The contributions of W.T. Tutte to matroid theory

Graham Farr and James Oxley

Abstract Bill Tutte was born on May 14, 1917 in Newmarket, England. In 1935, he began studying at Trinity College, Cambridge reading natural sciences specializing in chemistry. After completing a master's degree in chemistry in 1940, he was recruited to work at Bletchley Park as one of an elite group of codebreakers that included Alan Turing. While there, Tutte performed "one of the greatest intellectual feats of the Second World War." Returning to Cambridge in 1945, he completed a Ph.D. in mathematics in 1948. Thereafter, he worked in Canada, first in Toronto and then as a founding member of the Department of Combinatorics and Optimization at the University of Waterloo. His contributions to graph theory alone mark him as arguably the twentieth century's leading researcher in that subject. He also made groundbreaking contributions to matroid theory including proving the first excluded-minor theorems for matroids, one of which generalized Kuratowski's Theorem. He extended Menger's Theorem to matroids and laid the foundations for structural matroid theory. In addition, he introduced the Tutte polynomial for graphs and extended it and some relatives to matroids. This paper will highlight some of his many contributions focusing particularly on those to matroid theory.

1 Introduction

The task of summarizing Bill Tutte's mathematical contributions in a short paper is an impossible one. There are too many, they are too deep, and their implications are too far-reaching. This paper will discuss certain of these contributions giving

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particular emphasis to his work in matroid theory and the way in which that work links to graph theory. The terminology used here will follow Oxley [16].

This paper will attempt to give insight into the thoughts and motivations that guided Tutte's mathematical endeavours. To do this, we shall quote extensively from three sources. Dan Younger, Tutte's long-time colleague and friend at the University of Waterloo, wrote the paper *William Thomas Tutte 14 May 1917 – 2 May 2002* [53] in the *Biographical Memoirs of Fellows of the Royal Society*, and that paper includes many quotes from Tutte that are reproduced here. In 1999, Tutte presented the Richard Rado Lecture *The Coming of the Matroids* at the British Combinatorial Conference in Canterbury. We will also quote from Tutte's write-up of that lecture in the conference proceedings [47]. Finally, we draw on commentaries by Tutte on his own papers that appear in *Selected Papers of W.T. Tutte I, II* [45, 46], published in 1979 to mark Tutte's sixtieth birthday.

These *Selected Papers* were edited by D. McCarthy and R. G. Stanton. Ralph Stanton was a noted mathematician who had been the first Dean of Graduate Studies at the University of Waterloo and who recruited Tutte to Waterloo from Toronto in 1962. Stanton's foreward to the *Selected Papers* provides a context for the magnitude of Tutte's achievements:

> Not too many people are privileged to practically create a subject, but there have been several this century. Albert Einstein created Relativity . . . Similarly, modern Statistics owes its existence to Sir Ronald Fisher's exceptionally brilliant and creative work. And I think that Bill Tutte's place in Graph Theory is exactly like that of Einstein in Relativity and that of Fisher in Statistics. He has been both a great creative artist and a great developer.

Tutte's family moved several times when he was young but they returned to the Newmarket area, to the village of Cheveley, when Bill was about seven. Bill attended the local school. In May, 1927 and again a year later, he won a scholarship to the Cambridge and County High School for Boys, some eighteen miles from his home. The first time he won, his parents judged that it was too far for their son to travel and he was kept home. A year later his parents permitted him to attend the school despite the long daily commute each way, by bike and by train [53, p.287]. In the high school library, Bill came across Rouse Ball's book *Mathematical Recreations and Essays* [1], first published in 1892. That book included discussions of chessboard recreations, map colouring problems, and unicursal problems (including Euler tours and Hamiltonian cycles). Some parts of his chemistry classes were [45, p.1] pure graph theory and in his physics classes, he learned about electrical circuits and Kirchhoff's Laws. Tutte wrote [45, p.1],

> When I became an undergraduate at Trinity College, Cambridge, I already possessed much elementary graph-theoretical knowledge though I do not think I had this knowledge well-organized at the time.

In 1935, Tutte began studying at Trinity College, Cambridge. He read natural sciences, specializing in chemistry. From the beginning, he attended lectures of the Trinity Mathematical Society. Three other members of that Society, all of whom were first-year mathematics students, were R. Leonard Brooks, Cedric A.B. Smith, and Arthur H. Stone. This group had various names [47, p.4] including 'The Important Members, The Four Horsemen, The Gang of Four.' They became fast friends spending many hours discussing mathematical problems. Tutte wrote [53, p.288],

> As time went on, I yielded more and more to the seductions of Mathematics.

Tutte's first paper [22], in chemistry, was published in 1939 in the prestigious scientific journal *Nature*. His first mathematical paper, *The dissection of rectangles into squares*, was published with Brooks, Smith, and Stone in 1940 in the *Duke Mathematical Journal*. Their motivating problem was to divide a square into a finite number of unequal squares. In 1939, R. Sprague [21] from Berlin published a solution to this problem just as The Four were in the final stages of preparing their paper in which, ingeniously, they converted the original problem into one for electrical networks. Writing later about The Four's paper, Tutte said [45, p.3],

> I value the paper not so much for its ostensible geometrical results, which Sprague largely anticipated, as for its graph-theoretical methods and observations.

