Biquasiprimitive oriented graphs of valency four

Nemanja Poznanović and Cheryl E. Praeger

Abstract In this short note we describe a recently initiated research programme aiming to use a normal quotient reduction to analyse finite connected, oriented graphs of valency 4, admitting a vertex- and edge-transitive group of automorphisms which preserves the edge orientation. In the first article on this topic [1], a subfamily of these graphs was identified as 'basic' in the sense that all graphs in this family are normal covers of at least one 'basic' member. These basic members can be further divided into three types: quasiprimitive, biquasiprimitive and cycle type. The first and third of these types were analysed in some detail in the papers [1, 2, 3]. Recently, we have begun an analysis of the basic graphs of biquasiprimitive type. We describe our approach and mention some early results. This work is on-going. It began at the Tutte Memorial MATRIX Workshop.

A graph Γ is said to be *G-oriented* for some subgroup $G \leq \mathrm{Aut}(\Gamma)$, if G acts transitively on the vertices and edges of Γ, and Γ admits a G-invariant orientation of its edges. Any graph Γ admitting a group of automorphisms which acts transitively on its vertices and edges but which does not act transitively on its arcs can be viewed as a G-oriented graph. These graphs are usually said to be *G-half-arc-transitive*, and form a well-studied class of vertex-transitive graphs.

A G-oriented graph Γ necessarily has even valency, with exactly half of the edges incident to a vertex α being oriented away from α to one of its neighbours. Since such a graph is vertex-transitive, it follows that all of its connected components are isomorphic, and each connected component is itself a G-oriented graph. We therefore restrict our attention to the study of connected G-oriented graphs. For each even integer m, we let $\mathcal{OG}(m)$ denote the family of all graph-group pairs (Γ, G) such that Γ is a finite connected G-oriented graph of valency m.

Nemanja Poznanović
University of Melbourne, Australia, e-mail: nempoznanovic@gmail.com

Cheryl E. Praeger
University of Western Australia, Australia e-mail: cheryl.praeger@uwa.edu.au

Graphs contained in $\mathcal{OG}(2)$ are simply oriented cycles. The family $\mathcal{OG}(4)$ on the other hand, has been studied for several decades now, see for instance [4, 5, 6]. These papers suggest a framework for describing the structure of graphs Γ for pairs (Γ, G) in the class $\mathcal{OG}(4)$ by considering various types of quotients based on the structure of certain kinds of cycles of Γ. In this new approach we study quotients based on normal subgroups of the subgroup G.

Normal Quotients. Given a pair $(\Gamma, G) \in \mathcal{OG}(4)$ and a normal subgroup N of G, we define a new graph Γ_N as follows: the vertex set of Γ_N consists of all N-orbits on the vertices of Γ, and there is an edge between two N-orbits $\{B, C\}$ in Γ_N if and only if there is an edge of the form $\{\alpha, \beta\}$ in Γ, with $\alpha \in B$ and $\beta \in C$. The graph Γ_N is called a *G-normal-quotient* of Γ. The group G induces a subgroup G_N of automorphisms of Γ_N, namely $G_N = G/K$ for some normal subgroup K of G such that $N \leq K$. The K-orbits are the same as the N-orbits so $\Gamma_K = \Gamma_N$, although sometimes K may be strictly larger than N.

In general, the pair (Γ_N, G_N) need not lie in $\mathcal{OG}(4)$. For instance, if the normal subgroup N is transitive on the vertex set of Γ, then Γ_N consists of just a single vertex. If the graph Γ is bipartite and the two N-orbits form the bipartition, then Γ_N will be isomorphic to the complete graph on two vertices K_2. In other cases the quotient graph Γ_N may also be a cycle graph C_r, for $r \geq 3$. These three types of quotients are defined to be *degenerate* in the sense that in each of these cases Γ_N does not have valency 4 and so $(\Gamma_N, G_N) \notin \mathcal{OG}(4)$. It turns out that these cases are the only obstacles to the pair (Γ_N, G_N) lying in $\mathcal{OG}(4)$.

Theorem 1 ([1] Theorem 1.1). *Let $(\Gamma, G) \in \mathcal{OG}(4)$ with vertex set X, and let N be a normal subgroup of G. Then G induces a permutation group G_N on the set of N-orbits in X, and either*
(i) (Γ_N, G_N) is also in $\mathcal{OG}(4)$, Γ is a G-normal cover of Γ_N, N is semiregular on vertices, and $G_N = G/N$; or
(ii) (Γ_N, G_N) is a degenerate pair, (i.e. Γ_N is isomorphic to K_1, K_2 or C_r, for some $r \geq 3$).

