Cluster decorated geometric crystals, generalized geometric RSK-correspondences, and Donaldson-Thomas transformations

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Abstract For a simply connected, connected, semisimple complex algebraic group G, we define two geometric crystals on the $\mathscr{A}$-cluster variety of double Bruhat cell $B_- \cap Bw_0B$. These crystals are related by the $*$ duality . We define the graded Donaldson-Thomas correspondence as the crystal bijection between these crystals. We show that this correspondence is equal to the composition of the cluster chamber Ansatz, the inverse generalized geometric RSK-correspondence, and transposed twist map due to Berenstein and Zelevinsky.

1 Introduction

For reductive split algebraic groups, Berenstein and Kazhdan [3] defined decorated geometric crystals. One of important feature of such a crystal is a *decoration function*. For double Bruhat cells, in relation to mirror symmetry, this decoration function have been appeared in [10] as a pullback of a Landau-Ginsburg potential defined in the cluster setup in [12] with respect to a proper map of $\mathscr{A}$ -cluster variety to $\mathscr{X}$-cluster variety on the double Bruhat cells, for the Langlands dual groups.

We follow the recipes of [7, 9, 15], and endow the $\mathscr{A}$-cluster variety of double Bruhat cell $G^{w_0,e} := B_- \cap Bw_0B$ with two geometric crystals for Langlands dual group $G^\vee$, related by the $*$ duality. The Kashiwara crystal admits a duality operation $*$. One may regard the above $*$ duality as a geometric lift of the Kashiwara $*$ duality. The decoration function for the $*$ dual geometric crystal can be regarded as the pullback of the Landau-Ginzburg potential for the cluster algebra which is obtained by reversing all directions of edges in quivers of that considered in [10]. In this paper we will consider the case of simply-laced groups.

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There are two actions of Cartan torus H on $G^{w_0,e}$ from the left and from the right. Under the action H from the left , we regard $G^{w_0,e}$ as $H\times B_-^{w_0}$, where $B_-^{w_0} := B_-\cap Nw_0N$ is the reduced Bruhat cell. Berenstein and Kazhdan endowed such a reduced cell with decorated geometric crystal structure. Under the action H from the right, we regard $G^{w_0,e}$ as $N_-^{w_0}\times H$, where $N_-^{w_0} := N_-\cap Bw_0B$ is also a reduced cell. We endow $N_-^{w_0}$ with $*$ dual decorated geometric crystal structure. For the former crystal we let the frozen variables $\Delta_{w_0\omega_i,\omega_i}$, $i\in I$, be fixed (I denotes the set of vertices of the Dynkin diagram), while for the $*$ dual crystal we let be fixed another half of frozen variables $\Delta_{\omega_i,\omega_i}$, $i\in I$.

In order to obtain combinatorial crystal from geometric one, we have to consider toric charts of a positive structure and corresponding tropicalization [3].

There are several positive structures on $G^{w_0,e}$.

For one of such structures we use toric charts which constitute the Berenstein-Zelevinsky positive structures of $B_-^{w_0}$, BZ-variety, see [3] and [18]. The graded cones corresponding to the charts of such a positive structure are defined by tropicalization of the Berenstein-Kazhdan decoration function, and the cones turn out to be polyhedral realizations of the (graded) Kashiwara crystal due to Nakashima-Zelevinsky ([19]). Specifically, such charts and cones correspond to the same reduced decomposition $\mathbf{i}\in R(w_0)$ of the longest element w_0 of the Weyl group. For $\mathbf{i}\in R(w_0)$, we denote such a cone $gr\mathscr{N}\mathscr{Z}_{\mathbf{i}}$.

Another positive structure is related to the Lusztig variety on $N_-^{w_0}$ ([15, 1]). The charts of this variety are also defined for reduced decompositions of $R(w_0)$. For $\mathbf{i}\in R(w_0)$, the tropicaization with respect to the $*$ dual potential gives polyhedral realization of the combinatorial crystal with vertices being lattice vertices of the Kashiwara $*$ dual Lusztig graded cone, $gr\mathscr{L}_{\mathbf{i}}^*$.

One more positive structure is related to the $\mathscr{A}$-cluster variety. Specifically, we consider only a part of cluster toric charts of $\mathscr{A}$-cluster variety which correspond to the reduced decompositions of $R(w_0)$. We consider two families of positive charts, for one we let to be fixed frozen $\Delta_{w_0\omega_i,\omega_i}$, $i\in I$, we call fixing frozen *specialization*, and for another we make specialization at the frozen $\Delta_{\omega_i,\omega_i}$, $i\in I$. For $\mathbf{i}\in R(w_0)$, tropicalization with respect to the corresponding chart of the former one and Landau-Ginzburg potential provides us with the polyhedral realization of the Kashiwara crystal being unimodular isomorphic to the graded Lusztig cone $gr\mathscr{L}_{\mathbf{i}}$ and tropicalization with respect to the latter one and $*$ dual LG potential gives us the polyhedral realization of the Kashiwara $*$ dual being unimodular isomorphic to the graded Littelmann cone $gr\mathscr{S}_{\mathbf{i}}$, see [10].

We provide birational positive mappings between these positive structures. For that we use birational automorphisms tori $(\mathbb{C}^*)^{l(w_0)}$ called the generalized geometric RSK-correspondence, gRSK, and its inversion (Section 5), two mappings from cluster tori localized at frozen coordinates called Chamber Ansatz ([10]) and transposed twist map of [5].

We define the *graded Donaldson-Thomas transformation* as the map which, for each reduced decomposition $\mathbf{i}\in R(w_0)$, makes the following diagram with the positive structures on $G^{w_0,e}$ commutative and tropical graded DT-transformation as that wrt the tropicalization.

$$\begin{array}{ccc}
\boxed{\text{Lusztig-variety}\times H} & \xrightarrow{\eta^T_{w_0,e}} & \boxed{H\times\text{BZ-variety}} \\
\uparrow \alpha & & \uparrow \beta \\
\boxed{\text{cluster charts specialized at } \{\Delta_{\omega_i,\omega_i}\}_{i\in I}} & \xrightarrow{\mathscr{DT}} & \boxed{\text{cluster charts specialized at } \{\Delta_{w_0\omega_i,\omega_i}\}_{i\in I}}
\end{array} \tag{1}$$

where $\eta^T_{w_0,e}$ is transposition of the twist map defined in ([5], Definition 4.1), α is the composition of CA^- and inverse gRSK, β is the composition of CA^+ and gRSK, where CA^+ denotes the tuples maps $grCA_{\mathbf{i}}$, $\mathbf{i}\in R(w_0)$, and CA^- denotes the inverse of $grNA_{\mathbf{i}}$, defined in ([10], Definition 6.1 and 7.1), for details see Sections 5 and 6. The latter mappings are motivated by the Chamber Ansatz ([1]) for the Lusztig- and Berenstein-Zelevinsky - parametrizations of $N_-^{w_0}$ and $B_-^{w_0}$, respectively.

The twist $\eta^T_{w_0,e}$ is a crystal bijection sending $N_-\cap Bw_0B\times H$ to $H\times B_-\cap Nw_0N$. Since all vertical maps are crystal isomorphism we get that the graded Donaldson-Thomas transformation is an isomorphism of geometric cluster crystals.

In other words, the graded Donaldson-Thomas transformation is the composition of five maps, the inverse generalized geometric RSK and CA^-, sending cluster variety specialized at the half of frozen $\Delta_{\omega_i,\omega_i}$, $i\in I$, to the the graded Lusztig variety, both endowed with $*$-dual geometric crystal structure, then transposed twist map which sends the Lusztig variety to the Berenstein-Zelevinsky variety, where the latter is endowed with the geometric crystal as in [3, 18], and finally the inverse generalized geometric RSK and inverse CA^+ sending BZ-variety to the cluster variety specialized at another half of frozen $\Delta_{\omega_i,\omega_i}$, $i\in I$.

The tropical graded DT-transformation is as in [10]. Specifically, tropicalization of the above diagram leads to the definition of a *tropical DT-transformation* which makes the following diagram commutative

$$\begin{array}{ccc}
\boxed{gr\mathscr{L}^*_{\mathbf{i}}} & \xrightarrow{\text{tropical BZ-twist}} & \boxed{gr\mathscr{N}\mathscr{Z}_{\mathbf{i}}} \\
\uparrow \text{tropical inverse RSK} & & \uparrow \text{tropical RSK} \\
\boxed{gr\mathscr{S}_{\mathbf{i}}} & \xrightarrow{\text{tropical } \mathscr{D}T} & \boxed{gr\mathscr{L}_{\mathbf{i}}}
\end{array} \tag{2}$$

The mappings between the SW-NE and NW-SE corners of diagram (1) are geometric lifting of the Kashiwara $*$ dual crystal isomorphisms between corresponding corners of the diagram (2).

Thus we may regard $*$ duality (Kashiwara $*$ duality) as the composition of the transposed twist, the inverse generalized geometric RSK correspondence, and inverse CA^+ (tropical twist, tropical gRSK, and tropical inverse CA^+).

Goncharov and Shen [11] conjectured that the Donaldson-Thomas transformation, defined for $\mathscr{X}$-cluster variety, is the twist $\eta_{w_0,e}$ under specialization at all frozen variables. This conjecture is proven in [20].