Tutte went on to note [45, p.4] that, in this paper,

> two streams of graph theory from my early studies came together, Kirchhoff's Laws from my Physics lessons, and planar graphs from Rouse Ball's account of the Four Colour Problem.'

Tutte wrote a very readable account of this work in Martin Gardner's *Mathematical Games* column in *Scientific American* in November, 1958, and that account is now available online [32].

The Four's paper is remarkable not only for its solution to the squaring-the-square problem and its beautiful graph-theoretic ideas, but also for the extent to which it contains the seeds of Tutte's later work. In it, we find planarity, duality, flows, numbers of spanning trees, a deletion-contraction relation, symmetry, and above all the powerful application of linear algebra to graph theory.[1]

After completing his chemistry degree in 1938, Tutte worked as a postgraduate student in physical chemistry at Cambridge's famous Cavendish Laboratory completing a master's degree in 1940. Tutte's work in chemistry [53, p.288]

> convinced him that he would not succeed as an experimenter. He asked his tutor, Patrick Duff, to arrange his transfer from natural sciences to mathematics. This transfer took place at the end of 1940.

Tutte later wrote [47, p.4],

> I left Cambridge in 1941 with the idea that graph theory could be reduced to abstract algebra but that it might not be the conventional kind of algebra.

[1] Incidentally, it may also be regarded as Tutte's first paper on graph drawing. In that field, too, he is regarded as a pioneer, mostly because of his 1963 paper 'How to draw a graph' [35]. But it is still worth noting the graph-drawing aspect of his very first paper: the squared rectangles are a type of simultaneous "visibility drawing" of a planar graph and its dual.

Like so many of the brightest minds in Britain at the time, Tutte was recruited as a codebreaker and worked at the Bletchley Park Research Station — now famous, but then top secret — from 1941 till 1945. He wrote [47, p.5],

> at Bletchley I was learning an odd new kind of linear algebra.

Narrating the 2011 BBC documentary, *Code-Breakers: Bletchley Park's Lost Heroes*, the actress Keeley Hawes says,

> This is Bletchley Park. In 1939, it became the wartime headquarters of MI6. If you know anything, about what happened here, it will be that a man named Alan Turing broke the German Naval code known as 'Enigma' and saved the nation; and he did. But that's only half the story.

Then Captain Jerry Roberts, who had been a Senior Cryptographer at the Park during the war, speaks:

> There were three heroes of Bletchley Park. The first was Alan Turing; the second was Bill Tutte, who broke the Tunny system, a quite amazing feat; and the third was Tommy Flowers who, with no guidelines, built the first computer ever.

Tutte's work at Bletchley Park was truly profound. The problem he faced was to break into communications encoded by an *unknown* cypher machine, codenamed Tunny by the British. (Its real name was Lorenz SZ40.) This machine was much more secure and complex than the famous Enigma machine, reflecting its use at the highest levels of the Nazi regime including by Hitler himself. Furthermore, although the British knew the architecture of the Enigma machine, the design of the Tunny machine was a complete mystery to them. The problem Tutte faced was thus far harder than the Enigma problem which Turing is justly celebrated for solving. Tutte's first problem was the *diagnosis* of the Tunny machine (that is, determining how it worked), just from collected cyphertext; only then could he and his colleagues move on to *cryptanalysis*. Tutte made the crucial breakthrough in diagnosis, an astonishing achievement. He then went on to develop cryptanalysis algorithms. These were very computationally intensive. The Colossus cryptanalytic computers, designed by Tommy Flowers, were built to implement Tutte's algorithms and performed service of incalculable value for the remainder of the war.

The University of Waterloo's magazine for Spring, 2015 has an article *Keeping Secrets* about this work in which one reads,

> According to Bletchley Park's historians, General Dwight D. Eisenhower himself described Tutte's work as one of the greatest intellectual feats of the Second World War.

Some details of this work can be found in [53, 12, 8]. Tutte's own account of these efforts appear in [48, 47].

As a consequence of Tutte's top-secret code-breaking work at Bletchley Park, he was elected a Fellow of Trinity College in 1942. He wrote [53, p.291] of this,

> It seemed to me that the election might be criticized as a breach of security, but no harm came of it.

In 1945, after the war, Tutte returned to Cambridge for a Ph.D. in mathematics, supervised by Shaun Wylie, with whom Tutte had worked at Bletchley. Tutte completed his Ph.D. thesis, *An algebraic theory of graphs*, in 1948 despite Wylie's advice [53, p.291] to

> drop graph theory and take up something respectable, such as differential equations.

Tutte's thesis, which was xi + 417 pages, was an extraordinary accomplishment with the ideas in it forming the basis for much of his work for the next two decades. We discuss it in more detail in §8. He wrote [53, p.291] of his decision to stick with graph theory,

> If one assumes that graph theory was my métier, it was just as well that I had the prestige of a Fellow of Trinity.

2 Tutte's doctoral research

Tutte's first year of doctoral research was remarkably productive. He submitted six papers during the period from November 1945 to December 1946, including four that became classics of the field, though most of them bore no relation to his Ph.D. thesis.

