Simple group actions on arc-transitive graphs with prescribed transitive local action

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Abstract This paper gives a partial answer to a question asked by Pierre-Emmanuel Caprace at the Groups St Andrews conference at Birmingham (UK) in August 2017, and investigated at the 'Tutte Centenary Retreat' workshop held at MATRIX in November 2017. Caprace asked if there exists a 2-transitive permutation group P such that only finitely many simple groups act arc-transitively on a connected graph X with local action P (of the stabiliser of a vertex v on the neighbourhood of v). Some evidence is given to suggest that the answer is "No", even when '2-transitive' is replaced by 'transitive', and then by way of illustration, a follow-up question is answered by showing that all but finitely many alternating groups have such an action on a 6-valent connected graph with vertex-stabiliser A_6.

1 Introduction

At the Groups St Andrews conference held at Birmingham (UK) in August 2017, Pierre-Emmanuel Caprace asked if there exists a 2-transitive permutation group P such that only finitely many simple groups act arc-transitively on a connected graph X in such a way that the stabiliser (in the simple group) of a vertex v induces P on the neighbourhood of v. This question and a follow-up question about what happens when P is the alternating group A_6 were conveyed by Gabriel Verret and Michael Giudici at the 'Tutte Centenary Retreat' workshop held at MATRIX in November 2017. What follows is a partial answer to the main question, showing that even when '2-transitive' is replaced by 'transitive', no such group P can exist if a certain conjecture about alternating quotients of amalgamated free products is valid, and then a full answer to the sub-question, showing that P cannot be A_6, as well as noting that P cannot be one of a number of other permutation groups.

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2 The general question

One approach that can be taken to the general question is to consider the action of a group G on a graph X with the property that the stabiliser V in G of a vertex v of X is isomorphic to P (and induces P on the neighbourhood $X(v)$). First, let d be the degree of P (as a transitive permutation group). Then we may suppose that $d \geq 3$, since the automorphism groups of 2-valent connected arc-transitive graphs are dihedral and therefore soluble, and the question by Caprace is not relevant.

Now observe that if E is the stabiliser of an edge $e = \{v,w\}$ incident with v, and A is the stabiliser of the arc (v,w) (and hence isomorphic to a point-stabiliser in the given group P), then G is a homomorphic image of the free product $V *_A E$ with the subgroup $A = V \cap E$ amalgamated. Moreover, A has index d in V, and 2 in E.

Next, a conjecture made by Džambić and Jones [10] and supported by the author asserts that if V and E are any two finite groups with a common subgroup A with index $|V:A| \geq 3$ and index $|E:A| \geq 2$, then all but finitely many alternating groups A_n occur as homomorphic images of the amalgamated free product $V *_A E$. An even stronger version of this conjecture (believed to be true by the author) is as follows:

Conjecture 1. *Let V and E be any two finite groups with a common subgroup A with index $|V:A| \geq 3$ and index $|E:A| \geq 2$, and let K be the core of A in $V *_A E$. Then all but finitely many A_n occur as the image of the amalgamated free product $V *_A E$ under some homomorphism that takes V and E to subgroups (of A_n) isomorphic to V/K and E/K respectively. In particular, if the amalgamated subgroup A is core-free in $V *_A E$, then all but finitely many A_n occur as images of $V *_A E$ under homomorphisms that are faithful on each of V and E.*

It is easy to see this is stronger than the conjecture in [10], since for example any quotient of $C_2 *_{C_1} C_3 = C_2 * C_3$ (the modular group) is also a quotient of $C_4 *_{C_2} D_3$, but not vice versa. Also there is plenty of evidence in support of it. Indeed it is known to be true in many special cases (proved well before the original conjecture was made in [10]), such as those arising in the way described above from the study of finite arc-transitive and/or path-transitive 3-valent graphs [4, 8], or 7-arc-transitive 4-valent graphs [9], or similarly from the study of arc-transitive digraphs [6], chiral maps [2] or chiral polytopes [5], and even hyperbolic 3-manifolds [7].