Our graded Donaldson-Thomas transformation is the composition of transposed twist and maps α and β^{-1}.

2 Preliminary and Notations

2.1 Simply-connected algebraic groups

For a simply connected, connected, semisimple complex algebraic group G, let B and B_- be Borel subgroup and its opposite, N and N^- be unipotent radicals, a maximal torus $H = B \cap B_-$, and $W = NormG(H)/H$ be the Weyl group. Let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix, cardinality of I, $|I|$, equals the rank of G. The Weyl group W is canonically identified with the Coxeter group generated by the involutions $s_1, \ldots, s_{|I|}$, subject to the relations $(s_i s_j)^{d_{ij}} = e$, $d_{ij} = 0,3,4,6$ if $a_{ij} = 0,1,2,3$, respectively. A reduced decomposition of $w \in W$ is a word $\mathbf{i} = (i_1 \cdots i_l)$ in the alphabet I, such that $w = s_{i_1} \cdots s_{i_l}$ gives a factorization of smallest length. The length l is called the length of w and denoted by $l(w)$. For $w \in W$, the set of all reduced decompositions is denoted by $R(w)$. We denote by w_0 the element of maximal length in W. Any two reduced decompositions $\mathbf{i}$, $\mathbf{i}' \in R(w)$ are related by the Artin relations. For simply-laced cases, Artin relations are 2-mooves ad 3-moves. Specifically, a reduced word $\mathbf{j} = (j_1, \ldots, j_l)$ is defined to be obtained from $\mathbf{i} = (i_1, \ldots, i_l)$ by a 2-*move at position* $k \in [l-1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k, k+1\}$, $(i_{k+1}, i_k) = (j_k, j_{k+1})$ and $a_{i_k, i_{k+1}} = 0$.

A reduced word $\mathbf{j}$ is defined to be obtained from $\mathbf{i}$ by a 3-*move at position* $k \in [l-1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k-1, k, k+1\}$, $j_{k-1} = j_{k+1} = i_k$, $j_k = i_{k-1} = i_{k+1}$ and $a_{i_k, i_{k+1}} = -1$.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{h}$ the Cartan subalgebra. Let $\{\alpha_1, \ldots, \alpha_{|I|}\} \subset \mathfrak{h}^*$ be simple roots for which the corresponding root subgroups are contained in N. For $i \in I$, let ϕ_i be the homomorphism $SL_2 \to G$ corresponding to the ith simple root of G. Given $i \in I$ define

$$x_i(t) = \phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; y_i(t) = \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \overline{s}_i = \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$x_{-i}(p) = \phi_i \begin{pmatrix} \frac{1}{p} & 0 \\ 1 & p \end{pmatrix}.$$

$$\alpha_i^\vee(c) = \phi_i \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad \alpha_i(\mathbf{h}) = \frac{h_i}{h_{i+1}}, \, i \in I.$$

2.2 Cluster seeds associated to reduced decompositions

Recall that, for a dominant weight $\lambda : H \to \mathbb{G}_m$, the *principal minor* $\Delta_\lambda : G \to \mathbb{A}^1$ is the function defined on the open subset $N^-HN \subset G$ by

$$\Delta_\lambda(u^- \mathbf{h} u^+) := \lambda(\mathbf{h}) \quad u^- \in N^-, \mathbf{h} \in H, u^+ \in N.$$

Let γ, δ be extremal weights such that $\gamma = w_1\lambda$, $\delta = w_2\lambda$ for some $w_1, w_2 \in W$, $\lambda \in P^+$. The *generalized minor* associated to γ and δ is

$$\Delta_{\gamma,\delta}(g) := \Delta_\lambda(\overline{w}_1^{-1} g \overline{w}_2), \quad g \in G,$$

where $\overline{w}$ is a lift of W into $Norm_G H$ using $\overline{s}_i$, $i \in I$

The base affine space G/N is the partial compactification of the *open double Bruhat cell*

$$G^{w_0,e} := Bw_0B \cap B_-$$

obtained by allowing the generalized minors $\Delta_{\omega_a,\omega_a}$ and $\Delta_{w_0\omega_a,\omega_a}$ to vanish.

Here we need a small part of cluster seeds of the $\mathscr{A}$-variety. Namely, for a reduced decomposition $\mathbf{i} \in R(w_0)$, we consider the corresponding seed $\mathscr{S}(\mathbf{i})$ follow [2]. The vertices of the quiver $Q(\mathbf{i})$ are labeled by the fundamental weights ω_i, $i \in I$, and $\mathbf{i}|_{\leq k}\omega_{i_k}$, $k \in l(w_0)$, , $\mathbf{i}|_{\leq k}$ denotes the subword $\mathbf{i}$ of the first k letters.

The frozen vertices are labeled by the fundamental weights ω_i, $i \in I$, and $w_0\omega_i$, $i \in I$.

The cluster variables of $\mathscr{S}(\mathbf{i})$ are the generalized minors $\Delta_{\mathbf{i}|_{\leq k}\omega_{i_k},\omega_{i_k}}$ attached to vertices labeled by $\mathbf{i}|_{\leq k}\omega_{i_k}$.

Follow [2] we associate to every reduced word $\mathbf{i}$ a seed $\Sigma(\mathbf{i})$. The set of edges quiver $\Gamma_{\mathbf{i}}$ is described as follows. For $k \in [-n]$ we set $i_k = -k$. For $k \in [l(w_0)]$ we denote by $k^+ = k_{\mathbf{i}}^+$ the smallest ℓ such that $k < \ell$ and $i_\ell = i_k$. If no such ℓ exists, we set $k^+ = l(w_0) + 1$. For $k \in [l(w_0)]$, we further let k^- be the largest index ℓ with that $\ell < k$ and $i_\ell = i_k$.

There is an edge connecting v_k and v_ℓ with $k < \ell$ if at least one of the two vertices is mutable and one of the following conditions is satisfied:

1. $\ell = k^+$,
2. $\ell < k^+ < \ell^+$, $c_{k,\ell} < 0$ and $k, \ell \in [N]$.

Edges of type (1) are called *horizontal* and are directed from k to ℓ. Edges of type (2) are called *inclined* and are directed from ℓ to k.

We need the following fact. Let $\mathbf{j} \in R(w_0)$ be obtained from $\mathbf{i}$ by a 3-move in position k. Then the transposition $(k, k+1)$ is an isomorphism of quivers $\Gamma_{\mathbf{j}} \simeq \mu_k \Gamma_{\mathbf{i}}$, where μ_k is a mutation at the vertex labeled by $\mathbf{i}|_{\leq k}\omega_{i_k}$.

The new variables is obtained by the $\mathscr{A}$-cluster mutation

$$\mu_k A_\ell = \begin{cases} \dfrac{\prod_{\ell:(\ell,k)\in\Gamma(\mathbf{j})} A_\ell}{A_k} + \dfrac{\prod_{m:(k,m)\in\Gamma(\mathbf{j})} A_m}{A_k} & \text{if } \ell = k, \\ A_\ell & \text{else,} \end{cases}$$

For reduced seeds corresponding to reduced words, such cluster mutation take the form of Plúcker relations between generalized minors.

3 Cluster geometric crystals

On the $\mathscr{A}$-cluster variety $G^{w_0,e}$, we define two geometric crystals (for Langlands dual group) related by the Kashiwara $*$–involution.

3.1 Geometric crystal for $\mathscr{A}$-variety specialized at $\Delta_{w_0\omega_i,\omega_i}$'s

We consider simply-laced case and define the main ingredients of the geometric crystal on the $\mathscr{A}$-cluster variety G^r obtained of $G^{w_0,e}$ by the specialization at the frozen frozen variables $\Delta_{w_0\omega_i,\omega_i}$, $i \in I$.

For $k \in I$, we denote by $\mathbf{i}_k$ a reduced decomposition which starts with s_k, we call such a reduced decomposition *optimal from the head for k*.

For such an optimal reduced decomposition $\mathbf{i}_k$, we consider the corresponding seed $\mathscr{S}(\mathbf{i}_k)$.

We define the crystal actions $f_k(c,\cdots) : \mathbb{C}^* \times G^{w_0,e} \to G^{w_0,e}$, $k \in I$, by specifying it on the variables of the seed $\mathscr{S}(\mathbf{i}_k)$.Namely, we set

$$f_k(c,\cdot) : \Delta_{\omega_k,\omega_k} \to c\Delta_{\omega_k,\omega_k}, \tag{3}$$

and $f_k(c,\cdot)$ does not change other generalized minors labeling nodes of $\mathscr{S}(\mathbf{i}(k))$.

In order to get the action of another crystal operation $f_l(c,\cdot)$ on variables of this seed, firstly, we have to express the cluster variables of $\mathscr{S}(\mathbf{i}(k))$ as Laurent polynomials of cluster variables $\mathscr{S}(\mathbf{i}(l))$, secondly we have to apply $f_l(c,\cdot)$ to variables of these Laurent polynomials, and then to express such obtained Laurent polynomials in the variables of $\mathscr{S}(\mathbf{i}(k))$.