In the first of these classic papers [24], Tutte found a 46-vertex counterexample to an 1884 conjecture of Tait [23] that every cubic planar graph is Hamilitonian.

Tutte's paper *A ring in graph theory* is his first paper on the Tutte polynomial and one of his most profound. His polynomial is not given explicitly in any of its usual forms, and its presence is somewhat obscured by some technical details and the use of multivariate polynomials to develop much of the theory. But the main ingredients of Tutte-polynomial theory are all there. We return to it shortly, in §2.1.

His third classic paper [27] studied symmetry in cubic graphs. An *s-arc* in a graph is a walk with s edges in which consecutive edges are always distinct. Apart from this constraint, vertices and edges may occur repeatedly. Note that a walk and its reverse are considered to be different. A graph G is *s-arc-transitive* if it has at least one s-arc and, for any two s-arcs, there is an automorphism of G that maps one to the other. This is a very strong symmetry property indeed. Tutte showed that there are no s-arc-transitive cubic graphs with $s > 5$, gave an inequality relating girth to s, and characterized graphs where the inequality comes as close to equality as possible for a given girth; these are the *g-cages*, a finite family of graphs, the most complex being his 8-cage. This paper became enormously influential in the theory of symmetric graphs.

The fourth of these groundbreaking papers [25] proved the characterization of when a graph has a 1-factor, or perfect matching. This theorem is now a staple of most introductory courses on graph theory.

2.1 'A ring in graph theory'

The starting point and driving principle of this paper is the observation that certain functions on graphs obey *deletion-contraction relations*. As an example of such a function, Tutte considered the *complexity* $C(G)$ of a connected graph G, this being the number of spanning trees of G. When The Four were working on the problem of partitioning a rectangle into unequal squares, they observed that complexity obeys the following recursion.

Lemma 1. *In a graph G, let e be an edge that is neither a loop or a cut edge. Then*

$$C(G) = C(G \backslash e) + C(G/e).$$

Proof. Partition the set of spanning trees of G into

(i) those not using e; and
(ii) those using e.

There are $C(G\backslash e)$ spanning trees in (i); and the spanning trees in (ii) match up with the spanning trees of G/e.

Tutte wrote [45, p.51],

> I wondered if complexity, or tree number, could be characterized by the above identity alone and decided that it could not.

His paper considered the following.

Problem 1. What isomorphism-invariant functions W of graphs satisfy

$$W(G) = W(G\backslash e) + W(G/e)$$

for all non-loop edges e of G?

He called such a function, taking values in an abelian group, a *W-function*. A W-function is a *V-function* if

$$W(G_1 \cup G_2) = W(G_1)W(G_2)$$

for all disjoint graphs G_1 and G_2, where W now takes values in a commutative ring with unity.

For a graph G, let $P(G;\lambda)$ denote the number of proper λ-colourings of G. Tutte noted that $(-1)^{|V(G)|}$ times $P(G;\lambda)$ is an example of a V-function. This is an immediate consequence of the following lemma, which was first proved by Foster, in "Note added in proof" in [51, p.718].

Lemma 2. *For a non-loop edge e of a graph G,*

$$P(G;\lambda) = P(G\backslash e;\lambda) - P(G/e;\lambda).$$

Proof. Let e have distinct endpoints u and v. Partition the proper k-colourings of $G\backslash e$ into

(i) those in which u and v have different colours; and
(ii) those in which u and v have the same colour.

In (i), we are counting the number of proper k-colourings of G; while (ii) corresponds to the number of proper k-colourings of G/e.

Tutte's insightful breakthough here was to focus on the two recursions:

(i) $W(G) = W(G\backslash e) + W(G/e)$; and
(ii) $W(G_1 \cup G_2) = W(G_1)W(G_2)$.

Many readers will recognize here the origins of the Tutte polynomial. There are technical differences between the multivariate polynomials in this paper and the more familiar polynomials of Whitney and Tutte, which we will define in §3. But some simple adjustments — such as substitutions to give bivariate specializations, and dividing by $x^{k(G)}$ or $(x-1)^{k(G)}$ — reveal both the Whitney rank generating function and Tutte polynomial, albeit in period costume. The relationship between these two polynomials, which is just a coordinate translation of one step in each direction, is subsumed by a more general result in the paper. In fact, most of the main ingredients of Tutte-polynomial theory are here, with deletion-contraction relations at the core. The details of this paper are discussed in [9].

3 Graph polynomials

In 1954, Tutte published [29] *A contribution to the theory of chromatic polynomials*. By then, he was at the University of Toronto having been recruited there in 1948 by H.S.M. Coxeter, another famous graduate of Trinity College, Cambridge. In this paper, Tutte introduced what he called the *dichromate* of a graph, this now being known as the *Tutte polynomial* of the graph. The dichromate is a two-variable polynomial not to be confused with another two-variable polynomial Tutte labelled the *dichromatic polynomial* of a graph. The latter is now known as the *Whitney rank-generating function* of the graph. Welsh [50, p.44] draws attention to the rather confused history of these polynomials and their nomenclature. This history is clarified in [7, 9] where Tutte [49, p.8] is quoted concerning the use of the name 'Tutte polynomial' as saying,

> This may be unfair to Hassler Whitney who knew and used analogous coefficients without bothering to affix them to two variables.