Furthermore, if the above conjecture is valid, then the answer to Caprace's main question can be shown to be 'No', even when V is not 2-transitive:

Theorem 1. *If Conjecture 1 is valid, then for every transitive finite permutation group P, all but finitely many alternating groups A_n act arc-transitively on a connected graph X in such a way that the stabiliser in A_n of a vertex v induces P on the neighbourhood of v.*

Proof. Let V, E and A be as above, with E chosen as a group containing an index 2 subgroup isomorphic to A, and consider the amalgamated free product $V *_A E$. Note that because A is a point-stabiliser in the permutation group V $(= P)$, it is core-free in V and hence also A is core-free in $V *_A E$. Suppose further that $\theta : V *_A E \to G$ is any epimorphism to a finite non-abelian simple group G such that θ is faithful on each of V and E, and also let a be any element of the image of $E \setminus A$ in G, and let H be the θ-image of V (so that H is isomorphic to P).

Now let X be the double-coset graph $X(G,H,a)$, with vertices defined as the right cosets of H in G, with cosets Hx and Hy adjacent in X if and only if $xy^{-1} \in HaH$. This is a well-known construction attributed to Sabidussi, and described in detail in [4, 9] for example. The construction ensures that G acts as an arc-transitive group of automorphisms of the graph X (by right multiplication of cosets Hx in G), with vertex-stabiliser H acting transitively on the neighbourhood $X(H) = \{Hah : h \in H\}$ of the trivial coset H. Moreover, this action of H is equivalent to the action of V on cosets of A (by right multiplication), and hence the same as the natural action of the given permutation group P, as required.

Finally, if Conjecture 1 is valid then we can take G as the alternating group A_n for all but finitely many n, and this completes the proof. □

3 Some specific cases

The same argument as used in the above proof can be applied to many specific cases where Conjecture 1 is known to be valid.

For example, this is often known to happen when the amalgamated subgroup A in $V *_A E$ is trivial. The validity of Conjecture 1 for the free products $C_3 * C_2$ and $C_k * C_2$ for all $k \geq 7$ follows from the fact that all but finitely many alternating groups are quotients of the ordinary $(2,3,k)$ triangle group for any given such k (see [3]), and the same holds for $C_k * C_2$ for all $k \in \{4,5,6\}$ by the analogous properties of the $(2,k,m)$ triangle groups for $4 \leq k < m$ (see [11]).

Similarly, the fact that all but finitely many alternating groups are quotients of the extended $(2,3,k)$ triangle (see [3]) shows that the same thing holds for $D_k *_{C_2} V_4$ for $k = 3$ and all $k \geq 7$. Also in [10] it was shown that infinitely many alternating groups occur as quotients of $A_5 *_{C_5} D_5$.

Hence, in particular, the answer to Caprace's question is 'No' when P is a cyclic or dihedral group of degree 3 or more, or the group of degree 12 induced by A_5 on cosets of a subgroup of order 5. It is fairly clear that the same answer holds for many other permutation groups besides these, and we complete this paper (and answer the sub-question mentioned earlier) by considering the case where $P = A_6$.

From now on we take V as A_6 and A as its point-stabiliser A_5, and we choose E as $A_5 \times C_2$. Just as before, note that A is core-free in V and hence also core-free in $V *_A E$. We will show that all but finitely many alternating groups A_n occur as images of $V *_A E$ under homomorphisms that are faithful on each of V and E.

To do this, first we note that $V *_A E = A_6 *_{A_5} (A_5 \times C_2)$ is generated by three elements x, y and a with the following properties:

- x and y generate $V = A_6$ and satisfy the relations $x^2 = y^5 = (xy)^5 = (xy^2)^4 = 1$,
- y and $u = xy^{-1}xyx$ generate $A = A_5$ (and satisfy $u^2 = y^5 = (uy^2)^3 = 1$), and
- y, u and a generate $E = A_5 \times C_2$, and satisfy $a^2 = [u,a] = 1$ and $y^a = y^{-1}$.