Because of refinement of the Laurent phenomenon for cluster algebras ([8]), that claims that frozen variables do not appear in denominators of Laurent polynomials expressing a cluster variable of one seed in the variables of another, we get that the crystal operations take the form of Laurent polynomials, indeed.

Note that the frozen variables $\Delta_{w_0\omega_i,\omega_i}$, $i \in I$, do not change under any of such crystal actions. This is a reason to specialize the cluster algebra at these frozen variables.

We take the potential $\Phi_{BK} : G^{w_0,e} \to \mathbb{C}$, as the decoration function due to Berenstein and Kazhdan ([3])

$$\Phi_{BK}(M) = \sum_{i\in I} \frac{\Delta_{w_0\omega_i,s_i\omega_i}(M)}{\Delta_{w_0\omega_i,\omega_i}(M)} + \sum_{i\in I} \frac{\Delta_{w_0s_i\omega_i,\omega_i}(M)}{\Delta_{w_0\omega_i,\omega_i}(M)}, \quad M \in G^{w_0,e}. \tag{4}$$

For a group G with simply-laced Lie algebra, it follows from [10], that, for each k and any reduced decomposition $\mathbf{i}(k)$ optimal for k, we have

$$\begin{aligned}&\Phi_{BK}(f_k(c,M_{\mathbf{i}(k)}))-\Phi_{BK}(M_{\mathbf{i}(k)})\\=&(c-1)\frac{\Delta_{\omega_k,\omega_k}(M_{\mathbf{i}(k)})}{\Delta_{s_k\omega_k,\omega_k}(M_{\mathbf{i}(k)})}+(\frac{1}{c}-1)\frac{\Delta_{\omega_{k-1},\omega_{k-1}}(M_{\mathbf{i}(k)})\Delta_{\omega_{k+1},\omega_{k+1}}(M_{\mathbf{i}(k)})}{\Delta_{s_k\omega_k,\omega_k}(M_{\mathbf{i}(k)})\Delta_{\omega_k,\omega_k}(M_{\mathbf{i}(k)})}.\end{aligned} \tag{5}$$

where $M_{\mathbf{i}(k)}$ a toric chart of the $\mathscr{A}$ cluster variety $G^{w_0,e}$ written in cluster variables of $\mathscr{S}(\mathbf{i}(k))$.

For SL_n, this means the following. We consider matrix elements of $M \in G^{w_0,e}$, as Laurent polynomials which express $\frac{\Delta_{s_{i-1}\cdots s_j\omega_j,\omega_j}}{\Delta_{\omega_{j-1},\omega_{j-1}}}$, in variables of the cluster seed $\mathscr{S}(\mathbf{i}(k))$. $M_{\mathbf{i}(k)}$ denotes such a representation of matrix elements.

Because of that if we consider a point of the $\mathscr{A}$-variety, that is a collection of tuples, related by the cluster mutations, then each tuple of the collection defines the same matrix.

Because of Positivity Theorem [14], these matrix elements are Laurent polynomials with non-negative coefficients.

We define the functions φ, ε and γ being geometric lifting of the Kashiwara functions as follows.

For the seed $\mathscr{S}(\mathbf{i}(k))$, we set

$$\begin{aligned}\varphi_k(M_{\mathbf{i}(k)})&=\frac{\Delta_{s_k\omega_k,\omega_k}(M_{\mathbf{i}(k)})}{\Delta_{\omega_k,\omega_k}(M_{\mathbf{i}(k)})},\\ \varepsilon_k(M_{\mathbf{i}(k)})&=\frac{\Delta_{s_k\omega_k,\omega_k}(M_{\mathbf{i}(k)})\Delta_{\omega_k,\omega_k}(M_{\mathbf{i}(k)})}{\Delta_{\omega_{k-1},\omega_{k-1}}(M_{\mathbf{i}(k)})\Delta_{\omega_{k+1},\omega_{k+1}}(M_{\mathbf{i}(k)})},\text{ and}\\ \alpha_k(\gamma(M_{\mathbf{i}(k)})&:=\frac{\varphi_k}{\varepsilon_k}.\end{aligned} \tag{6}$$

Thus we have,

$$\alpha_k(\gamma(M_{\mathbf{i}(k)}))=\frac{\Delta_{\omega_k,\omega_k}(M_{\mathbf{i}(k)})^2}{\Delta_{\omega_{k-1},\omega_{k-1}}(M_{\mathbf{i}(k)})\Delta_{\omega_{k+1},\omega_{k+1}}(M_{\mathbf{i}(k)})}.$$

Because of the above formula and that all seeds before action of operations f_k's has the same frozen variables, we get that γ does not depend on a seed and

$$\alpha_k(\gamma(M))=\frac{\Delta_{\omega_k,\omega_k}(M)^2}{\Delta_{\omega_{k-1},\omega_{k-1}}(M)\Delta_{\omega_{k+1},\omega_{k+1}}(M)},\quad M\in G^{w_0,e}. \tag{7}$$

Note that we can also regard functions φ and ε independently of cluster seeds,

$$\varphi_k(M) = \frac{\Delta_{s_k\omega_k,\omega_k}(M)}{\Delta_{\omega_k,\omega_k}(M)},$$
$$\varepsilon_k(M) = \frac{\Delta_{s_k\omega_k,\omega_k}(M)\Delta_{\omega_k,\omega_k}(M)}{\Delta_{\omega_{k-1},\omega_{k-1}}(M)\Delta_{\omega_{k+1},\omega_{k+1}}(M)}.$$

From the refined Laurent phenomenon, we get that, for any cluster seed, the functions φ and ε are Laurent polynomials in variables of that seed.

3.2

For example for SL_3 and a cluster seed, corresponding to a reduced word 121, let us denote the cluster variables $t_1 = \Delta_{\omega_1,\omega_1}$, $t_2 = \Delta_{s_1\omega_1,\omega_1}$, $t_3 = \Delta_{s_2s_1\omega_1,\omega_1}$, $t_{12} = \Delta_{\omega_2,\omega_2}$, $t_{23} = \Delta_{w_0\omega_2,\omega_2}$.

Then elements of the corresponding cluster chart are matrices of the form

$$M_{121} := \begin{pmatrix} t_1 & 0 & 0 \\ t_2 & \frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_1t_{23}+t_3t_{12}}{t_1t_2} & \frac{1}{t_{12}} \end{pmatrix}$$

and, since the 121 is optimal for s_1, the action $f_1(c,\cdot)$ is

$$f_1(c, M_{121}) = \begin{pmatrix} ct_1 & 0 & 0 \\ t_2 & \frac{t_{12}}{ct_1} & 0 \\ t_3 & \frac{ct_1t_{23}+t_3t_{12}}{ct_1t_2} & \frac{1}{t_{12}} \end{pmatrix}$$

Then the potential Φ_{BK} computed in variables of this cluster chart is

$$\Phi_{BK}^{121} = \frac{t_{12}}{t_1t_2} + \frac{t_2}{t_{12}t_{23}} + \frac{t_{23}}{t_2t_3} + \frac{t_1}{t_2} + \frac{t_{12}t_3}{t_{23}t_2} + \frac{t_2}{t_3}.$$

The functions $\varphi_1(M_{121}) = \frac{t_2}{t_1}$, $\varepsilon_1(M_{121}) = \frac{t_1t_2}{t_{12}}$ and $\gamma_1 = \frac{t_1^2}{t_{12}}$.

In order to have the action $f_2(c,\cdot)$ and to compute $\varphi_2(M_{121})$ and $\varepsilon_2(M_{121})$, we have to represent M_{121} in the cluster coordinates of the chart for the reduced decomposition 212. For 212, we get

$$M_{212} := \begin{pmatrix} t_1 & 0 & 0 \\ \frac{t_1t_{23}+t_3t_{12}}{t_{13}} & \frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_{13}}{t_1} & \frac{1}{t_{12}} \end{pmatrix},$$

where $t_{13} = \Delta_{s_2\omega_2,\omega_2}$ and there due to the Plúcker we have $t_{13}t_2 = t_1t_{23} + t_3t_{12}$, where t_1, t_2, t_3, t_{12}, , t_{23} are as above, and t_1, t_3, t_3, t_{12}, , t_{23} are the cluster variables of $\mathscr{S}(212)$.