Tutte cites Whitney's 1932 paper [51] as his source. Formally, let G be a graph with edge set E. For a subset X of E, let $G[X]$ be the subgraph of G induced by X, and let $r(X)$, the *rank* of X, be the difference between the number of vertices and the number of connected components of $G[X]$. The *Whitney rank-generating function* $R(G;x,y)$ of G is

$$R(G;x,y) = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{|X|-r(X)}.$$

The *Tutte polynomial* $T(G;x,y)$ is the translation of $R(G;x,y)$ defined by

$$T(G;x,y) = R(G;x-1,y-1).$$

In particular, when G is connected, $T(G;1,1)$ is the complexity of G, that is, its number of spanning trees. When G has $k(G)$ components, $P(G;\lambda)$, the number of proper λ-colourings of G is $\lambda^{k(G)}(-1)^{r(E)}T(G;1-\lambda,0)$. Another important evaluation of the Tutte polynomial involves flows.

To define a flow in a graph G, first assign directions to every edge of G. A *nowhere-zero k-flow* assigns a flow value $f(e)$ from $\mathbb{Z}_k - \{0\}$ to every edge e of G such that, at every vertex v, the sum of the flows on the edges directed into v equals the sum of the flows on the edges directed out from v. Intuitively, Kirchhoff's Current Law holds at each vertex of G. For example, G has a nowhere-zero 2-flow if and only if every vertex has even degree. When G is connected, this is, of course, equivalent to G being Eulerian.

It is straightforward to show that if G has a nowhere-zero k-flow, then G has no cut edges. Moreover, a plane graph G without cut edges has a nowhere-zero k-flow if and only if its dual G^* is k-colourable.

Let A be an additive abelian group. A *nowhere-zero A-flow* takes flow values from $A - \{0\}$ such that, at every vertex, the flow into the vertex equals the flow out from that vertex. Thus a nowhere-zero k-flow is just a nowhere-zero $\mathbb{Z}_k$-flow. Remarkably, Tutte [29] showed that the number of nowhere A-flows on a graph depends only on the cardinality of A.

Proposition 1 (Tutte, 1954). *For $n \geq 2$, let A be an abelian group with n elements and G be a graph without cut edges. Then the number of nowhere-zero A-flows on G equals the number of nowhere-zero n-flows on G.*

Tutte [29] made two striking conjectures about flows.

Conjecture 1 (Tutte, 1954). There is a fixed number t such that every graph without cut edges has a nowhere-zero t-flow.

This conjecture was not settled for over twenty years until Jaeger [14] proved the following.

Theorem 1 (Jaeger, 1976). *Every graph without cut edges has a nowhere-zero* 8-*flow.*

Tutte's second flow conjecture is even more elusive and still remains open.

Conjecture 2 (Tutte, 1954). Every graph without cut edges has a nowhere-zero 5-flow.

The best partial result towards this 5-Flow Conjecture was proved by Seymour [19]. Just as Jaeger's proof relied on the fact that 8 is 2 cubed, Seymour's proof relies on 6 being the product of 3 and 2.

Theorem 2 (Seymour, 1981). *Every graph without cut edges has a nowhere-zero 6-flow.*

4 Matroids

Before discussing Tutte's contributions to matroid theory, we briefly introduce matroids to readers unfamiliar with them.

Let A be a matrix having E as its set of column labels. Let $\mathscr{I}$ be the collection of subsets X of E such that X labels a linearly independent set of columns. The pair $(E,\mathscr{I})$ is an example of a matroid M with the members of the set $\mathscr{I}$ being its *independent sets*. We denote this matroid by $M[A]$. In general, $(E,\mathscr{I})$ is a *matroid* M with *ground set* E if $\mathscr{I}$ is a non-empty hereditary collection of subsets of the finite set E with the property that, whenever X and Y are in $\mathscr{I}$ and $|X| > |Y|$, there is an element x of $X - Y$ such that $Y \cup \{x\} \in \mathscr{I}$. Subsets of E that are not in $\mathscr{I}$ are *dependent* and the minimal dependent sets are the *circuits* of M. Evidently, M is uniquely determined by its collection of circuits. If G is a graph, there is a matroid $M(G)$ having $E(G)$ as its ground set and the set of edge sets of cycles of G as its set of circuits. The matroid $M(G)$ is the *cycle matroid* of G.

For a field $\mathbb{F}$, a matroid M is $\mathbb{F}$-*representable* if there is a matrix A over $\mathbb{F}$ such that $M = M[A]$. A $GF(2)$-representable matroid is called *binary*. For example, over $GF(2)$, let

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Then $M[A]$ is a matroid with ground set $\{1,2,\ldots,7\}$ whose circuits include $\{4,5,6\}$ since, over $GF(2)$, the three corresponding vectors are linearly dependent although any two of them are linearly independent. This matroid is usually called the *Fano matroid* and is denoted by F_7. A geometric representation of this matroid is shown in Figure 1. In such a picture, three collinear points form a circuit as do four coplanar points of which no three are collinear. The dual, F_7^*, of the Fano matroid is the matroid $M[A^*]$ where

$$A^* = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

In general, if the n-element matroid $M = M[I_r|D]$, its dual M^* is $M[D^T|I_{n-r}]$. More generally, suppose M is a matroid on the set E having $\mathscr{B}$ as its set of maximal independent sets (*bases*). The collection $\{E-B : B \in \mathscr{B}\}$ can be shown to be the set of bases of a matroid on E; this matroid M^* is the *dual* of M.