This leads to a framework for studying the family $\mathcal{OG}(4)$ using normal quotient reduction. The first goal of this approach is to develop a theory to describe the 'basic' pairs in $\mathcal{OG}(4)$. A graph-group pair $(\Gamma, G) \in \mathcal{OG}(4)$ is said to be *basic* if all of its G-normal quotients relative to non-trivial normal subgroups are degenerate. Since Theorem 1 ensures that every member of $\mathcal{OG}(4)$ is a normal cover of a basic pair, the second aim of this framework is to develop a theory to describe the G-normal covers of these basic pairs. This approach has been successfully used in the study of other families of graphs with prescribed symmetry properties, see for instance [7, 8, 9].

The basic pairs may be further divided into three types. A pair $(\Gamma, G) \in \mathcal{OG}(4)$ is said to be basic of *quasiprimitive type* if all G-normal quotients Γ_N of Γ are isomorphic to K_1. This occurs precisely when all non-trivial normal subgroups of G are transitive on the vertices of Γ. (Such a permutation group is said to be quasiprimitive.)

If the only normal quotients of a basic pair $(\Gamma, G) \in \mathcal{OG}(4)$ are the graphs K_1 or K_2, and Γ has at least one G-normal quotient isomorphic to K_2, then (Γ, G) is said to be basic of *biquasiprimitive type*. (The group G here is biquasiprimitive: it is not quasiprimitive but each nontrivial normal subgroup has at most two orbits.) The other basic pairs in $\mathcal{OG}(4)$ must have at least one normal quotient isomorphic to a cycle graph C_r, and these basic pairs are said to be of *cycle type*.

The basic pairs of quasiprimitive type have been analysed in [1], and further analysis was conducted on basic pairs of cycle type in [2] and [3]. Although more remains to be done to describe the structure of basic pairs of cycle type, the main focus of our work is the biquasiprimitive case.

Basic Pairs of Biquasiprimitive Type: Early Results. Our current work aims to develop a theory to describe the basic pairs of biquasiprimitive type. Following the work done in [1] describing quasiprimitive basic pairs, we aim to produce similar structural results and constructions for the biquasiprimitive case. In [10] there is a group theoretic tool available for studying finite biquasiprimitive groups analogous to the O'Nan-Scott Theorem for finite primitive and quasiprimitive permutation groups. We outline our general approach below, though this work is still in progress.

Let Γ be a graph with vertex set X and suppose that $(\Gamma, G) \in \mathcal{OG}(4)$ is basic of biquasiprimitive type for some group G. Then there exists a normal subgroup N of G with exactly two orbits on X, and all normal subgroups of G have at most two orbits. It is easy to see that Γ is bipartite: since Γ is connected there is an edge joining vertices in different N-orbits, and since G normalises N and is edge-transitive, each edge joins vertices in different N-orbits. Thus the two orbits of N form a bipartition of Γ.

Let $\{\Delta, \Delta'\}$ denote the bipartition of the vertices of Γ, and let G^+ be the index 2 subgroup of G fixing Δ (and Δ') setwise. Since Γ is G-vertex-transitive it follows that G^+ is transitive on both Δ and Δ'. As we just saw, any non-trivial intransitive normal subgroup N of G must have the sets Δ, Δ' as its two orbits on X, and hence $N \leq G^+$. It can also be shown that the action of G^+ on Δ is faithful.

Consider now a minimal normal subgroup M of G^+. If M is also normal in G then the M-orbits on X are Δ and Δ' and M is a minimal normal subgroup of G.

On the other hand, if M is not normal in G, then for any element $x \in G \backslash G^+$, we see that M^x is also a minimal normal subgroup of $(G^+)^x = G^+$, and furthermore, $M \neq M^x$ since otherwise G would normalise M. It follows from the minimality of M that $M \cap M^x = 1$ and hence that $M \times M^x$ is contained in G^+ and is normal in G. Letting $N := M \times M^x$, we see that the N-orbits on X are Δ and Δ' since N is normal in G.

In summary, we always have a normal subgroup N of G contained in G^+ with Δ and Δ' the N-orbits in X, and such that either

(a) N is a minimal normal subgroup of G^+ and $N = T^k$ for some simple group T and $k \geq 1$; or
(b) $N = M \times M^x$ where $x \in G \backslash G^+$, and $M = T^\ell$ is a minimal normal subgroup of G^+ with T a simple group and $\ell \geq 1$. In particular, $N \cong T^k$ with $k = 2\ell$.

For a vertex $\alpha \in \Delta$, the vertex stabilisers G_α and G^+_α are equal and $G^+ \cong NG_\alpha$. Moreover, since the vertex stabilisers of 4-valent G-oriented graphs are 2-groups, it follows that $G^+/N \cong G_\alpha/N_\alpha$ is also a 2-group.