Then the action $f_2(c,\cdot)$ at the chart M_{212}, corresponding to the seed $\mathscr{S}(212)$, takes the form

$$f_2(c,M_{212}) := \begin{pmatrix} t_1 & 0 & 0 \\ \frac{t_1t_{23}+ct_3t_{12}}{t_{13}} & c\frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_{13}}{t_1} & \frac{1}{ct_{12}} \end{pmatrix},$$

and, hence, $f_2(c,\cdot)$ acts in the chart for 121 as follows

$$f_2(c,M_{121}) = \begin{pmatrix} t_1 & 0 & 0 \\ t_2+\frac{(c-1)t_3t_{12}t_2}{t_1t_{23}+t_3t_{12}} & c\frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_1t_{23}+t_3t_{12}}{t_1t_2} & \frac{1}{ct_{12}} \end{pmatrix}.$$

Hence we get $\varphi_2(M_{121}) = \frac{t_1t_{23}+t_3t_{12}}{t_{12}t_2}$, $\varepsilon_2(M_{121}) = \frac{t_{12}(t_1t_{23}+t_3t_{12})}{t_1t_2}$, and $\gamma_2(M) = \frac{t_{12}^2}{t_1}$. Note the potential Φ_{BK} computed in variables of cluster chart for 212 is

$$\Phi_{BK}^{212} = \frac{t_1}{t_{12}t_{13}} + \frac{t_{13}}{t_1t_2} + \frac{t_3}{t_{13}t_{23}} + \frac{t_1}{t_3} + \frac{t_1t_{23}}{t_3t_{13}} + \frac{t_{12}}{t_{13}}.$$

3.3

Denote by $G^r(\mathbf{h})$ a leaf of $G^{w_0,e}$ obtained by fixing $\Delta_{w_0\omega_i,\omega_i} =: h_i$, $i\in I$. Namely, for a cluster seed, corresponding to a reduced word $\mathbf{i}\in R(w_0)$, we regard matrix $M_{\mathbf{i}}$ as product

$$M_{\mathbf{i}} = \alpha_1^\vee(\frac{1}{\Delta_{w_0\omega_{|I|}}})\alpha_2^\vee(\frac{1}{\Delta_{w_0\omega_{|I|-1},\omega_{|I|-1}}})\cdots\alpha_{|I|-1}^\vee(\frac{1}{\Delta_{w_0\omega_1,\omega_1}})\tilde{M}_{\mathbf{i}},$$

where $\tilde{M}_{\mathbf{i}}\in G^r(\mathbf{1})$.

Theorem 1. *For any $\mathbf{h}\in H$, the crystal operations defined by the rule (3), the decoration function defined by (4), the functions φ, ε, γ defined by (6), define a geometric crystal on the $\mathscr{A}$-cluster variety $G^r(\mathbf{h})$, in the sense of [3].*

Proof. We have to verify that the crystal operations satisfy the Verma relations and proper behavior of the above functions under the crystal actions.

a) The claim that the crystal actions f_k, $k\in I$, satisfy the Verma relations (quantum Yang-Baxter equation[1])

$f_k^a(f_{k'}^{ab}(f_k^b)) = f_{k'}^b(f_k^{ab}(f_{k'}^a))$ if k and k' are joined by an edge in the Dynkin diagram

and commute elsewhere,

[1] Integrable system related to this quantum Yang-Baxter equation is Toda lattice, and we will come to this issue in another paper.

can be reduced to that claim for the case of SL_3 (see [7]). The later case is straightforward computation.

b) The relation of the decoration function Φ_{BK} and the Kashiwara functions φ and ε, takes the form

$$\Phi_{BK}(f_k(c,M)) - \Phi_{BK}(M) = \frac{c-1}{\varphi_k(M)} + \frac{1/c-1}{\varepsilon_k(M)}. \quad (8)$$

For simply-laced groups, $k \in I$ and a seed $\mathscr{S}(\mathbf{i}(k))$, $\mathbf{i}(k) \in R(w_0)$ and is optimal for k, (8) follows from [9, 10].

The claim that relations between ε, φ and γ fulfill the requirements of [3] also follows from the above claim. □

4 $*$ Dual geometric crystal

We define the $*$ dual geometric crystal on $\mathscr{A}$-variety G^l, obtained from $G^{w_0,e}$ by the specialization at the frozen variables $\Delta_{\omega_i,\omega_i}$, $i \in I$.

The Kashiwara crystal admits a duality operation $*$ (see, for example, [13]), and one may regard such $*$ dual geometric crystal as a geometrization the Kashiwara duality.

Namely, for $a \in I$, a reduced decomposition $\mathbf{i}^a$ is *optimal from the tail for a*, if i_a is the last element of $\mathbf{i}^a$. One can consider $\mathbf{i}^a$ as reversed $\mathbf{i}(a)$ with $s_{w_0(j)}$ replacing s_j.

Let us consider the corresponding seed $\mathscr{S}(\mathbf{i}^a)$.

For such a reduced decomposition $\mathbf{i}^a$, we define the action of $f^*_{w_0(a)}(c,\cdot)$ on the variables of the seed $\mathscr{S}(\mathbf{i}^a)$ by acting only on frozen variable

$$\Delta_{w_0\omega_a,\omega_a} \to c\Delta_{w_0\omega_a,\omega_a},$$

of this seed and does not changing other cluster variables of $\mathscr{S}(\mathbf{i}^a)$.

To define $f^*_{w_0(a)}(c,\cdots)$ in another seed $\mathscr{S}$, we have to mutate from $\mathscr{S}$ to $\mathscr{S}(\mathbf{i}^a)$, than apply $f_{w_0(a)}(c,\cdot)$ on $\mathscr{S}(\mathbf{i}^a)$, and than mutate back to $\mathscr{S}$.

Remark that the frozen variables $\Delta_{\omega_i,\omega_i}$, $i \in I$, do not change under all such crystal actions. Because of that we make specialization at these frozen.

To define all functions for a geometric crystal, we firstly define the decoration function

$$\Psi_{*K}(M) := \sum_{i\in I} \frac{\Delta_{\omega_i,s_i\omega_i}(M)}{\Delta_{\omega_i,\omega_i}(M)} + \sum_{i\in I} \frac{\Delta_{w_0\omega_i,s_i\omega_i}(M)}{\Delta_{w_0\omega_i,\omega_i}(M)}, \quad M \in G^{w_0,e}. \quad (9)$$

Then, in the seed $\mathscr{S}(\mathbf{i}^a)$ we get the following functions

$$\varphi^*_{w_0(a)}(M_{\mathbf{i}^a}) = \frac{\Delta_{w_0 s_a \omega_a, \omega_a}(M_{\mathbf{i}^a})}{\Delta_{w_0 \omega_a, \omega_a}(M_{\mathbf{i}^a})}, \tag{10}$$

$$\varepsilon^*_{w_0(a)}(M_{\mathbf{i}^a}) = \frac{\Delta_{w_0 s_a \omega_a, \omega_a}(X)\Delta_{w_0 \omega_a, \omega_a}(M_{\mathbf{i}^a})}{\Delta_{w_0 \omega_{a-1}, \omega_{a-1}(M_{\mathbf{i}^a})}\Delta_{w_0 \omega_{a+1}, \omega_{a+1}}(M_{\mathbf{i}^a})} \tag{11}$$

$$\alpha_{w_0(k)}(\gamma^*(M)) = \frac{\Delta_{w_0 \omega_k, \omega_k}(M)^2}{\Delta_{w_0 \omega_{k-1}, \omega_{k-1}}(M)\Delta_{w_0 \omega_{k+1}, \omega_{k+1}}(M)}, \quad M \in X. \tag{12}$$

Note that γ^* is the 'highest weight' for the Kashiwara geometric crystal with the potential Φ_{BK}.

For SL_n and $\mathbf{i} \in R(w_0)$, we have the following relations, which shows symmetry of weights and highest weights on the language of geometric crystals,

$$\alpha_k(\gamma(M_{\mathbf{i}}))\alpha_k(\gamma^*(M_{\mathbf{i}})) = \prod_{\rho \in T(k)} t_m^{\mathrm{sign}\rho \cdot \chi(m,\rho)} \prod_{\rho \in T(k+1)} t_m^{\mathrm{sign}\rho \cdot \chi(m,\rho)}, \tag{13}$$

where $T(k)$ is a train track colored by k in the rhombus tiling for $\mathbf{i}$, $\chi(m,\rho)$ is the delta function of positive roots labeled by m-th cluster variable and the tile ρ, and $\mathbf{t} := CA^+(\mathscr{S}(\mathbf{i}))$ (for details see [9]). Note that symmetry between γ and γ^* breaks when we choose from what side the Cartan torus acts on B_-. Another relations between weights and highest weights is

$$\alpha_k(\gamma(M_{\mathbf{i}}))\alpha_{w_0(k)}(\gamma^*(M_{\mathbf{i}})) = \prod_{m \in I(k)} q_{i_m}^2 \prod_{m' \in I(k') : a_{i'_m, k} = -1} q_{i'_m}, \tag{14}$$

where, for $\mathbf{i}$, $I(k) = \{j : i_j = k\}$, $\mathbf{q} := CA^-(\mathscr{S}(\mathbf{i}))$.

Denote by $G^l(\hat{\mathbf{h}})$ a leaf of $G^{w_0,e}$ with fixed $\Delta_{\omega_i,\omega_i} =: \hat{h}_i$, $i \in I$.

Theorem 2. *For each $\hat{\mathbf{h}}$, the above defined crystal actions f_a^*, $a \in I$, the decoration (9), and the functions (10)–(12) define a geometric crystal on the $\mathscr{A}$-cluster variety $G^l(\hat{\mathbf{h}})$.*

Proof. For the Verma relations, it suffices to check for SL_3, and this is a rather straightforward. Then the relations among the actions, decorations and the functions in order to fulfill the axioms of the geometric crystal, follows from the above property of the potential Ψ_{*K} in each seed tail optimal for $a \in I$. □

4.1

For SL_3, the cluster seed for 121 is tail optimal for 1 and hence for the second action f_2^*.