The deletion of the element 1 from F_7 is the matroid of the matrix that is obtained from A by deleting the first column. The reader may wish to check that this deletion is actually equal to $M(K_4)$ where $\{2,3,\ldots,7\}$ is the edge set of K_4. The contraction of 1 from F_7 is the matroid of the matrix that is obtained from A by deleting the first row and the first column. This contraction is the cycle matroid of the *doubled triangle* graph, obtained from a triangle with edge set $\{2,6,3\}$ by adding $4,7$, and 5 in parallel with $2,6$, and 3, respectively In general, for a matroid M, the *deletion* of the element e from M is the matroid $M\backslash e$ having ground set $E-\{e\}$ and set of independent sets $\{I \in \mathscr{I} : e \notin I\}$. Moreover, provided $\{e\}$ is independent, the *contraction* M/e of e from M is the matroid with ground set $E-\{e\}$ and set of independent sets $\{I' \subseteq E-\{e\} : I' \cup \{e\} \in \mathscr{I}\}$. When $\{e\}$ is dependent, we define M/e to be $M\backslash e$. A *minor* of M is any matroid that can be obtained from M by a sequence of deletions and contractions. As partially outlined above, every minor of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable. This means that the class of $\mathbb{F}$-representable matroids can be characterized by the matroids that are themselves not $\mathbb{F}$-representable but for which every minor is $\mathbb{F}$-representable. These minor-minimal matroids that are not $\mathbb{F}$-representable are the *excluded minors* for the class of $\mathbb{F}$-representable matroids.

Tutte wrote [53, p.292] that his Ph.D. thesis

> attempted to reduce Graph Theory to Linear Algebra. It showed that many graph-theoretical results could be generalized to algebraic theorems about structures I called 'chain-groups'. Essentially, I was discussing a theory of matrices in which elementary operations could be applied to rows but not columns.

As Dan Younger noted in his wonderful memoir of Tutte [53, p.292]:

> This is matroid theory.

His chain-groups, called *nets* in his thesis, are essentially row spaces of representative matrices of representable matroids. In a sense, they may be regarded as represent*ed* matroids. But it would be pedantic to make much of the difference between these and represent*able* matroids.

In essence, then, Tutte developed a theory of representable matroids as generalizations of graphs. Some of his work is valid for arbitrary matroids, in that some

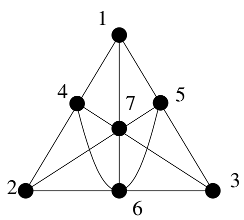

Fig. 1 The Fano matroid, F_7.

definitions and arguments only use matroid ideas (such as rank) in a way that does not depend on representability. But the thesis does not mention arbitrary matroids, and does not cite Whitney's seminal 1935 paper on matroids [52].

5 The excluded-minor theorems

In a commentary on one of his matroid papers, Tutte wrote [46, p.497],

> If a theorem about graphs can be stated in terms of edges and circuits only it probably exemplifies a more general theorem about matroids.

The application of this principle is evident in much of Tutte's work and has guided the efforts of a number of other researchers in matroid theory. Two of the most well-known graphs are K_5 and $K_{3,3}$, the latter being the *three-houses-three-utilities graph*. These graphs are forever linked by their appearance in Kuratowski's famous characterizations [15] of planar graphs in terms of excluded (topological) minors. Tutte introduced the operation of contraction for matroids and also the notion of a minor of a matroid. In a very productive period in the late 1950s, Tutte published three important papers that included excluded-minor characterizations of various classes of matroids. This section will discuss these theorems.

Looking back on his thesis, Tutte wrote [47, p.6],

> I went on happily developing a theory of chain-groups and their elementary chains, these latter of course being defined by minimal supports. The method was to select theorems about graphs and try to generalize them to chain-groups. This was not too difficult for theorems expressible in terms of circuits. But theorems about 1-factors imposed problems. As I look back on this episode I am grieved to recall that I still did not appreciate the work of Whitney [on matroids]. Yet these chain-groups were half-way to matroids and their minimal supports were Whitney's matroid circuits.

Later in the same paper, Tutte wrote [47, p.7],

> By 1958 ... I had learned to appreciate matroids. I put the work in my thesis into matroid terminology and generalized from chain-groups to matroids. ... Then from the thesis-theorems I got the now well-known excluded minor conditions for a binary matroid to be regular and for a regular matroid to be graphic.

The uniform matroid $U_{2,4}$, which geometrically corresponds to four collinear points, is the matroid $M[A]$ where A is the real matrix

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

It is straightforward to see that $U_{2,4}$ is not binary. Tutte's first excluded-minor theorem [31], which is relatively straightforward to prove, establishes that $U_{2,4}$ is the unique excluded minor for the class of binary matroids.