These properties may be seen by taking $x = (3,6)(4,5)$ and $y = (1,2,3,4,5)$ in A_6, with $u = (1,4)(3,5)$, and by viewing a as the inner automorphism of A_5 induced by conjugation by $(1,3)(4,5)$.

Next, we consider six particular transitive permutation representations of $V *_A E$, of degrees 1, 12, 42, 62, 21 and 31, as given below. In each case we give also the permutations induced by $u = xy^{-1}xyx$ and $w = xa$, and identify the fixed points of the subgroup E (generated by y, u and a), and call these fixed points 'link points', for reasons that should soon become clear.

Representation R_1 (degree 1)

$x \mapsto ()$,
$y \mapsto ()$,
$u \mapsto ()$,
$a \mapsto ()$,
$w \mapsto ()$.

Link point 1

Representation R_2 (degree 12)

$x \mapsto (1,2)(3,4)(9,12)(10,11)$,
$y \mapsto (2,3,4,5,6)(7,8,9,10,11)$,
$u \mapsto (2,4)(3,5)(7,10)(9,11)$,
$a \mapsto (2,7)(3,11)(4,10)(5,9)(6,8)$,
$w \mapsto (1,7,2)(3,10)(4,11)(5,9,12)(6,8)$.

Link points 1 and 12

Representation R_3 (degree 42)

$x \mapsto (1,2)(3,4)(7,12)(8,20)(9,32)(10,27)(11,24)(13,29)(14,33)(15,18)(16,31)(17,25)(21,34)(23,36)(26,28)(30,35)(39,42)(40,41)$,
$y \mapsto (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16)(17,18,19,20,21)(22,23,24,25,26)(27,28,29,30,31)(32,33,34,35,36)(37,38,39,40,41)$,
$u \mapsto (2,4)(3,5)(7,10)(9,11)(12,17)(13,22)(14,27)(15,28)(16,23)(18,21)(19,31)(20,29)(24,26)(25,30)(32,34)(33,35)(37,40)(39,41)$,
$a \mapsto (2,7)(3,11)(4,10)(5,9)(6,8)(13,16)(14,15)(18,21)(19,20)(22,23)(24,26)(27,28)(29,31)(32,37)(33,41)(34,40)(35,39)(36,38)$,
$w \mapsto (1,7,12,2)(3,10,28,24)(4,11,26,27)(5,9,37,32)(6,8,19,20)(13,31)(14,41,34,18)(15,21,40,33)(16,29)(17,25)(22,23,38,36)(30,39,42,35)$.

Link points 1 and 42

Representation R_4 (degree 62)

$x \mapsto (1,2)(3,4)(7,12)(8,20)(10,14)(11,21)(13,16)(15,18)(23,32)(24,31)(25,35)(26,40)(27,41)(28,33)(29,39)(36,38)(42,52)(43,50)(44,46)(45,54)(47,55)(48,53)(57,61)(60,62)$,
$y \mapsto (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16)(17,18,19,20,21)(22,23,24,25,26)(27,28,29,30,31)(32,33,34,35,36)(37,38,39,40,41)(42,43,44,45,46)(47,48,49,50,51)(52,53,54,55,56)(57,58,59,60,61)$,

$u \mapsto (2,4)(3,5)(7,10)(9,11)(12,17)(13,19)(16,20)(18,21)(22,27)(23,29)$
$(26,30)(28,31)(33,37)(34,39)(35,41)(38,40)(44,47)(45,49)(46,51)$
$(48,50)(52,54)(53,56)(57,60)(58,61)$,
$a \mapsto (2,7)(3,11)(4,10)(5,9)(6,8)(12,22)(13,26)(14,25)(15,24)(16,23)$
$(17,27)(18,31)(19,30)(20,29)(21,28)(32,42)(33,46)(34,45)(35,44)$
$(36,43)(37,51)(38,50)(39,49)(40,48)(41,47)(52,57)(53,61)(54,60)$
$(55,59)(56,58)$,
$w \mapsto (1,7,22,12,2)(3,10,25,44,33,21)(4,11,28,46,35,14)(5,9)(6,8,29,49,$
$39,20)(13,23,42,57,53,40)(15,31)(16,26,48,61,52,32)(17,27,47,59,$
$55,41)(18,24)(19,30)(34,45,60,62,54)(36,50)(37,51)(38,43)(56,58)$.
Link points 1 and 62

