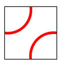
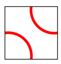
The loop-weight changing operator in the completely packed loop model

Bernard Nienhuis and Kayed al Qasimi

Abstract Loop models are a statistical ensemble of closed paths on a lattice. The most well-known among them has a variety of names such as the dense O(n) loop model, the Temperley-Lieb (TL) model. This note concerns the model in which the weight of the loop $n = 1$, and a local operator which changes the weight of all the loops that surround the position of the operator to some other value. A conjecture of the expectation value of the one-point function of this operator was formulated fifteen years ago. In this note we sketch the proof.

1 Introduction

It has long been recognized that loop models can represent many different local spin models in statistical mechanics. The model we deal with in this note, was introduced [1] as a representation of the Potts model. It naturally has a free parameter, the weight of a loop, which is the square root of the number of states of the Potts model. The case that this weight is unity corresponds to the bond percolation model, and is the case we deal with here. The configurations of the model are a tiling of the square lattice, in which each face of the lattice is covered with one of two tiles

 and 

Thus in every configuration the red arcs in the tiles form paths on the lattice,

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which are either closed (hence loops), or end at the boundary, if there is one. The partition sum of the model is trivial, it is the product over all faces of the sum of the two weights of the faces. Non-trivial are observables, which give weights to the configurations according to some specific properties of the paths. The loop-weight changing operator (LWCO) inserted at a given vertex, gives the loops surrounding that vertex a new weight w possibly different from that of the other loops, i.e. from one. The expectation value of the one-point function of this operator is the generating function of the probabilities of having a specific number of loops surrounding the point of insertion. For brevity we will use the short-hand LWCO primarily for one-point function of this operator, trusting the context will make it unambiguous.

A new approach to the study of this model originates in the work of Razumov and Stroganov [2] who found a connection between the XXZ model and combinatoric problems as Alternating Sign Matrices (ASM) and Plane Partitions (PP) [3]. A connections with loop models followed quickly [4] and led to the famous Razumov-Stroganov conjecture featuring a connection between two types of loop models on different geometries [5]. It was proven by Cantini and Sportiello [6]. These connections led to a wealth of explicit formulae for expectation values of observables and of indicator functions, some proven others conjectured. The value of these is that the formulae are (supposedly) exact with finite distances and geometries, rather than only in the scaling limit. One of these observables is the LWCO that turns the loop weight of surrounding loops into w, on a cylinder of infinite length and circumference L. Mitra and Nienhuis [7] conjectured the value of its one-point function $P(L,w) = F(L,w)/F(L,1)$, with

$$F(L,\, a^2+a^{-2}) = (a+a^{-1})^{-(L \bmod 2)} \qquad \det_{r,s=0}^{L-1} a^{-1}\binom{r+s}{s} + a\,\delta_{r,s} \tag{1}$$

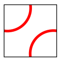
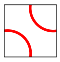
This expression was not based on any theoretical understanding, let alone a derivation or proof. It was completely guessed from the recognition of the coefficients in the polynomial, and subsequently verified for large but finite L. We remark that the symmetry for $a \to 1/a$ is manifest in the LHS, it is not in the RHS of eq. (1).

A discussion with Christian Hagendorf at the Matrix workshop Statistical Mechanics, Combinatorics and Conformal Field Theory in 2017, eventually led to a proof, which we will sketch in this note. The line of argument is as follows. We will first generalize the model to be inhomogeneous, thus introducing a number of variables on which the LWCO depends. We will show that the LWCO is a rational function of these variables. A family of recursion relations in the size of the system can be used to fix the value of the numerator and the denominator, for a number of values of one of the variables. The number of values suffices to completely determine these functions, by the polynomial interpolation formula. A publication of Hagendorf and Morin-Duchesne [8] suggested the inhomogeneous generalization of eq. (1).

It then remained to show that this expression satisfies the same recursion relation as the LWCO.