Theorem 3 (Tutte, 1958). *A matroid is binary if and only if it has no $U_{2,4}$-minor.*

A real matrix A is *totally unimodular* if the determinant of every square submatrix of A is in $\{0,1,-1\}$. A matroid M is *regular* if there is a totally unimodular matrix A such that $M = M[A]$. As an example, take a graph G and arbitrarily orient its edges. Then take the vertex-edge incidence matrix A of this directed graph. In this real matrix, each non-zero column has one 1 and one -1. By a result of Poincaré [17], A is totally unimodular. The matroid $M[A]$ of this matrix can be shown to be equal to the cycle matroid $M(G)$ of G. In general, a matroid is graphic it it equals the cycle matroid of some graph. Thus every graphic matroid is regular. Tutte [31] proved the following.

Lemma 3. *A matroid M is regular if and only if M is $\mathbb{F}$-representable for all fields $\mathbb{F}$.*

Tutte's second excluded-minor characterization [31] is significantly more difficult than his first.

Theorem 4 (Tutte, 1958). *A matroid is regular if and only if it has none of $U_{2,4}$, F_7, or F_7^* as a minor.*

The last theorem was proved in two papers in the *Transactions of the American Mathematical Society* called *A homotopy theorem for matroids I, II*. In a 1959 paper *Matroids and graphs* in the same journal, Tutte [33] characterized graphic matroids in terms of excluded minors. For a graph G, the dual of its cycle matroid $M(G)$ is denoted by $M^*(G)$. Recognizing the link between cycles in a plane graph and bonds in the dual graph, the reader may not be surprised to learn that the circuits of $M^*(G)$ coincide with the bonds in G. One attractive feature of $M^*(G)$ is that it is defined whether or not G is planar. Thus, although non-planar graphs do not have graphic duals, the cycle matroids of such graphs do have matroid duals.

Theorem 5 (Tutte, 1959). *A regular matroid is graphic if and only if it has neither $M^*(K_{3,3})$ nor $M^*(K_5)$ as a minor.*

Tutte wrote [47, p.8] of this theorem that it

> was guided, in the usual vague graph-to-matroid way, by Kuratowski's Theorem and my favourite proof thereof.

6 Higher connectivity for matroids

Whitney [52] had introduced the notion of a *non-separable* matroid as one with the property that, for every two distinct elements, there is a circuit containing both. Such a matroid is now more commonly called *connected*. A loopless graph G has the property that every two edges lie in a cycle if and only if G is 2-connected, provided G has at least three vertices and has no isolated vertices. Given the importance of higher connectivity for graphs, it was natural to seek a matroid analogue. Tutte [40] did this. One feature of Tutte's definition was the desire for the connectivity of a matroid and its dual to be equal. Of course, a 3-connected graph cannot have a bond of size at most two. Dually, Tutte felt that a 3-connected graph should have no cycles of size at most two; in other words, it should be simple. Tutte began his work in this area by proving the following result for graphs [34].

Theorem 6 (Tutte, 1961). *A* 3*-connected simple graph G has an edge e such that $G\backslash e$ or G/e is* 3*-connected and simple unless G is a wheel.*

Five years later, in the paper *Connectivity in matroids*, Tutte [40] generalized this theorem to matroids. Indeed, it is in his commentary [46, p.487] on this paper that Tutte made the statement about generalizing graph results to matroids quoted at the beginning of Section 5. Let M be a matroid with ground set E. For a subset X of E, the *rank* $r(X)$ of X is the cardinality of the largest independent set that is contained in X. Earlier, we defined the rank of a set of edges in a graph G. That rank is precisely the rank of X in the cycle matroid of G.

Tutte defined the matroid M to be 2*-connected* if

$$r(X)+r(E-X)-r(M)\geq 1$$

for all $X\subseteq E$ with $|X|,|E-X|\geq 1$. He then defined a 2-connected matroid M to be 3*-connected* if

$$r(X)+r(E-X)-r(M)\geq 2$$

for all $X\subseteq E$ with $|X|,|E-X|\geq 2$.

The following result is elementary.

Proposition 2. *Let G be a graph with at least four vertices. Then*

(i) $M(G)$ is 2*-connected if and only if G is* 2*-connected and loopless; and*
(ii) $M(G)$ is 3*-connected if and only if G is* 3*-connected and simple.*

In the cycle matroid $M(\mathscr{W}_r)$ of the r-spoked wheel $\mathscr{W}_r$, the rim R is a cycle whose complement is a bond. The set R has the same size as the bases of $M(\mathscr{W}_r)$, that is, as the spanning trees of $\mathscr{W}_r$. Indeed, Tutte defined a new matroid $\mathscr{W}^r$, the *rank-r whirl*, on the set of edges of $\mathscr{W}_r$ having as its bases all of the bases of $M(\mathscr{W}_r)$ together with the set R.

Theorem 7 (Tutte, 1966). *A* 3*-connected matroid M has an element e such that $M\backslash e$ or M/e is* 3*-connected unless M has rank at least three and is a whirl or the cycle matroid of a wheel.*

In 1980, Seymour [18] generalized this theorem by proving the following.