Representation R_5 (degree 21)

$x \mapsto (1,2)(3,4)(7,12)(8,20)(10,14)(11,21)(13,16)(15,18)$,
$y \mapsto (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16)(17,18,19,20,21)$,
$u \mapsto (2,4)(3,5)(7,10)(9,11)(12,17)(13,19)(16,20)(18,21)$,
$a \mapsto (2,7)(3,11)(4,10)(5,9)(6,8)(13,16)(14,15)(18,21)(19,20)$,
$w \mapsto (1,7,12,2)(3,10,15,21)(4,11,18,14)(5,9)(6,8,19,20)$.
Link point 1

Representation R_6 (degree 31)

$x \mapsto (1,2)(3,4)(7,12)(8,20)(10,14)(11,21)(13,16)(15,18)(23,25)(24,31)$
$(26,28)(27,29)$,
$y \mapsto (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16)(17,18,19,20,21)$
$(22,23,24,25,26)(27,28,29,30,31)$,
$u \mapsto (2,4)(3,5)(7,10)(9,11)(12,17)(13,19)(16,20)(18,21)(22,27)(23,29)$
$(26,30)(28,31)$,
$a \mapsto (2,7)(3,11)(4,10)(5,9)(6,8)(12,22)(13,26)(14,25)(15,24)(16,23)$
$(17,27)(18,31)(19,30)(20,29)(21,28)$,
$w \mapsto (1,7,22,12,2)(3,10,25,16,26,21)(4,11,28,13,23,14)(5,9)$
$(6,8,29,17,27,20)(15,31)(18,24)(19,30)$.
Link point 1

Note that in each case, the permutations induced by x, y and u are necessarily even, since they generate a subgroup isomorphic to A_6 or the trivial group. On the other hand, the permutations induced by the involution a have 0, 5, 18, 30, 9 and 15 transpositions respectively, and hence the permutations induced by a and $w = xa$ are even in representations R_1, R_3 and R_4, but are odd in representations R_2, R_5 and R_6. Indeed, the cycle structure of the permutation induced by $w = xa$ in representations R_2 to R_6 is $2^3\,3^2$, $2^3\,4^9$, $2^8\,5^2\,6^6$, $1^3\,2^1\,4^4$ and $2^4\,5^1\,6^3$, respectively.

We will use these six representations as 'building blocks' for constructing transitive permutation representations of $V *_A E$ of arbitrarily large degree, by using the link points to join representations together.

To help to explain that, we observe how the image of each representation of $V *_A E$ splits into orbits of the subgroups $V = \langle x,y\rangle \cong A_6$, $E = \langle y,u,a\rangle \cong A_5 \times C_2$

and $A = V \cap E = \langle y, u \rangle \cong A_5$. For example, the image of R_3 (of degree 42) splits into three orbits of V, of lengths 6, 6 and 30, namely $\{1,2,\ldots,6\}$, $\{7,8,\ldots,36\}$ and $\{37,38,\ldots,42\}$, and these in turn split into seven orbits of A, of lengths 1, 5, 5, 20, 5, 5 and 1, namely $\{1\}$, $\{2,3,\ldots,6\}$, $\{7,8,\ldots,11\}$, $\{12,13,\ldots,31\}$, $\{32,33,\ldots,36\}$, $\{37,38,\ldots,41\}$ and $\{42\}$. Every orbit of the subgroup $E = \langle A, a \rangle$ is then either an orbit of A preserved by a, or a union of two orbits of A that are interchanged by a. For example (again), in R_3 the subgroup E has five orbits, of lengths 1, 10, 20, 10 and 1, namely $\{1\}$, $\{2,3,\ldots,11\}$, $\{12,13,\ldots,31\}$, $\{32,33,\ldots,41\}$ and $\{42\}$.