Thus, we have

$$f_2^*(c, M_{121}) = \begin{pmatrix} t_1 & 0 & 0 \\ t_2 & \frac{t_{12}}{t_1} & 0 \\ ct_3 & \frac{t_1 t_{23} + ct_3 t_{12}}{t_1 t_2} & \frac{1}{t_{12}} \end{pmatrix}$$

The cluster seed for 212 is tail-optimal for 2 and hence, is optimal for the crystal action f_1^*, we have

$$f_1^*(c, M_{212}) := \begin{pmatrix} t_1 & 0 & 0 \\ \frac{ct_1t_{23}+t_3t_{12}}{t_{13}} & \frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_{13}}{t_1} & \frac{1}{ct_{12}} \end{pmatrix},$$

and, hence, $f_1^*(c, \cdot)$ acts in the chart for 121 as follows

$$f_1^*(c, M_{121}) = \begin{pmatrix} t_1 & 0 & 0 \\ t_2 + \frac{(c-1)t_1t_{23}t_2}{t_1t_{23}+t_3t_{12}} & \frac{t_{12}}{t_1} & 0 \\ t_3 & \frac{t_1t_{23}+t_3t_{12}}{t_1t_2} & \frac{1}{ct_{12}} \end{pmatrix}.$$

Note that in the cluster chart for 121, the specialization lead to the right action of H of the form

$$\begin{pmatrix} \frac{t_1}{t_3} & 0 & 0 \\ \frac{t_2}{t_3} & \frac{t_{12}t_3}{t_1t_{23}} & 0 \\ 1 & \frac{(t_1t_{23}+t_3t_{12})t_3}{t_1t_2t_{23}} & \frac{t_{23}}{t_{12}} \end{pmatrix} \cdot \begin{pmatrix} t_3 & 0 & 0 \\ 0 & \frac{t_{23}}{t_3} & 0 \\ 0 & 0 & \frac{1}{t_{23}} \end{pmatrix}.$$

For cluster chart for 121, the decoration Ψ_{*K} is of the form

$$\Psi_{*K}^{121} = \frac{t_4}{t_1t_2} + \frac{t_2}{t_4t_5} + \frac{t_5}{t_2t_3} + \frac{t_2}{t_1} + \frac{t_5t_1}{t_4t_2} + \frac{t_3}{t_2}.$$

Restricted to the leaf for $t_1 =: \hat{h}_1$, $t_4 =: \hat{h}_2$, we get

$$\Psi^{121}(\hat{h}_1, \hat{h}_2) = \frac{\hat{h}_2}{\hat{h}_1t_2} + \frac{t_2}{t_5\hat{h}_2} + \frac{t_5}{t_2t_3} + \frac{t_2}{\hat{h}_1} + \frac{t_5\hat{h}_1}{\hat{h}_2t_2} + \frac{t_3}{t_2}.$$

For the cluster seed, corresponding to a reduced word 212, we have the cluster variables $t'_1 = \Delta_{\omega_1,\omega_1}$, $t'_4 = \Delta_{s_2\omega_2,\omega_2}$, $t'_2 = \Delta_{s_2s_1\omega_1,\omega_1}$, $t'_3 = \Delta_{\omega_2,\omega_2}$, $t'_5 = \Delta_{w_0\omega_2,\omega_2}$. For this seed, Ψ_{*K}^{212} is of the form

$$\Psi_{*K}^{212} = \frac{t'_1}{t'_4t'_3} + \frac{t'_4}{t'_1t'_2} + \frac{t'_2}{t'_4t'_5} + \frac{t'_4}{t'_3} + \frac{t'_2t'_3}{t'_1t'_4} + \frac{t'_5}{t'_4}.$$

Note that Ψ_{*K}^{212} coincides with Φ_{BK} computed in the chart for 121 under 'reversing' of the variables $t_k = t'_{w_0(k)}$, $k = 1, \ldots, 5$.

As a generalization this remark we get the following

Proposition 3 *For a reduced decomposition* $\mathbf{i} \in R(w_0)$ *and the cluster chart* $\mathscr{S}(\mathbf{i})$*, we have*

$$\Phi_{BK}^{\mathbf{i}}(\mathscr{S}(\mathbf{i})) = \Psi_{*K}^{\mathbf{i}^*}((\mathscr{S}(\mathbf{i}^*)^{op}). \tag{15}$$

We establish an explicit crystal bijection between the geometric crystal and $*$ dual geometric crystal below.

5 Piece-wise linear combinatorics and RSK-correspondences

5.1 Elementary maps from which we make geometric RSK-correspondences

For $w \in W$ and a reduced decomposition $\mathbf{i} \in R(w)$, we define the geometric $\mathbf{i}$-RSK as the composition of $l(w)$ primitive maps, where $l(w)$ is the length of $\mathbf{i}$. (For simplicity we regard $\mathbf{i}$ as a word of $I^{l(w)}$.)

For any $\mathbf{i} \in I^{l(w)}$ and $k = 1, \ldots, l(w)$ we define a primitive map as the rational map $\kappa_k = \kappa_k^{A,\mathbf{i}} : \mathbb{T}^{l(w)} \to \mathbb{T}^{l(w)}$, $\mathbb{T}^{l(w)} := (\mathbb{G}_m)^{l(w)}$, by

$$\kappa_k(\mathbf{t})_{k'} = \begin{cases} t_{k'} & \text{if } k' > k \\ \sigma_{0,k}(\mathbf{t}) & \text{if } k' = k \\ t_{k'} \cdot \sigma_{k',k}(\mathbf{t})^{-a_{i_{k'},i_k}} & \text{if } k' < k,\, i_{k'} \neq i_k \\ \frac{t_{k'}}{\sigma_{k',k}(\mathbf{t}) \cdot (t_{k'} + \sigma_{k',k}(\mathbf{t}))} & \text{if } k' < k,\, i_{k'} = i_k \end{cases},$$

where we abbreviated $\sigma_{k',k}(\mathbf{t}) := \sum_{k' < \ell \leq k : i_\ell = i_k} t_\ell$. Clearly, each κ_k is a positive birational isomorphism of $\mathbb{T}^{l(w)}$.

For a word $\mathbf{i} \in R(w)$, the coordinates of $\mathbb{T}^{l(w)}$ are labeled (colored) by simple roots follow to $\mathbf{i}$,

$$\begin{matrix} t_1 & t_2 & t_3 & \cdots & t_k & \cdots & t_{l(w)-1} & t_{l(w)} \\ \alpha_{i_1} & \alpha_{i_2} & \alpha_{i_3} & \cdots & \alpha_{i_k} & \cdots & \alpha_{i_{l(w)-1}} & \alpha_{i_{l(w)}} \end{matrix}$$

Suppose, for example, that $s_{i_2} = s_{i_k} = s_{i_{l(w)}}$ and $s_{i_j} \neq s_{i_{l(w)}}$ for other j, than $\kappa_{l(w)}(\mathbf{t})$ is the following map

$$\begin{matrix} t_1 & t_2 & t_3 & \cdots \\ \downarrow & \downarrow & \downarrow & \cdots \\ t_1(t_2 + t_k + t_{l(w)})^{-a_{i_1,i_{l(w)}}} & \frac{t_2}{(t_2+t_k+t_{l(w)})(t_k+t_{l(w)})} & t_3(t_k + t_{l(w)})^{-a_{i_3,i_{l(w)}}} & \cdots \end{matrix}$$

$$\begin{matrix} \cdot & t_k & t_{k+1} & \cdots & t_{l(w)-1} & t_{l(w)} \\ \cdot & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \cdot & \frac{t_k}{t_{l(w)}(t_k+t_{l(w)})} & t_{k+1} t_{l(w)}^{-a_{i_{k+1},i_{l(w)}}} & \cdots & t_{l(w)-1} t_{l(w)}^{-a_{i_{l(w)-1},i_{l(w)}}} & t_2 + t_k + t_{i_{l(w)}} \end{matrix}$$

Definition 1. For a Cartan matrix A, a reduced decomposition $\mathbf{i}$ of $w \in W$, the composition of maps

$$\mathbf{K}_{\mathbf{i}}^A := \kappa_1 \circ \cdots \circ \kappa_{l(w)} \tag{16}$$

is a *geometric* $\mathbf{i}$-*RSK*.

(This definition is a slight generalization of that introduced in [4].)

Geometric $\mathbf{i}$-RSK is a positive birational isomorphism of $\mathbb{T}^{l(w)}$ which depends on $\mathbf{i}$.

Example. For SL_3, and the word 121, we get

$$K_{121}(t_1,t_2,t_3) = (\frac{t_1t_2}{t_1+t_3}, t_2t_3, t_1+t_3).$$

In this example, we have $K_{121} = K_{212}$, but this is because $212 = w_0(1)w_0(2)w_0(1)$.

5.2 Inverse geometric RSK

The composition of the following maps provide us with the inverse map for RSK.

Let $w \in W$ and $\mathbf{i} \in R(w)$. Then the map $(\kappa_k^{-1})^{A,\mathbf{i}} : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m$ sends the vector $(p_1,\dots,p_m)$ to the vector defined as follows.

Denote by $I(i_k) = \{j \in [k]\, i_j = i_k\}$, and let $j_1 < j_2 < \dots < j_{|I(i_k)|} = k$ be elements of this set.

