\tilde{\Omega}$. Hence

$$\chi(\Omega) = 1, \quad \chi(\tilde{\Omega}) \leq 1.$$

Following the argument on p. 91–92 of [16],

$$\frac{\chi(\Omega)}{6} - \frac{n}{12} + \sum_{k=1}^{n} \frac{\pi^2 + \theta_k^2}{24\pi\theta_k} > \frac{1}{6} \geq \frac{\chi(\tilde{\Omega})}{6},$$

which violates (38). □

4 Microlocal analysis in the curvilinear case

It turns out that the heat trace expansions above are also valid for curvilinear polygons, once terms accounting for the curvature of the boundary away from the corners have been included. Although this has been demonstrated in [16] for the Dirichlet boundary condition using monotonicity, it becomes a much more subtle matter for the Neumann and Robin boundary conditions.

The main problem is that for curvilinear polygons, we no longer have an exact geometric match. Hence, we can no longer use the locality principle to compute the heat trace expansion, because there are no known expressions for the heat kernels. For classical polygons, one may compute the Neumann heat trace using the Dirichlet heat trace together with the trace of a Euclidean surface with conical singularities created by doubling the polygon. However, this technique fails once the edges of the polygon are no longer necessarily straight near the corners. Therefore, in order to compute the short time asymptotic expansion of the heat trace without exact geometric matches, we turn to the robust techniques of geometric microlocal analysis. This allows us to give a full description of the Dirichlet, Neumann, and Robin heat kernels on a curvilinear polygon in all asymptotic regimes. Restricting to the diagonal and integrating yields the heat trace.

In order to describe the heat kernel in all asymptotic regimes, we build a space, called the *heat space* or *double heat space*, on which the heat kernel is well-behaved. This space is built by blowing up various p-submanifolds of $\Omega \times \Omega \times [0,\infty)$. To see why this is needed, first consider the heat kernel (1) on $\mathbb{R}^n$. At the diagonal in $\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)$, the heat kernel behaves as $O(t^{-n/2})$ as $t \downarrow 0$. However, as long as $d(z,z') \geq \varepsilon > 0$, the heat kernel behaves as $O(t^{\infty})$ as $t \downarrow 0$. So the heat kernel fails to be well-behaved at $\{z = z', t = 0\}$. This is the motivation for "blowing up" the diagonal $\{z = z'\}$ at $t = 0$, which means replacing this diagonal with its inward pointing spherical normal bundle, corresponding to the introduction of "polar coordinates". The precise meaning of "blowing up" is explained in [20], and in this particular case of blowing up $\{z = z'\}$ at $t = 0$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)$, see [20, Chapter 7].

For the case of a curvilinear polygonal domain $\Omega \subset \mathbb{R}^2$, we begin with $\Omega \times \Omega \times [0,\infty)$ and perform a sequence of blow-ups. Our construction is inspired by the construction of the heat kernel on manifolds with wedge singularities performed by Mazzeo and Vertman in [19]. We leave the details to our forthcoming work [23].

Once the double heat space has been constructed, the heat kernel may be built in a similar spirit to the Duhamel's principle construction of the Robin heat kernel in the proof of Theorem 5. We start with a parametrix, or initial guess, and then use Duhamel's principle to iterate away the error. This requires the proof of a composition result for operators whose kernels are well-behaved on our double heat space, and that in turn requires some fairly involved technical machinery (a proof "by hand" without using this machinery would be entirely unreadable). However, it works out and gives us a very precise description of the heat kernel on a curvilinear polygon, with any combination of Dirichlet, Neumann, and Robin conditions. Moreover, we are able to generalize our techniques and results to surfaces which have boundary, edges, corners, and conical singularities.

The details of this sort of geometric microlocal analysis construction are intricate, but its utility is undeniable. In settings such as this, where exact geometric matches are lacking, but instead, one has *asymptotic geometric matches,* these microlocal techniques may be helpful. For the full story in the case of curvilinear polygons and their heat kernels, please stay tuned for our forthcoming work [23]. We have seen here that the heat kernels for circular sectors gives the same angular contribution, arising from the so-called "C term" for both Neumann and Robin boundary conditions. Moreover, this is the same in the Dirichlet case as well [29]. Interestingly, it appears that for mixed boundary conditions, there is a sudden change in this corner contribution. We are in the process of obtaining a small collection of negative isospectrality results in these general settings in the spirit of Theorem 1, including a generalization of Theorem 1 which removes the hypothesis that the corners are exact; see [23] for the full story.