2 Inhomogeneous TL model

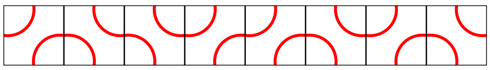
An important step towards a partial proof of the RS conjecture by Di Francesco and Zinn-Justin [9], was making the TL model inhomogeneous, by associating a variable to each column and each row of faces in the lattice. We imagine the axis of the cylinder to be vertical, so the columns run along the length of the cylinder, and the rows form rings around the cylinder. These variables are often called rapidities due to their role in the relativistic field theory which is the scaling limit of the model. The Boltzmann weights at a particular face then depend on the two rapidities associated with the column and row the face is in. Specifically the Boltzmann weight of a face can be written as

$$R(w,z) = \frac{qz - q^{-1}w}{qw - q^{-1}z}$$

$$+ \frac{z-w}{qw - q^{-1}z}$$

(2)

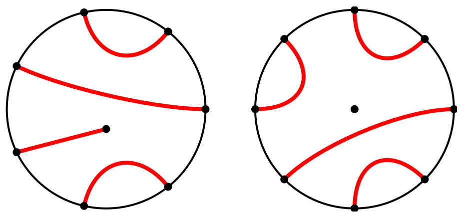
where z and w are the variables associated with the column and row, respectively, that the face belongs to, and $q = \mathrm{e}^{2\pi \mathrm{i}/3}$. The two coefficients add up to one, and are real positive when $z/w = \mathrm{e}^{\mathrm{i}\phi}$ with $\phi \in (0, 2\pi/3)$. The transfer matrix can be written as

$$T(w, \mathbf{z}) \equiv T(w; z_1, z_2, \ldots, z_L) = \prod_{i=1}^{L} R(z_i, w)\ , \tag{3}$$

where we use $\mathbf{z}$ as a shorthand for $\{z_1, z_2, \ldots, z_L\}$. A term in the expansion of this product corresponds to the graph



This transfer matrix acts as a stochastic matrix in the space of so-called link patterns. In each link pattern the edges cut by the rim of the cylinder are connected pairwise, by paths that do not intersect, as in the following example for $L = 7$ and $L = 8$



For odd L the link pattern includes an unpaired edge, which is the connected to a path all along the half-infinite cylinder. The dot at the end of the For even L we mark the disk with a puncture, to distinguish if a path between two edges at the rim is along one side of the cylinder or the other. When two edges that are connected in a link pattern are reconnected by the transfer matrix or another operator, the resulting loop is removed.

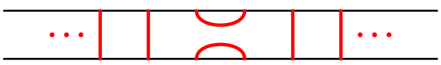
Due to the Yang-Baxter equation $[T(w,\mathbf{z}),T(v,\mathbf{z})]=0$, and consequently the eigenvectors of $T(w,\mathbf{z})$ do not depend on w. With $\Psi(\mathbf{z})$ we denote the ground state, i.e. the eigenvector with eigenvalue 1. In the regime where all transfer matrix elements are non-negative this is the largest (Perron-Frobenius) eigenvalue. Because the transfer matrix is a rational function of the variables z_j, also $\Psi(\mathbf{z})$ is rational, and with suitable normalization polynomial. As shown in [9] $\Psi(\mathbf{z})$ satisfies

$$\check{R}_i(z_i,z_{i+1})\,\Psi(z_1,\ldots,z_i,z_{i+1},\ldots,z_L) \;=\; \Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_L), \tag{4}$$

where the operator

$$\check{R}(w,z)=\frac{qz-q^{-1}w}{qw-q^{-1}z}\,\mathbb{1}\;+\;\frac{z-w}{qw-q^{-1}z}\,e_i \tag{5}$$

and the operator e_i acts on position i and $i{+}1$ of a link pattern as



i.e. connecting the partners of position i and $i{+}1$, and creating an arc connecting i and $i{+}1$ themselves. These equations (4) are called quantum Knizhnik-Zamolodchikov (qKZ) equations as they are analogous to q-deformed versions of the Knizhnik-Zamolodchikov equations [10] on correlation functions in conformal field theory.

We write the ground state vector

$$\Psi(\mathbf{z})=\sum_{\alpha}\psi_\alpha(\mathbf{z})\,\alpha \tag{6}$$

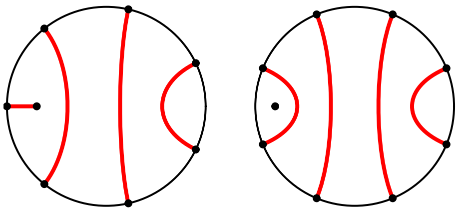
where α is a link pattern, and the sum is over all link patterns of a given size (i.e. circumference of the cylinder). For the weights ψ_α of link patterns α in which the positions i and $i{+}1$ are *not connected* by a small (minimal) arc, eq. (4) leads to

$$(qz_{i+1}-q^{-1}z_i)\,\psi_\alpha(\ldots,z_i,z_{i+1},\ldots)=(qz_i-q^{-1}z_{i+1})\,\psi_\alpha(\ldots,z_{i+1},z_i,\ldots) \tag{7}$$