Theorem 8 (Seymour, 1980). *Let M and N be* 3*-connected matroids such that N is a proper minor of M. Then M has an element e such that $M\backslash e$ or M/e is* 3*-connected having a minor isomorphic to N unless M is a wheel or a whirl.*

7 The first conference on matroids

In 1964, Jack Edmonds was working at the National Bureau of Standards in Washington. He and his colleagues there organized the first conference on matroids. Tutte gave a series of *Lectures on Matroids* [36]. These appeared in the conference proceedings, which were published in the *Journal of Research of the National Bureau of Standards* in 1965. Tutte wrote [47, p.8] about that 1964 meeting,

> To me that was the year of the Coming of the Matroids. Then and there the theory of matroids was proclaimed to the mathematical world. And outside the halls of lecture there arose the repeated cry: 'What the hell is a matroid?'

The 1965 *Journal of Research of the National Bureau of Standards* included Tutte's paper, *Menger's Theorem for matroids* [37]. That important paper was largely ignored for about 35 years until, in 2002, Geelen, Gerards, and Whittle [10] recognized its utility. The theorem has been used extensively since then.

For disjoint sets X and Y in a matroid M, define the *connectivity* between X and Y by

$$\kappa_M(X,Y) = \min\{r(S) + r(E-S) - r(M) : X \subseteq S \subseteq E - Y\}.$$

Theorem 9 (Tutte, 1965). *Let X and Y be disjoint sets in a matroid M. Then $\kappa_M(X,Y)$ is the maximum value of $\kappa_N(X,Y)$ over all minors N of M with ground set $X \cup Y$.*

Subsequently, Geelen, Gerards, and Whittle [11, Theorem 4.2] proved that this maximum could be restricted to minors N of M with $E(N) = X \cup Y$ such that $N|X = M|X$ and $N|Y = M|Y$. As an example, let $\{1,2,3\}$ and $\{4,5,6\}$ be the disjoint triangles in a triangular prism graph P. Then, by contracting the three edges of P that are not in triangles, we get a doubled triangle with edge set $\{1,2,\ldots,6\}$. A consequence of Theorem 9 is that, in an arbitrary 3-connected binary matroid M, if $\{1,2,3\}$ and $\{4,5,6\}$ are disjoint 3-element circuits, then M has a minor on $\{1,2,\ldots,6\}$ consisting of the cycle matroid of a doubled triangle.

8 Tutte's Ph.D. thesis

So far, we have mostly described Tutte's *published* work on matroids. But many of his discoveries were made much earlier and were included in his remarkable Ph.D. thesis, completed in 1948 [28]. In this section. we discuss some particulars of the thesis.

In reading the thesis, it must be borne in mind that Tutte's viewpoint for matroids is dual to the usual one, so that, for example, his "circuits" in nets generalize bonds (or minimal edge cuts) of graphs, and a matroid is "graphic" if its dual is graphic in the sense defined above. Similarly, the terminology for deletion and contraction aligns with standard usage for graphs but is swapped around for nets; see the discussion in [9]. There is also much nonstandard terminology, for example, "codendroids" for bases, "dendroids" for cobases, and "cyclic elements" for blocks in graphs and components in matroids.

In Chapter III of the thesis, Tutte presents his extension of Menger's Theorem to matroids although it is not until Chapter VII that he deduces Menger's Theorem for graphs from his generalization. His paper 'Menger's theorem for matroids' was not published until 1965 [37].

Chapter IV introduces regular matroids, under the name "simple nets", approaching them from an unusual direction. Tutte then shows that a matroid of rank r on n elements is regular (according to his definition) if and only if it has an $r \times n$ representative matrix over $\mathbb{Z}$ such that the determinant of every $r \times r$ submatrix is in $\{0, 1, -1\}$. It is routine to show that this condition is equivalent to total unimodularity of the matrix. Parts of this chapter were published and extended in [30].

Chapter V, the shortest in the thesis, is about his polynomials. It is the only chapter of the thesis that contains results he published before the thesis was completed in 1948. Its results are generalizations of a subset of those in 'A ring in graph theory' (published in 1947) [26]. Whereas [26] is restricted to graphs, this chapter of the thesis introduces polynomials for representable matroids. Instead of the V-functions of [26], we now have *chromatic functions*, which are called *Tutte invariants* or *Tutte-Grothendieck invariants* by later writers. Tutte's definition of chromatic functions only needs deletion, contraction, and the notion of a matroid component. He then extends the Whitney rank generating function to matroids, which only needs a rank function. Thus these definitions make no real use of representability, and it is reasonable to regard them as the first extension to matroids of any polynomials in the Tutte-Whitney family. It would be another twenty years until Henry Crapo [4] formally defined the Tutte polynomial for matroids.

Tutte gives a recipe theorem for the (matroidal) Whitney rank generating function and defines, without name, the (matroidal) Tutte polynomial. An appropriate evaluation gives the number of bases, generalizing his observation for the number of spanning trees of a graph in [26]. Other evaluations give a representable-matroid analogue of counting q-colourings in a graph. Care is need with Tutte's terminology in this chapter, as discussed in [9].