Then, for $s \in [j_1 - 1]$,

$$\kappa_k^{-1}(p_s) = p_s p_k^{a_{i_s,i_k}},$$

for $s = j_1$,

$$\kappa_k^{-1}(p_s) = \frac{p_s p_k^2}{p_k p_s + 1},$$

and we redefine $p_k := p_k(1)$ as $p_k(2) := \frac{p_k}{p_k p_s + 1}$;

for $s \in [j_l - 1] \setminus [j_{l-1}]$,

$$\kappa_k^{-1}(p_s) = p_s(p_k(l))^{a_{i_s,i_k}},$$

for $s = j_l$, $l < |I(i_k)|$,

$$\kappa_k^{-1}(p_s) = \frac{p_s p_k(l)^2}{p_s p_k(l) + 1},$$

and we define

$$p_k(l+1) := \frac{p_k(l)}{p_k(l) p_{j_l} + 1};$$

and continue as above for the next interval $[j_{l+1}] \setminus [j_l]$;

then, for $s = j_{|I(i_k)|}$, we set

$$\kappa_k^{-1}(p_k) = p_k(|I(i_k)|),$$

and for $s > k$, $\kappa_k^{-1}(p_s) = p_s$.

We define $\mathbf{K}_{\mathbf{i}}^{-1} : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m$ by the rule

$$\mathbf{K}_{\mathbf{i}}^{-1}(p_1,\dots,p_m) := \kappa_m^{-1} \circ \dots \circ \kappa_2^{-1} \circ \kappa_1^{-1}(p_1,\dots,p_m).$$

Note that $\kappa_1^{-1}(p_1,\dots,p_m) = (p_1,\dots,p_m)$ is the identical map.

For example, for SL_3 and a reduced word $s_1s_2s_1$, we get

$$\kappa_2^{-1}(p_1,p_2,p_3) = (\frac{p_1}{p_2}, p_2, p_3),$$

$$\kappa_3^{-1}(q_1,q_2,q_3) = (\frac{q_1q_3^2}{q_1q_3+1}, q_2(\frac{q_3}{q_1q_3+1})^{-1}, \frac{q_3}{q_1q_3+1}),$$

The composition of these maps is

$$\mathbf{K}_{121}^{-1} : (p_1,p_2,p_3) \to (\frac{p_1p_3^2}{p_1p_3+p_2}, \frac{p_1p_3+p_2}{p_3}, \frac{p_2p_3}{p_1p_3+p_2}).$$

5.3 Geometric Lusztig mutations

Piece-wise linear combinatorics of canonical bases was defined by Lusztig [15, 5, 17] as tropicalization the following birational mappings between tori $(\mathbb{G}_m)^l$ coordinates of which are colored by corresponding transpositions of a reduced decomposition $\mathbf{i} \in R(w)$, $w \in W$, l is the length of w.

Here we give the rule for simply-laced groups: Positive birational mappings between tori for different reduced decompositions $\mathbf{i}$ and $\mathbf{i}'$ are either swapping coordinates for 2-move, if the decompositions are related by the corresponding 2-move, or

$$(\ldots,p,q,r,\ldots) \to (\ldots, \frac{qr}{p+r}, p+r, \frac{pq}{p+r}, \ldots)$$

for corresponding 3-move of the decompositions, and is the identical map on the torus T.

5.4 Commutativity elementary maps κ_l and Lusztig moves

Proposition 4 *For any* $\mathbf{i}$*, the mapping* κ_l *and any geometric Lusztig move are commutative.*

Proof. The statement is clear for 2-moves. It suffices to check the statement for a 3-move and a mapping κ_l with $i_l \in \{i_s, i_{s+1}, i_{s+2}\}$, where the latter set of indexes corresponds to the triple of the 3-move, and $s+2 \leq l$.

For $i_l = i_s$, we have

$$\kappa_l(\cdots,a,b,c,\cdots) = (\frac{a}{(t+c)(t+a+c)}, b(t+c), \frac{c}{t(t+c)}),$$

where we denote by t the 'running value' of t_{i_l} at the $k-(s+3)+1$-step.

Then the composition of the Lusztig 3-move and κ_l is

$$(\cdots, \frac{bc(t+a+c)}{a+c}, \frac{a+c}{t(t+a+c)}, \frac{abt}{a+c}, \cdots),$$

and 'running value' at $k-s+1$ steps is $t+a+c$.

On the other hand side we have, the Lusztig map sends

$$(\ldots,a,b,c,\ldots)\to(\ldots,\frac{bc}{a+c},a+c,\frac{ab}{a+c},\ldots),$$

and

$$\kappa_l(\ldots,\frac{bc}{a+c},a+c,\frac{ab}{a+c},\ldots)=(\cdots,\frac{bc(t+a+c)}{a+c},\frac{a+c}{t(t+a+c)},\frac{abt}{a+c},\cdots),$$

and 'running value' at $k-s+1$ steps is $t+a+c$.

Checking of other possible cases we leave to the reader. □

Remark. Let us note that in diagram (1), we can consider an expanded version by replacing the geometric RSK and its inverse by the elementary maps of which they are composed. On this way we will obtain new family of tori and corresponding tropicalizations of corresponding potentials. Explaining of meaning the corresponding potentials and crystal structures will be in done in another paper.

5.5 Lusztig Variety and the map CA^+

Consider the part of cluster variety, corresponding to seeds labeled by reduced decompositions, $\mathscr{S}(\mathbf{i}),\mathbf{i}\in R(w_0)$.

For a reduced word $\mathbf{i}$, the mutations of the tuples $CA^+_{\mathbf{i}}(\Delta_{\mathbf{i}})$ of the the cluster variables of seeds $\mathscr{S}(\mathbf{i})$ (specialized at the frozen $\Delta_{w_0\omega_i,\omega_i}$, $i\in I$), $\mathbf{i}\in R(w_0)$, at vertices corresponding to 3-moves follow the Lusztig rule ([10]). Recall that, for a reduced decomposition $\mathbf{i}\in R(w_0)$, the Chamber variables are defined as

$$t_k(\mathbf{i})=\frac{\prod_{l;i_k^-<i_l<i_k}\Delta^{-a(i_k,i_l)}_{\mathbf{i}|_{\le i_l}\omega_{i_l},\omega_{i_l}}}{\Delta_{\mathbf{i}|_{\le i_k^-}\omega_{i_k},\omega_{i_k}}\Delta_{\mathbf{i}|_{\le i_k}\omega_{i_k},\omega_{i_k}}},\quad k\in[l(w_0)], \tag{17}$$

plus the frozen variables $\Delta_{w_0\omega_i,\omega_i}$, $i\in I$.

We denoted that map CA^+ in the diagram (1), specifically, for a cluster seed $\mathscr{S}(\mathbf{i})$, this transformation sends cluster variables $\Delta_{\mathbf{i}}$ to $(t_k(\mathbf{i}))_{k=1,\ldots,l(w_0)}=:CA^+(\Delta(\mathbf{i}))$ and leaves unchanged the half of the frozen variables $\Delta_{w_0\omega_i,\omega_i}$, $i\in I$.

Proposition 5 *The tuples* $\{t_k(\mathbf{i}),k\in[l(w_0)]\}$, $\mathbf{i}\in R(w_0)$, *form the Lusztig variety in the sense of Definition 2.2.1 [1].*

Proof. See, for example [10]. □

This Lusztig variety has the following implementation using elementary matrices. The following proposition a cluster version of the Chamber Ansatz of ([1]).

Proposition 6 *For each* $\mathbf{i}\in R(w_0)$, *the matrix*

$$x_{-\mathbf{i}}(K^A_{\mathbf{i}}(\{t_k(\mathbf{i}), k \in [l(w_0)]\})) \tag{18}$$

coincides with $M_{\mathbf{i}}$ under change of cluster variables (17).

5.6 Berenstein-Zelevinsky variety

In [5] it was considered the following positive birational maps for tori $(\mathbb{C}^*)^{l(w_0)}$ labeled by reduced decompositions $\mathbf{i}$ and $\mathbf{i}'$: swapping coordinates for $\mathbf{i}$ and $\mathbf{i}'$ related by a 2-move and the birational positive transformations of the form

$$(\ldots, p, q, r, \ldots) \to (\ldots, \frac{q}{p + \frac{q}{r}}, pr, p + \frac{q}{r}, \ldots)$$

for that related by the corresponding 3-move.

The *BZ-variety* is the collection of tori labeled by elements of $R(w_0)$ and glued together follows 3-moves by the above BZ-map. For an element $\{p^{\mathbf{i}}_k, k \in [l(w_0)]\}$ of the BZ-variety, the product

$$x_{-i_1}(p^{\mathbf{i}}_1) \cdots x_{-i_{l_{w_0}}}(p^{\mathbf{i}}_{l_{w_0}})$$

does not depend on the choice of a reduced word $\mathbf{i} \in R(w_0)$, see [5].

Moreover, the BZ-variety endow $G^{w_0,e}$ with a positive structure. This result essentially appears in [6].