For the polynomial solution this implies that $\psi_\alpha(\ldots,z_i,z_{i+1},\ldots)$ must contain the factor $(qz_i-q^{-1}z_{i+1})$, and is otherwise symmetric for interchange of z_i and z_{i+1}. If this argument is used recursively on a link pattern not containing a minimal arch in a sequence of edges $\{k,\ldots,n\}$, it must contain the factor

$$\prod_{i=k}^{n-1} \prod_{j=i+1}^{n} (qz_j - q^{-1}z_i)$$

and be otherwise symmetric in the variables $\{z_k, \ldots, z_n\}$. For the most nested link pattern, μ, (again for $L = 7$ and $L = 8$)



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in which the small arc not containing the puncture connects the position 1 and L, contains the factor $\prod_{i=1}^{L-1} \prod_{j=i+1}^{L} (qz_j - q^{-1}z_i)$. Because all other weights can be derived from this one with the qKZ equations (4), from which functions symmetric in z_i and z_{i+1} can be factored out, the weight of the most nested link pattern μ is in fact given by

$$\psi_\mu(\mathbf{z}) = \prod_{i=1}^{L-1} \prod_{j=i+1}^{L} (qz_j - q^{-1}z_i) \tag{8}$$

with no further factors symmetric in $\mathbf{z}$. Clearly $\psi_\mu(\mathbf{z})$ is homogeneous and of joint degree $L(L-1)/2$. As polynomial of a single z_i it is of degree $L-1$. These properties transcend to all ψ_α, as they are conserved by eq. (4).

3 Recursions in system size

Ref. [9] also shows that (4) implies a recursion relation between the ground states of systems different in size by 2. For this it is useful to introduce operators that mediate between link patterns of different sizes. The operator σ_i introduces two additional positions between positions $i-1$ and i, and a small arc that connects them. Conversely, the operator τ_i connects (the partners of) $i-1$ and i, and then removes the positions themselves. Thus, the operators σ acting on a link patterns of size L results in a link pattern of size $L+2$, and τ_i results in link pattern of size $L-2$. These operators satisfy

$$\sigma_i\, \tau_i = e_i \qquad \text{and} \qquad \tau_i\, \sigma_i = \mathbb{1} \tag{9}$$

and acting on the ground state vectors gives

$$\psi_{\sigma_i\alpha}(\ldots, z_{i-1}, zq, zq^2, z_i, \ldots) =$$

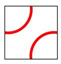
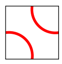
$$\psi_\alpha(\ldots,z_{i-1},z_i,\ldots)\;(q^{-1}-q)\;zprod_{i=1}^{L}-(z-z_i)^2 \tag{10}$$

and

$$\tau_i\,\Psi(\ldots,z_{i-1},zq^2,zq,z_i,\ldots)\;= \\ \Psi(\ldots,z_{i-1},z_i,\ldots)\;(q^{-1}-q)\;z\prod_{i=1}^{L}-(z-z_i)^2 \tag{11}$$

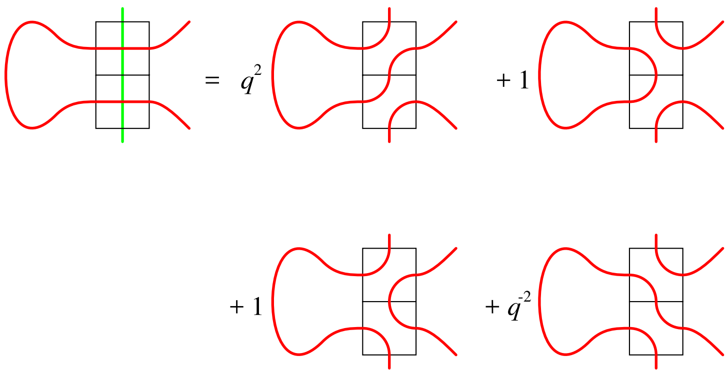
These relations for ground state elements can be read as recursion relations in the system size, as the LHS refers to a system of size $L+2$, and the RHS has size L. We call these relation the *fusion recursion relations*.