Hence by analysing the minimal normal subgroups of G and G^+ as above, and considering the various possibilities for the direct factors T of N, we can reduce the possibilities for basic pairs of biquasiprimitive type to several cases. In fact, our main result so far uses combinatorial arguments to bound the values of ℓ and k in cases (a) and (b) above, though this is still a work in progress.

We give an infinite family of examples of basic biquasiprimitive pairs. These graphs have order $2p^2$, with p prime, and G^+ has an elementary abelian normal subgroup. There were no analogues of these examples in the basic quasiprimitive case since the minimal normal subgroups in that case are nonabelian, [1, Theorem 1.3].

Example 1. Let p be a prime such that $p \equiv 3 \pmod 4$, let $\Delta = \{(x,y)_0 \mid x,y \in C_p\}$ and $\Delta' = \{(x,y)_1 \mid x,y \in C_p\}$, two copies of the additive group $N = C_p^2$, and let $X = \Delta \cup \Delta'$. Also define $\delta \in \mathrm{Sym}(X)$ by $(x,y)_\varepsilon^\delta = (y,-x)_{1-\varepsilon}$, for $x,y \in C_p$ and $\varepsilon \in \{0,1\}$, and let $G := N \rtimes \langle \delta \rangle$. Note that δ has order 4 and normalises N. Also for $\alpha = (0,0)_0$, $G_\alpha = N_\alpha = \langle \delta^2 \rangle \cong C_2$.

Define the G-oriented graph Γ to have vertex set X and, for each $x,y \in C_p$, edges oriented from $(x,y)_0$ to $(x,y\pm 1)_1$ and from $(x\pm 1,y)_1$ to $(x,y)_0$.

Then Γ has valency 4, G is vertex- and edge-transitive, and G preserves the edge orientation. Thus $(\Gamma,G) \in \mathcal{OG}(4)$. Also Γ is bipartite, and G is biquasiprimitive (verifying the latter property uses the fact that $p \equiv 3 \pmod 4$). Hence (Γ,G) is basic of biquasiprimitive type.

Our goal is to refine our restrictions on k and ℓ to such an extent that we can give constructions of families of examples for all possible values of these parameters. Example 1 defines the graphs as BiCayley graphs, and other BiCayley examples arise naturally in our context. However, in the case where G^+ has no normal subgroup which is regular on the two G^+ orbits Δ and Δ', different constructions will be required.

As noted above, these results develop the work initiated in [1] and developed further in [2, 3].

Acknowledgements Both authors are grateful for the opportunity to participate in the Tutte Memorial MATRIX retreat which gave them to chance to commence work on this problem. The authors also thank Georgina Liversidge for some useful discussions at the retreat. The first author acknowledges support of a Research Training Program Scholarship at the University of Melbourne. The second author is grateful for project funding from the Deanship of Scientific Research, King Abdulaziz University (grant no. HiCi/H1433/363-1) which provided the opportunity to focus on this research problem.

References

1. Al-bar, J. A., Al-kenani, A. N., Muthana, N. M., Praeger, C. E., and Spiga, P. (2016). Finite edge-transitive oriented graphs of valency four: a global approach. Electr. J. Combin. 23 (1), #P1.10. arXiv: 1507.02170.
2. Al-Bar, J. A., Al-Kenani, A. N., Muthana, N. M., and Praeger, C. E. (2017). Finite edge-transitive oriented graphs of valency four with cyclic normal quotients. Journal of Algebraic Combinatorics, 46(1), 109–133.
3. Al-bar, J. A., Al-kenani, A. N., Muthana, N. M., and Praeger, C. E. (2017). A normal quotient analysis for some families of oriented four-valent graphs. Ars Mathematica Contemporanea, 12(2), 361–381.
4. Marušič, D. (1998). Half-transitive group actions on finite graphs of valency 4. Journal of Combinatorial Theory, Series B, 73(1), 41-76.
5. Marušič, D., and Praeger, C. E. (1999). Tetravalent graphs admitting half-transitive group actions: alternating cycles. Journal of Combinatorial Theory, Series B, 75(2), 188-205.
6. Marušič, D. and Šparl, P. (2008). On quartic half-arc-transitive metacirculants. J. Algebr. Comb. 28, 365–395.
7. Morris, J., Praeger, C. E., and Spiga, P. (2009). Strongly regular edge-transitive graphs. Ars Mathematica Contemporanea, 2(2).
8. Praeger, C. E. (1993). An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs. Journal of the London Mathematical Society, 2(2), 227-239.
9. Praeger, C. E. (1999). Finite normal edge-transitive Cayley graphs. Bulletin of the Australian Mathematical Society, 60(2), 207-220.
10. Praeger, C. E. (2003). Finite transitive permutation groups and bipartite vertex-transitive graphs. Illinois Journal of Mathematics, 47(1-2), 461-475.