5.7

Theorem 7. *1. For $\mathbf{i} \in R(w_0)$, the tropicalization of Φ_{BK} corresponding to the BZ-torus, labeled by $\mathbf{i}$, defines $gr\mathscr{N}\mathscr{Z}_{\mathbf{i}}$, the graded Nakashima-Zelevinsky cone for $\mathbf{i}$[2];*

2. Tropicalization of the BK-potential Φ_{BK} defined by the torus obtained as the composition of geometric RSK $K^A_{\mathbf{i}}$ and the map (17) is $gr\mathscr{L}_{\mathbf{i}}$, the graded Lusztig cone for $\mathbf{i}$;

3. The geometric RSK $K^A_{\mathbf{i}}$ sends the Lusztig (geometric) crystal actions $f^c_{\alpha_k}$ defined on the variables (17) to the Berenstein-Kazhdan geometric crystal actions ([3]) defined on the BZ-variety.

---

[2] The graded Nakashima-Zelevinsky cone for $\mathbf{i}$ is obtained of the realization of highest weight Kashiwara crystal with the highest weight $\sum_{a\in I} c_a \omega_a$ of the form the integer points of a polytope obtained as the intersection of the string cone $\mathscr{S}(\mathbf{i})$ and the polyhedron defined by inequalities: for each $a \in I$ $\mathbf{r}\gamma \leq c_a$, while γ runs the set of the crossings $\mathbf{R}^l_a(\mathbf{i})$ wrt the left boundary (see [9]) and $\mathbf{r}\gamma$ denotes the corresponding Reineke vector.

From Theorem 7 follows that the composition of the CA^+ and the geometric RSK-correspondence provides birational maps between the cluster positive structure and the BZ-positive structure for the same geometric crystal on $G^{w_0,e}$.

Before proving we give an example.

Example. For SL_3 and a reduced word 121, we have

$$x_{-1}(\frac{t_1t_2}{t_1+t_3})x_{-2}(t_2t_3)x_{-1}(t_1+t_3) = \begin{pmatrix} \frac{1}{t_1t_2} & 0 & 0 \\ \frac{1}{t_3} & \frac{t_1}{t_3} & 0 \\ 1 & t_1+t_3 & t_2t_3 \end{pmatrix}$$

Here $t_1 := \frac{\Delta_{12}}{\Delta_1\Delta_2}$, $t_2 := \frac{\Delta_2}{\Delta_{12}\Delta_{23}}$, $t_3 := \frac{\Delta_{23}}{\Delta_2\Delta_3}$, and hence, we have the following form of the above matrix

$$\begin{pmatrix} \Delta_1\Delta_{23} & 0 & 0 \\ \frac{\Delta_2\Delta_3}{\Delta_{23}} & \frac{\Delta_3\Delta_{12}}{\Delta_1\Delta_{23}} & 0 \\ 1 & \frac{\Delta_{12}\Delta_3+\Delta_{23}\Delta_1}{\Delta_2\Delta_1\Delta_3} & \frac{1}{\Delta_3\Delta_{12}} \end{pmatrix}$$

Note that de-specialization is obtained by multiplication on the left by the diagonal matrix

$$\begin{pmatrix} \frac{1}{\Delta_{23}} & 0 & 0 \\ 0 & \frac{\Delta_{23}}{\Delta_3} & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix}$$

that is

$$\begin{pmatrix} \frac{1}{\Delta_{23}} & 0 & 0 \\ 0 & \frac{\Delta_{23}}{\Delta_3} & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix} \cdot \begin{pmatrix} \Delta_1\Delta_{23} & 0 & 0 \\ \frac{\Delta_2\Delta_3}{\Delta_{23}} & \frac{\Delta_3\Delta_{12}}{\Delta_1\Delta_{23}} & 0 \\ 1 & \frac{\Delta_{12}\Delta_3+\Delta_{23}\Delta_1}{\Delta_2\Delta_1\Delta_3} & \frac{1}{\Delta_3\Delta_{12}} \end{pmatrix} = \begin{pmatrix} \Delta_1 & 0 & 0 \\ \Delta_2 & \frac{\Delta_{12}}{\Delta_1} & 0 \\ \Delta_3 & \frac{\Delta_{12}\Delta_3+\Delta_{23}\Delta_1}{\Delta_2\Delta_1} & \frac{1}{\Delta_{12}} \end{pmatrix}$$

the latter is nothing but the cluster torus M_{121} for the reduced word 121.

Thus, the Berenstein-Kazhdan potential computed in coordinates t_i's and h_i's this torus

$$\begin{pmatrix} h_1\frac{1}{t_1t_2} & 0 & 0 \\ \frac{h_2}{h_1}\frac{1}{t_3} & \frac{h_2}{h_1}\frac{t_1}{t_3} & 0 \\ \frac{1}{h_2} & \frac{1}{h_2}(t_1+t_3) & \frac{1}{h_2}t_2t_3 \end{pmatrix}$$

is

$$\Phi_{BK}(\mathbf{t},\mathbf{h}) := t_1+t_3+t_2+\frac{h_2^2}{h_1}\frac{1}{t_3}+\frac{h_1^2}{h_2}\frac{t_1+t_3}{t_1t_2}.$$

Recall that the potential Φ_{BK} computed at the torus in coordinates p_i's and h_i's

$$\alpha_1(h_1)\alpha_2(h_2)x_{-1}(p_1)x_{-2}(p_2)x_{-1}(p_3) = \begin{pmatrix} h_1\frac{1}{p_1p_3} & 0 & 0 \\ \frac{h_2}{h_1}(\frac{p_1}{p_2}+\frac{1}{p_3}) & \frac{h_2}{h_1}\frac{p_1p_3}{p_2} & 0 \\ \frac{1}{h_2} & \frac{1}{h_2}p_3 & \frac{1}{h_2}p_2 \end{pmatrix}$$

is

$$\Phi_{BK}(\mathbf{p},\mathbf{h}) = p_3 + p_1 + \frac{p_2}{p_3} + \frac{h_2^2}{h_1}\left(\frac{p_1}{p_2} + \frac{1}{p_3}\right) + \frac{h_1^2}{h_2}\frac{1}{p_1}.$$

Formal tropicalization of $\Phi_{BK}(\mathbf{t},\mathbf{h})$ defines the graded Lusztig cone for 121 and that of $\Phi_{BK}(\mathbf{p},\mathbf{h})$ defines the graded Nakashima-Zelevinsky cone for 121.

5.8 Proof of Theorem 7

We consider SL_n. Items 1 and 2 are slight generalization of the Chamber Ansatz of ([5]).

Item 3: because of Proposition 4, we can make proof for the lexmin reduced decomposition $\mathbf{i}_{\min} := 1(21)(321)\ldots(n-1\,n-2\cdots 1)$. We have explicit form of geometric crystal actions. Let us check the statement for $f_{\alpha_{n-1}}$. Namely, we have to show that the composition $K_{\mathbf{i}}(f_{\alpha_{n-1}}(c, CA^+(\Delta_{\mathbf{i}}))$ turns into multiplication by c the coordinate $(n(n-1)/2-n-2)$th coordinate of $K_{\mathbf{i}}(CA^+(\Delta_{\mathbf{i}}))$. The latter coordinate corresponds to s_{n-1} of the lexmin reduced decomposition.

With help of $(n-1)$ 3-moves and $(n-3)(n-3)/2$ 2-moves, we can get from the lexmin decomposition, the following one $\mathbf{i}_{\min}(n-1) := 1(21)(321)\ldots(n-3\,n-4\cdots 1)(n-1\,n-2n-1\,n-3n-2\,n-4n-3\cdots 12)$ of $I(n-1)$. The corresponding sequence of moves the following point of the Lusztig variety corresponding to $\mathbf{t}^{\min}$, a tuple of coordinates for the lexmin reduced decomposition. We denote $\mathbf{t} := \mathbf{t}^{\min}$ for simplicity. Denote by $b_s := t_{l(w_0)-(n-1)-(n-2-s)}$, $s = n-2,\ldots 1$, the coordinates of $\mathbf{t}$ which correspond to the segment $(n-2n-3\cdots 1)$of $\mathbf{i}_{\min}$ and by $a_s := t_{l(w_0)-(n-1-s)}$, , $s = n-1,\ldots 1$, the coordinates of $\mathbf{t}$ which correspond to the segment $(n-1n-2\cdots 1)$ of $\mathbf{i}_{\min}$. Then we have

$$\mathbf{t}^{\mathbf{i}_{min}(n-1)} = \begin{cases} t_k, & k < \frac{n(n-1)}{2} - (2n-3) \\ \dfrac{1}{\frac{1}{a_{n-1}} + \frac{b_{n-2}}{a_{n-2}a_{n-1}} + \frac{b_{n-3}b_{n-2}}{a_{n-3}a_{n-2}a_{n-1}} + \ldots + \frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}} & k = \frac{(n-2)(n-3)}{2} \\ b_{n-2} + \dfrac{1}{\frac{b_{n-3}b_{n-2}}{a_{n-3}a_{n-2}a_{n-1}} + \ldots + \frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}} & k = \frac{(n-2)(n-3)}{2} + 1 \\ b_{n-s} + \dfrac{1}{\frac{b_1\cdots b_{n-s-1}}{a_{n-3}a_{n-2}a_{n-1}} + \ldots + \frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}} & k = \frac{(n-2)(n-3)}{2} + s, s \leq n-2 \\ b_{n-s+1}a_{n-s}\Big(b_{n-s} + \dfrac{1}{\frac{b_1\cdots b_{n-s-1}}{a_{n-3}a_{n-2}a_{n-1}} + \ldots + \frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}}\Big)^{-1} & k = \frac{(n-2)(n-1)}{2} + s, s \leq n-1 \end{cases}$$