Later [11] introduced another recursion in the system size, not by fixing the ratio between two variables, but by sending one to zero. Since we extend their results we treat this in some more detail. When one of the variables z_i is zero, the Boltzmann weights in the corresponding column are from eq. (2)

$R(w,0) = -q$  $-\,q^{-1}$  (12)

We wish to relate a system with size L to a system with size $L+1$, with the same variables, and one additional variable equal to zero. We will refer to this recursion relation as the braid recursion relation, as the $\check{R}$-operator reduces to a so-called braid operator that satisfies the Reidemeister moves of the braid group.

Consider a path in the size-L system, that crosses the location of the rapidity-zero column, wanders around and crosses back on a face adjacent to the other crossing. In the size-(L+1) system, we consider the same configuration, but with all possible configurations in the rapidity-zero column. For the weight of these two cases we get the following equation:



(13)

The green line is the line with variable 0, and the red curves are the paths, while the two faces where the crossing occurs are indicated with a black box.

We see that in the system of size $L+1$, there are only two possible connectivities in four configurations. In one case with weight 1, the path continues as in the size-L system, but avoiding the D-tour, while the paths entering the two boxes vertically are connected along the D-tour that is avoided by the original path. In the other cases, the branches of the path connect up or down to the vertical. The total weight of the last connectivity is $1+q^2+q^{-2}=0$, so it can be disregarded.

If this applies to paths which cross in two adjacent faces, the same must apply to paths which cross the zero rapidity line at more distant faces, as all the crossings in between these two faces, form a nested set of double crossings.

In conclusion one can say that any path that crosses the rapidity-zero column once and back, has the same connectivity between the systems of size L and that of $L+1$.

When L is odd, the system also one path (the defect path) that is not closed, but runs along the infinite cylinder from one end to the other. The system with size $L+1$ then does not contain such path, so when the system is extended with a rapidity zero, the defect path must join up with the additional rapidity zero column. However, this can be done in two ways, as now the puncture can be placed on either side of the path. Figure 1 shows the two possibilities. Since there is no way to exclude one, we accept both, with

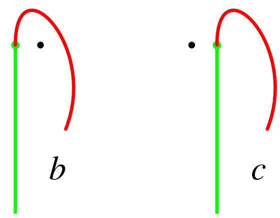


Fig. 1 The puncture relative to the path formed by the unmatched path and the zero rapidity line. The defect path is shown as red, the zero rapidity line is indicated by green, and the puncture is black.

weights b if the puncture is to the right of the juncture, and c if it is to the left as indicated in the figure. The result is that the puncture is "inside" the path with weight b and "outside" with weight c.

The defect line can intersect the zero rapidity column any number of times, even or odd. Therefore an additional crossing should not make any difference in the weights. Figure 2 shows the configurations if the defect path crosses one more time before it joins up with the zero-rapidity line. Now we see that the puncture is inside with weight $-cq^{-1}$ and outside with weight $-bq-cq-bq^{-1}$. This is consistent only when both

$$c\,q^{-1}=-b \qquad \text{and} \qquad b\,q+c\,q+b\,q^{-1}=-c \tag{14}$$

These (overdetermined) equations are solved by

$$b = (-q)^{-1/2} \qquad \text{and} \qquad c = (-q)^{1/2} \tag{15}$$

where we chose $c = b^*$. If the defect path is connected to the zero-rapidity line in the opposite direction, the same solution is found. In conclusion, since an intersection of the defect line with the zero-rapidity line does not affect the weight of a configuration, we may argue without loss of generality as if the defect line never crosses the zero-rapidity line.

4 Recursion relations for the LWCO

In the previous section we showed that the qKZ equations induce recursion relations in the system size for the elements of the ground state vector, and for the probability of certain events. In this section we will present how these recursions lead to recursion relations for the LWCO in the inhomogeneous TL model on an infinite cylinder. First we remind the reader that the elements of the ground state vector, that is the relative configurational weight of a half-infinite cylinder, are polynomials of degree $L-1$ in each of the rapidities. Thus the configuration weights of the infinite cylinder, made up of two half-infinite cylinder has degree $2(L-1)$. Thus the LWCO is a rational function of degree $2(L-1)$ for both the numerator and denominator. We can determine these polynomials completely, from knowing their value for $2L-1$ values of one of the variables. Let z_i to be the variable of choice. We can apply the fusion recursion relation when $z_i = q^{-1}\, z_{i-1}$ or $z_i = q\, z_{i+1}$, and the braid recursion relation when $z_i = 0$. However, the qKZ equations (4) ensure that both the numerator and denominator are symmetric functions of the variables. Therefore we can use the fusion recursion relation for $z_i = q^{-1}\, z_j$ or $z_i = q\, z_j$ for any $j \neq i$. Together with the braid recursion relation this gives precisely enough values.