This orbit decomposition is depicted for all six of our 'building block' representations in Figure 1, with each small box indicating an orbit of A (and the number inside it indicating the length of that orbit), and each thin horizontal line indicating a connection between a pair of orbits of A that are interchanged by a. In particular, each small box with a '1' inside it contains a link point, fixed by $E = \langle y, u, a \rangle$.

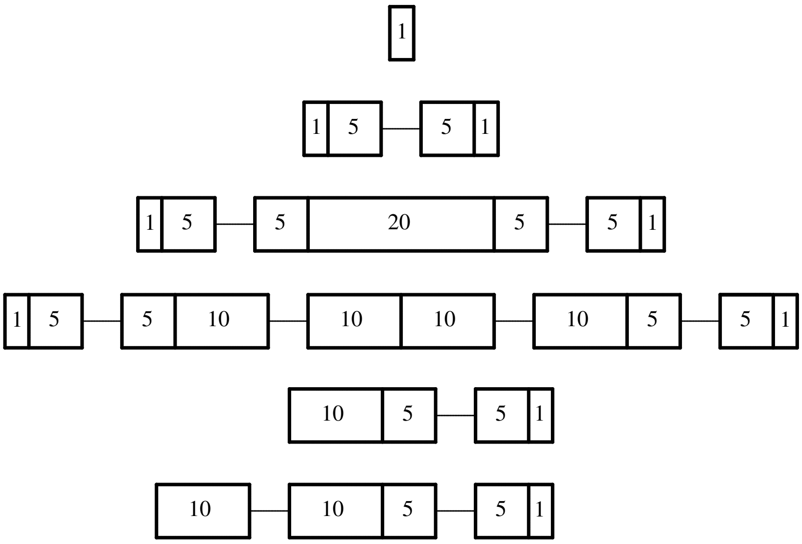

Fig. 1 Our 'building block' representations 1 to 6 (on 1, 12, 42, 62, 21 and 31 points respectively)

Next, if we take any two transitive permutation representations of $V *_A E$, say of degrees n_1 and n_2, such that each representation contains at least one link point, then we can join them together to form a larger one of degree $n_1 + n_2$, by simply concatenating the permutations induced by each of x, y and a, and then adding a transposition to a that swaps the two chosen link points.

For example, we can join the first two representations together by re-labelling the single point of R_1 as '13', and then adding a new transposition $(12, 13)$ to the permutation induced by a. This gives a transitive representation on 13 points, in which x, y and u induce the same permutations as given in R_2, while a induces the involution $(2,7)(3,11)(4,10)(5,9)(6,8)(12,13)$ and $w = xa$ induces the permutation $(1,7,2)(3,10)(4,11)(5,9,13,12)(6,8)$.

Here, and in general when a pair of transitive representations are joined together in this way, the images of x, y and a still satisfy the same relations as in $V *_A E$, and hence (by the universal property of amalgamated products), the definition of the images extends to a new permutation representation of $V *_A E$. The only significant change is made to the permutation induced by a, and this simply joins two single-point orbits of $E = \langle y,u,a\rangle$ into a single two-point orbit of E. Similarly, the cycles of $w = xa$ containing the two link points are merged into a single cycle.

For another example, suppose we join together a copy of each of R_5 and R_2 by adding a new transposition that swaps the link point 1 of R_5 with link point 1 of R_2 (suitably re-labelled). Then we obtain a transitive permutation representation on $21 + 12 = 33$ points. Before the join, the permutations induced by w have cycle structures $1^3\,2^1\,4^4$ and $2^3\,3^2$, with link point 1 of R_5 lying in a cycle of length 4 and link point 12 of R_2 lying in a cycle of length 3. The effect of the join is to merge those two cycles into a single cycle of length 7, leaving other cycles unchanged.