By definition, $f_{\alpha_{n-1}}$ changes only one coordinate

$$\frac{1}{\frac{1}{a_{n-1}} + \frac{b_{n-2}}{a_{n-2}a_{n-1}} + \frac{b_{n-3}b_{n-2}}{a_{n-3}a_{n-2}a_{n-1}} + \ldots + \frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}}$$

of $\mathbf{t}^{\mathbf{i}_{\min}(n-1)}$ to

$$\frac{c}{\frac{1}{a_{n-1}}+\frac{b_{n-2}}{a_{n-2}a_{n-1}}+\frac{b_{n-3}b_{n-2}}{a_{n-3}a_{n-2}a_{n-1}}+\ldots+\frac{b_1\cdots b_{n-2}b_{n-1}}{a_1\cdots a_{n-3}a_{n-2}a_{n-1}}}.$$

We have

$$\mathbf{t}_k^{\mathbf{i}_{\min}(n-1)}+\mathbf{t}_{k+n-2}^{\mathbf{i}_{\min}(n-1)}=b_s+a_s,\ k=\frac{n(n-1)}{2}-(2n-3)+s, s\leq n-1, b_{n-1}:=0.$$

Because of this property and since each elementary κ_l has the same conservation law, we get the statement. □

6 Inverse geometric RSK and $*$ dual geometric crystals

One can see that elementary maps of which the reverse geometric RSK is composed are also commute with transformations of the Lusztig variety.

6.1 Lusztig variety and the map CA^-

For each seed $\mathscr{S}(\mathbf{i})$, with $\mathbf{i}\in R(w_0)$, we make the following change of cluster variables wrt the specialization of the frozen variables $\Delta_{\omega_i,\omega_i}$, denoted by $CA^-(\Delta(\mathbf{i}))$, that is inversing the map $grNA_{\mathbf{i}}$ of ([10]), and defined by

$$q_{l(w_0)-l+1}:=\frac{\Delta_{\mathbf{i}|_{\leq l}\omega_{i_l},\omega_{i_l}}}{\Delta_{\mathbf{i}|_{<l}\omega_{i_l},\omega_{i_l}}}, l\in[l(w_0)]. \tag{19}$$

Then for a reduced word $\mathbf{i}(k)$ which is tail optimal for k, we get that the $*$-dual $f_k*(c,\cdot)$ action changes only one variable q_1, sending it to $c\cdot q_1$.

Note, that in coordinates q_l's, the cluster transformations between seeds labeled by reduced decompositions are nothing else but the inverse BZ-moves for 3-braid moves between the corresponding words. Inverse means the following

$$(\ldots,p,q,r,\ldots)\to(\ldots,p+\frac{q}{r},pr,\frac{q}{p+\frac{q}{r}},\ldots).$$

For a reduced word $\mathbf{i}\in R(w_0)$, we have the corresponding variant of the Chamber Ansatz

Proposition 8 *The factorization*

$$y_{\mathbf{i}}(K_{\mathbf{i}}^{-1}(\mathbf{q})) \tag{20}$$

defined the cluster torus factorization $M_{\mathbf{i}}$ written in coordinates (19).

Here is an example.

Example. Consider SL_3 and reduced word 121. Then the reverse RSK sends

$$(q_1,q_2,q_3) \to (\frac{q_1 q_3^2}{q_1q_3+q_2}, \frac{q_1q_3+q_2}{q_3}, \frac{q_2q_3}{q_1q_3+q_2}),$$

and

$$y_1(\frac{q_1 q_3^2}{q_1q_3+q_2})y_2(\frac{q_1q_3+q_2}{q_3})y_1(\frac{q_2q_3}{q_1q_3+q_2}) = \begin{pmatrix} 1 & 0 & 0 \\ q_3 & 1 & 0 \\ q_1q_3 & \frac{q_1q_3+q_2}{q_3} & 1 \end{pmatrix}.$$

Recalling that $q_3 := \frac{\Delta_2}{\Delta_1}$, $q_2 := \frac{\Delta_{23}}{\Delta_{12}}$, $q_1 := \frac{\Delta_3}{\Delta_2}$, we get the latter matrix as

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\Delta_2}{\Delta_1} & 1 & 0 \\ \frac{\Delta_3}{\Delta_1} & \frac{\Delta_3\Delta_{12}+\Delta_1\Delta_{23}}{\Delta_2\Delta_{12}} & 1 \end{pmatrix}.$$

Multiplying the latter matrix on the right by diagonal matrix $\begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \frac{\Delta_{12}}{\Delta_1} & 0 \\ 0 & 0 & \frac{1}{\Delta_{12}} \end{pmatrix}$ we get

$$\begin{pmatrix} \Delta_1 & 0 & 0 \\ \Delta_2 & \frac{\Delta_{12}}{\Delta_1} & 0 \\ \Delta_3 & \frac{\Delta_3\Delta_{12}+\Delta_1\Delta_{23}}{\Delta_1\Delta_2} & \frac{1}{\Delta_{12}} \end{pmatrix},$$

that is M_{121}. □

6.2 Lusztig variety and decoration Ψ_{*K}

For SL_n, we have

$$\Psi_{*K}(y_{\mathbf{i}}(\mathbf{q})\alpha_1(h_1)\cdots\alpha_{n-1}(h_{n-1})) = \sum_{i\in I(w_0)} q_i + \sum_{k\in I} \frac{h_k^2}{h_{k-1}h_{k+1}} \sum_{\gamma\in\mathbf{R}_k^l(\mathbf{i})} \mathbf{q}^{-\mathbf{s}\gamma}, \qquad (21)$$

where $\mathbf{R}_k^l(\mathbf{i})$ denotes the set of crossings wrt the left boundary [9], and $\mathbf{s}\gamma$ is the Reineke statistics.

Thus the tropicalization of this potential defines the $*$-Kashiwara dual Lusztig cone.

In such a case, we have

Theorem 9. *For any* $\mathbf{i} \in R(w_0)$,

$$\Psi_{*K}(y_{\mathbf{i}}(\mathbf{K}_{\mathbf{i}}^{-1}(\mathbf{q}))\alpha_1(h_1)\cdots\alpha_{n-1}(h_{n-1})) =$$

$$\sum_{k\in I}\sum_{\gamma\in\mathbf{R}^l_k(\mathbf{i})}\mathbf{q}^{\mathbf{r}\gamma}+\sum_{k\in I}\frac{h_k^2}{h_{k-1}h_{k+1}}\sum_{s\in I(k)}\mathbf{q}^{x_{i_k}+2\sum_{m\geq k}x_{i_m}-\sum_{(i\pm1)_l\geq i_k}x_{(i\pm1)_l}}$$

Note that the tropicalization of the latter potential defines the Littelmann graded cone $gr\mathscr{S}_{\mathbf{i}}$.

We also have the following theorem

Theorem 10. *1. For* $\mathbf{i}\in R(w_0)$*, tropicalization of the* Ψ_{*K} *corresponding to the torus of the Lusztig variety labeled by* $\mathbf{i}$*, defines* $*$*-Kashiwara dual graded Lusztig cone,* $gr\mathscr{L}^*_{\mathbf{i}}$*.*
2. The inverse geometric RSK $K_{\mathbf{i}}^{-1}$ *sends the dual Kashiwara (geometric) crystal action* $f^c_{\alpha_k}$*, defined on the variables (19), to the geometric* $*$*-dual Lusztig crystal action on the Lusztig variety.*

6.3 Transposed BZ-twist

Berenstein and Zelevinsky ([5], Definition 4.1) defined twist map between reduced double Bruhat cells. We use this map for $G^{w_0,e}$, and in such a case it is

$$\eta_{w_0,e}:N\cap B_-w_0B\to B\cap N_-w_0N_-,\quad \eta_{w_0,e}(x)=[(x^\iota)^{-1}]_+([\overline{w}^{-1}x]_+)^\iota,$$

where $x\to x^\iota$ is the involutive antiautomorphism of G given by

$$\mathbf{h}^\iota=\mathbf{h}^{-1},\mathbf{h}\in H,\, x_i(t)^\iota=x_i(t), y_i(t)^\iota=y_i(t).$$

By transposing the BZ-twist we get a map

$$N_-\cap Bw_0B\to B\cap N_-w_0N_-$$

which is a crystal isomorphism between the $*$-dual geometric crystals for the Lusztig-variety and geometric crystal for the BZ-variety. This result is a reformulation of Theorem 5.10 of ([5]).

Thus all the maps are in place in the diagrams (1) and (2) in order to define the Donaldson-Thomas transformation through commutativity.

Note that all these maps are crystal isomorphism between corresponding cluster geometric crystals.

Thus we have as a corollary

Theorem 11. *The Donaldson-Thomas transformation defined above is an isomorphism of geometric crystals.*

In particular, the tropical DT-transfomation is a crystal isomorphism between the Littelmann graded cone $gr\mathscr{S}_{\mathbf{i}}$ and the Lusztig graded cone $gr\mathscr{L}_{\mathbf{i}}$.

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