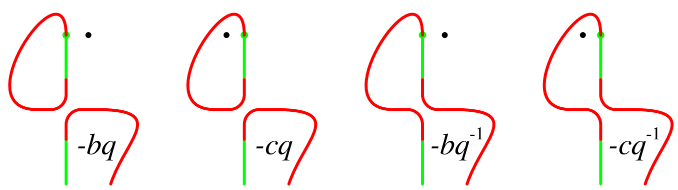


Fig. 2 The configurations formed by placing the puncture on either side of the path and uncrossing the intersection between the path and the rapidity-zero line in the two possible ways. As in Fig. 1 the green line represents the (unresolved) zero-rapidity line, and the red curves represent paths.

To establish some notation, let us use $\Phi(w,\mathbf{z})$ for the (one-point function of the) LWCO, with altered loop weight w, and $\Phi_{\rm n}(w,\mathbf{z})$ and $\Phi_{\rm d}(\mathbf{z})$ for its polynomial numerator and denominator:

$$\Phi(w,\mathbf{z}) = \frac{\Phi_{\rm n}(w,\mathbf{z})}{\Phi_{\rm d}(\mathbf{z})} \tag{16}$$

Because $\Phi_{\rm d}(\mathbf{z}) = \Phi_{\rm n}(1,\mathbf{z})$ it suffices to study $\Phi_{\rm n}$. To denote the recursions, we will use $\mathbf{z}$ for the list $\{z_1, z_2, \ldots, z_L\}$ as before, $(\mathbf{z} \mid z_i \to v)$ to indicate that z_i takes a specific value v, and $(\mathbf{z} \setminus i)$ for the list $\mathbf{z}$ from which z_i is omitted, and similarly $(\mathbf{z} \setminus i,j)$ from which both z_i and z_j are omitted. The system size is implicit as the length of the last argument. We choose $\mathbf{z}$ to have length L always, so that e.g. $(\mathbf{z} \setminus i)$ has length $(L-1)$.

The weight of a specific combination of link patterns in the two half-infinite cylinders, is the product of two ground state elements. But the dependence on the variables is different, as the order of one is reversed relative to the other. The list of variable in reversed order is denoted as $\rho\mathbf{z}$. For example the sum of weights of all link patterns for both halves of the cylinder, should be equal to $\Phi_{\rm d}(\mathbf{z})$, so

$$\Phi_{\rm d}(\mathbf{z}) = \sum_{\alpha,\beta} \psi_\alpha(\mathbf{z})\, \psi_\beta(\rho\mathbf{z}) = \left(\sum_\alpha \psi_\alpha(\mathbf{z})\right)^2 \tag{17}$$

The order of the variables is immaterial because the qKZ equations ensure that the sum of all elements of the ground state vector is a symmetric function.

4.1 The fusion recursion relation

From eq. (10) and (11) it is clear that

$$\Phi_{\rm d}(\mathbf{z} \,|\, z_i \to q z_j) = \Phi_{\rm d}(\mathbf{z} \setminus i,j)\; (-3q)\; z_j^2 \prod_{k\neq i,j}^{L} -(q^{-1} z_j - z_k)^4, \tag{18}$$

and

$$\Phi_{\rm d}(\mathbf{z} \,|\, z_i \to q^{-1} z_j) = \Phi_{\rm d}(\mathbf{z} \setminus i,j)\; (-3q^{-1})\; z_j^2 \prod_{k\neq i,j}^{L} -(q z_j - z_k)^4. \tag{19}$$

In order to see what the analogous recursion is for $\Phi_{\rm n}$, we first use the symmetry of $\Phi_{\rm n}$ for permutation of the $\mathbf{z}$ to place z_i and z_j adjacent. As shown in [9], the corresponding double column simply connects the paths on the left to those on the right in the same row. This implies that the topology of the paths in the system of size L and that of size $L-2$ is the same. As long as the LWCO is not inserted between the two adjacent columns carrying the variables z_i and z_j, the number of loops surrounding the point of insertion is the same in the two models. This implies that Φ_{n} satisfies precisely the same fusion recursion relations (18) and (19) as Φ_{d}, irrespective of the value of the altered loop weight. Clearly, the difference between the two functions Φ_{n} and Φ_{d} must come from the braid recursion relation.