Chapter VI concerns connectivity in binary matroids, extending to them the notion of a 2-separation of a graph. For graphs, some of the theory appears in [39, Ch. 11].

Having worked entirely at the level of representable matroids for Chapters II–VI, Tutte establishes the relationship with graphs in Chapter VII. He develops the theory of cycle matroids and cocycle matroids and applies the theory of the previous chapters to them. Graphic matroids and their duals are shown to be regular. The Tutte polynomial evaluations of Chapter V are specialized to counting colourings and spanning trees. The theory of Chapter VI is applied to 2-connected graphs.

Chapters VIII-IX, occupying 140 pages, give Tutte's excluded-minor characterization of (the duals of) graphic matroids among binary matroids. The four excluded minors are called "gnarls" and he calls his result the "gnarl theorem". It has been the foundation and inspiration of matroid structure theory ever since, and is a fitting climax for one of the greatest doctoral theses of twentieth century mathematics.

9 The move away from matroids

Although Tutte did publish some matroid papers after 1966, these later papers were in conference proceedings [43, 44] reiterating results from earlier journal papers or were supplements to earlier papers [41, 42]. Tutte's 1966 paper *On the algebraic theory of graph colorings* [38] proposed a conjecture for binary matroids now called *Tutte's Tangential* 2*-Block Conjecture*, which can be viewed as an analogue of Hadwiger's Conjecture. The same paper included [38, p.22] the following conjecture on 4-flows, now known as '*Tutte's* 4*-Flow Conjecture*'. This conjecture remains open in general.

Conjecture 3. A graph without cut edges or nowhere-zero 4-flows has a Petersen-graph minor.

For cubic graphs, the last conjecture is equivalent to the assertion that every cubic graph without a cut edge or a Petersen-graph minor is 3-edge-colourable. A proof of this has been announced by Robertson, Sanders, Seymour, and Thomas. It appears in a series of papers including [6], which provides details of the other papers.

In 1981, Seymour [20] reduced Tutte's Tangential 2-Block Conjecture to the 4-Flow Conjecture by using his decomposition theorem for regular matroids [18].

By 1967, Tutte had essentially stopped publishing new results in matroid theory. Why? Looking back on his homotopy theorem for matroids, the excluded-minor characterization of regular matroids noted above (Theorem 4), Tutte wrote [47, p.8],

> One aspect of this work rather upset me. I had valued matroids as generalizations of graphs. All graph theory, I had supposed would be derivable from matroid theory and so there would be no need to do independent graph theory any more. Yet what was this homotopy theorem, with its plucking of bits of circuit across elementary configurations, but a result in pure graph theory? Was I reducing matroid theory to graph theory in an attempt to do the opposite? Perhaps it was this jolt that diverted me from matroids back to graphs.

10 Tutte's contributions

Tutte's contributions to mathematics were immense. MathSciNet credits him with 160 publications. As of May 17, 2018, MathSciNet also lists 3656 citations for his papers although it should be noted that this source primarily constructs its list for the years 2000 onwards. From 1967, he was the Editor-in-Chief of the *Journal of Combinatorial Theory*.

> Under his leadership the journal flourished. It became such a desirable place to publish that in time it was partitioned into two, series A and B, with Tutte retaining the leadership of the latter until his retirement as professor from the University of Waterloo in 1985 [53, pp.294–95]

To this day, that journal remains preeminent in combinatorics. Tutte was a founding member of the Department of Combinatorics and Optimization at the University of Waterloo, and he had eight Ph.D. students most notably Ron Mullin and Neil Robertson.

In 2012, British Prime Minister David Cameron wrote a letter to Tutte's niece Jeanne Youlden [53, p.286] expressing the gratitude of the United Kingdom for Tutte's codebreaking work. Cameron wrote [53, p.286],

> We should never forget how lucky we were to have men like Professor Tutte in our darkest hour and the extent to which their work not only helped protect Britain itself but also shorten the war by an estimated two years, saving countless lives.

One aspect of Tutte's creative work has yet to be touched on here. The Four invented a mathematical poetess named 'Blanche Descartes'. Any one of them could add works under her name but Tutte was believed to be the primary contributor.

> The Four carefully refused to admit Blanche was their creation. Visiting Tutte's office in 1968, [Tutte's fifth Ph.D. student Arthur] Hobbs had the following conversation with him:
> **Hobbs:** "Sir, I notice you have two copies of that proceedings. I wonder if I could buy your extra copy?"
> **Tutte:** "Oh, no, I couldn't sell that. It belongs to Blanche Descartes." [13, p.4]

At the conference banquet celebrating Tutte's eightieth birthday, he recited the following poem written by Ms Descartes especially for the occasion. The second author, on requesting a copy of the poem from Professor Tutte, was handed the original handwritten version.

The Three Houses Problem

In central Spain in mainly rain
Three houses stood upon the plain.

The houses of our mystery
To which from realms of industry
Came pipes and wires to light and heat
And other pipes with water sweet.

The owners said, "Where these things cross
Burn, leak or short, we'll suffer loss
So let a graphman living near
Plan each from each to keep them clear."

Tell them, graphman, come in vain,
They'll bear the cross that must remain
Explain the planeness of the plain.

Blanche Descartes

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