We have now dealt with enough properties of the building blocks and their conjunction to prove the following:

Theorem 2. *For all but finitely many positive integers n, both the alternating group A_n and the symmetric group S_n are homomorphic images of the amalgamated free product $A_6 *_{A_5} (A_5 \times C_2)$, and hence act faithfully as an arc-transitive group of automorphisms of some* 6*-valent graph with vertex-stabiliser isomorphic to A_6.*

Proof. For any positive integers k and m, let $n = 21 + 12k + 62m$, and observe that every odd positive integer $n \geq 395$ is expressible in this way.

Now construct a transitive permutation representation of $A_6 *_{A_5} (A_5 \times C_2)$ of odd degree n by stringing together a single copy of R_5 with k copies of R_2, and then m copies of R_4. Then the permutation induced by a has $9 + 6k + 31m$ transpositions (with $k+m$ of these coming from the linkages), and so the permutations induced by a and w are even when m is odd, but odd when m is even. Indeed, the permutation induced by w has cycle structure $1^3\,2^{1+3k+8m}\,4^3\,5^1\,6^{k-1+6m}\,7^1\,8^1\,10^{m-1}$.

The single 7-cycle comes from the linkage between the copy of R_5 and the first copy of R_2. Also the length of every other cycle of w divides 120, so w^{120} is a single 7-cycle. Moreover, this 7-cycle contains a pair of points interchanged by x, a fixed point of y, and a pair of points interchanged by a. It follows that the image of this new representation is primitive (for otherwise there would be a block B of imprimitivity containing all 7 points of the 7-cycle, but then B would be preserved by each of x, y and a and hence by the whole group). And now by a theorem of Jordan [12, Theorem 13.9], this 7-cycle ensures that the permutations generate A_n for large $n \equiv 3 \bmod 4$ when m is odd, and S_n for large $n \equiv 1 \bmod 4$ when m is even.

Next, we can add a copy of R_1 to the final copy of R_4, and get a transitive representation of $A_6 *_{A_5} (A_5 \times C_2)$ of even degree $n = 21 + 12k + 62m + 1$, and the same argument works, except that the parity of the permutations a and w changes, with a 5-cycle of $w = xa$ becoming another 6-cycle. In this case the permutations generate S_n with $n \equiv 0 \bmod 4$ when m is odd, and A_n with $n \equiv 2 \bmod 4$ when m is even.

Finally, we can replace the single copy of R_5 by a copy of R_6, and insert a single copy of R_3 between the k copies of R_2 and the m copies of R_4, and get transitive permutation representations of odd degree $n = 31+12k+42+62m$ and even degree $n = 31+12k+42+62m+1$, in which the permutations induced by a and w are even if and only if m is even in the first case, and are even if and only if m is odd in the second case. The cycle structure of xa is altered by addition of a 9-cycle, plus some changes in the 1-, 2-, 4-, 6- and 8-cycles, and replacement of the single 7-cycle by a new single 7-cycle coming from the linkage between R_2 and R_3, but the same arguments apply as earlier. In this case the induced permutations generate A_n for large $n \equiv 1$ mod 4 and S_n for large $n \equiv 3$ mod 4 when m is even, and S_n for large $n \equiv 2$ mod 4 and A_n for large $n \equiv 0$ mod 4 when m is odd.

These constructions cover all residue classes mod 4 for the degree n, for both A_n and S_n for large enough n (indeed for all $n \geq 447$), as required. □

Incidentally, we also obtain the following, because if a is any involution in $E \setminus A$, then the index 2 subgroup $S = \langle V, V^a \rangle$ in the group $V *_A E = A_6 *_{A_5} (A_5 \times C_2)$ used above is isomorphic to $A_6 *_{A_5} A_6$, and also maps onto A_n for large n. This strengthens an observation made by Peter Neumann and Cheryl Praeger at the Groups St Andrews conference that $A_6 *_{A_5} A_6$ has infinitely many alternating quotients.

Corollary 1. *All but finitely many alternating groups occur as quotients of the amalgamated free product* $A_6 *_{A_5} A_6$.

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