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References

1. C. Aldana, J. Rowlett, *A Polyakov formula for sectors,* J Geom Anal (2017) https://doi.org/10.1007/s12220-017-9888-y
2. J. D. Bondurant and S. A. Fulling, *The Dirichlet-to-Robin Transform,* preprint arXiv:math-ph/0408054v1 (2004).
3. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids,* second edition, Oxford, Clarendon, (1959).
4. D.Daners, *Heat kernel estimates for operators with boundary conditions,* Mathematische Nachrichten, vol 217, no. 1, (2000), 13–41.
5. C. Durso, *On the inverse spectral problem for polygonal domains,* Ph. D. dissertation, Massachusetts Institute of Technology, Cambridge, 1988.
6. A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of integral transforms,* McGraw-Hill, (1954).
7. B. Fedosov, *Asymptotic formulas for eigenvalues of the Laplacian in a polyhedron.* Doklady Akad. Nauk SSSR, **157** (1964), 536–538.
8. C. Gordon, D. L. Webb, S. Wolpert, *One cannot hear the shape of a drum,* Bull. Amer. Math. Soc. (N.S.) **27** no. 1 (1992) 134–138.
9. —, —, —, *Isospectral plane domains and surfaces via Riemannian orbifolds,* Invent. Math. **110** no. 1 (1992) 1–22.
10. I. S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products,* eighth edition, Elsevier/Academic Press, Amsterdam, (2015).
11. D. Grieser, S. Maronna, *Hearing the shape of a triangle,* Notices Amer. Math. Soc. **60** no. 11 (2013) 1440–1447.
12. H. Hezari, Z. Lu, and J. Rowlett, *The Neumann Isospectral Problem for Trapezoids,* Ann. Henri Poincar 18, no. 12, (2017), 3759–3792.
13. M. Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly **73** no. 1 (1966) 1–23.
14. L. Li & A. Strohmaier, *Heat kernel estimates for general boundary problems,* J. Spectral Theory, 6, (2016) 903–919, (arxiv:1604.00784v1).
15. Z. Lu and J. Rowlett, *The sound of symmetry,* Amer. Math. Monthly 122, no. 9, (2015), 815–835.
16. Z. Lu and J. Rowlett, *One can hear the corners of a drum,* Bull. Lond. Math. Soc. 48, no. 1, (2016), 85–93.
17. W. Lück and T. Schick, *L^2-torsion of hyperbolic manifolds of finite volume,* Geom. Funct. Anal., 9, no. 3, (1999), 518–567.
18. R. Mazzeo and J. Rowlett, *A heat trace anomaly on polygons,* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 159 no. 02 (2015) 303–319.
19. R. Mazzeo and B. Vertman, *Analytic torsion on manifolds with edges.* Adv. Math. 231, no. 2, (2012), 1000–1040.
20. R. Melrose, *The Atiyah-Patodi-Singer Index Theorem,* Research Notes in Mathematics 4. A K Peters, Ltd., (1993).
21. H. P. McKean Jr., I. M. Singer, *Curvature and the eigenvalues of the Laplacian,* J. Diff. Geom. **1** no. 1 (1967) 43–69.
22. J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds,* Proc. Nat. Acad. Sci. U.S.A. **51** (1964) 542.
23. M. Nursultanov, J. Rowlett, and D. Sher, *The heat kernel on curvilinear polygons, surfaces, and singular geometric spectral invariants,* preprint.
24. F. Oberhettinger and L. Badii, *Tables of Laplace transforms,* Springer-Verlag, New York-Heidelberg, (1973).
25. V. G. Papanicolaou, *The probabilistic solution of the third boundary value problem for second order elliptic equations,* Probab. Theory Related Fields, 87, no. 1, (1990), 27–77.
26. Å. Pleijel, *A study of certain Green's functions with applications in the theory of vibrating membranes,* Ark. Mat. **2** (1954) 553–569.

27. D. Sher, *Conic degeneration and the determinant of the Laplacian,* J. Anal. Math. **126** no. 2 (2015) 175–226.
28. T. Sunada, *Riemannian coverings and isospectral manifolds,* Ann. of Math. (2) **121** no. 1 (1985) 169–186.
29. M. van den Berg, S. Srisatkunarajah, *Heat equation for a region in $\mathbb{R}^2$ with polygonal boundary,* J. London Math. Soc. (2) 2 no. 1 (1988) 119–127.
30. M. Taylor, *Partial differential equations I, Basic theory,* Applied Math. Sciences, vol. 115, second edition, Springer, New York, (2011).
31. F.Y. Wang & L. Yan, *Gradient estimate on the Neumann semigroup and applications,* arxiv:1009.1965v2.
32. F. Y. Wang, *Gradient estimates and the first Neumann eigenvalue on manifolds with boundary,* Stoch. Proc. Appl. 115 (2005), 1475–1486.
33. F. Y. Wang, *Semigroup properties for the second fundamental form,* Docum. Math. 15, (2010), 543–559.
34. F. Y. Wang, *Gradient and Harnack inequalities on noncompact manifolds with boundary,* Pacific Journal of Math. 245, (2010), 185–200.
35. F. Y. Wang, *private communication.*
36. G. N. Watson, *A treatise on the theory of Bessel functions, Cambridge Mathematical Library,* Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.
37. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung),* Math. Ann. **71** no. 4 (1912) 441–479.
38. E. Witt, *Eine Identität zwischen Modulformen zweiten Grades,* Abh. Math. Sem. Univ. Hamburg, **14** no. 1 (1941) 323–337.
39. E. M. Zayed, *Short time asymptotics of the heat kernel of the Laplacian of a bounded domain with Robin boundary conditions,* Houston J. Math, vol. 24, no. 2, (1998), 377–385.
40. S. Zelditch, *Spectral determination of analytic bi-axisymmetric plane domains,* Geom. Funct. Anal. 10, no. 3, (2000), 628–677.