4.2 The braid recursion relation

We consider an inhomogeneous TL model on a cylinder with perimeter $L-1$ and another one on a cylinder with perimeter L, with the same variables supplemented with one variable equal to zero. In this we consider a contractible loop in the size-$(L-1)$ system, that intersects the position of the zero rapidity column in two adjacent faces, and study how this configuration is resolved in the size-L system. This is shown in Figure 3. We observe that if the LWCO is *not* inserted in the original loop, the weight in the size-L system is equal to that in the size-$(L-1)$ system, multiplied with the sum of the four weights, that is $1+1+q^2+q^{-2}=1$. If the LWCO is inserted in the original loop, to the left of the zero-rapidity line, then it sits in a loop in the upper-right

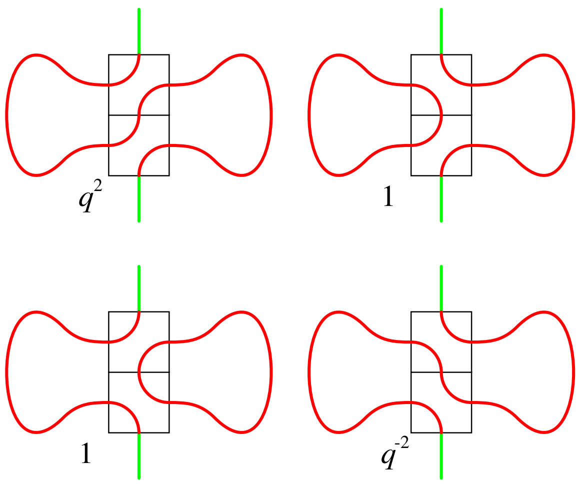


Fig. 3 A closed, contractible loop that cuts the rapidity-zero line in consecutive faces. The path as resolved in the size-L system is drawn red, and the continuation of the zero-rapidity line is shown in green. The total weight of the two faces is given for each configuration.

figure, and not in the remaining configurations, whose weights adds up to $1+q^2+q^{-2}=0$. In the surviving (upper-right) configuration, the two ends of the zero rapidity line are connected by a path. Likewise if the operator is inserted inside the loop, to the right of the zero rapidity line, it sits in a loop in the lower-left figure, and again not in the remaining figures. As far as this type of configuration is concerned, the weight of the LWCO is the same between the system of size L and that of size $(L-1)$.

When a contractible loop intersects the zero-rapidity line in two arbitrary faces, other loops inside it intersect the zero-rapidity line in a nested fashion, in which the loops deepest in the nesting cut the zero-rapidity line in adjacent faces. We can resolve this recursively, starting with the loops deepest in the nest, and then the loops enclosing them, and so on. We conclude that in a system of size L, in which one of the variables is zero, the relative weight of configurations with any given number of loops surrounding an operator insertion is completely the same as in the system of size $(L-1)$ with only the $(L-1)$ non-zero rapidities. This suggests that also the braid recursion relation is the same for $\Phi_{\rm n}$ as for $\Phi_{\rm d}$. However, this is not the case. Since the braid recursion relation relates even and odd sized systems, a defect line may appear or disappear or turn into a loop.

In eq. (1) the determinantal expression is divided by $(a+1/a)$ for odd L. So in the determinant itself, while the loops that surround the operator insertion have weight (a^2+a^{-2}), the defect line has weight $(a+1/a)$. It is convenient to take this operator to replace the original LWCO: in the numerator the surrounding loops have weight (a^2+a^{-2}), and the defect line has co-varying weight $(a+1/a)$. In analogy, in the denominator all loops have weight 1, but the defect line has weight $\sqrt{3}$. So, from now on, we multiply the LWCO in odd-L systems by the simple factor $(a+1/a)/\sqrt{3}$, and rename the resulting one-point function $\overline{\Phi}$ and similarly its numerator and denominator $\overline{\Phi}_{\rm n}$ and $\overline{\Phi}_{\rm d}$. The results for the fusion recursion relations are not altered, as the factors appear on the RHS and LHS of the recursion equally.

First we will consider a system of odd size L, and send one of its rapidities to zero. We have already seen that for a single insertion of the LWCO, all the loops in the corresponding system of size $(L-1)$ that surround it, translate into a surrounding loop in the system of size L, with precisely the same weight. What changes is that in all configurations in the larger system a defect line appears, of which the weight is always $(a+1/a)$. Thus we find for odd L

$$\overline{\Phi}_{\rm n}(a^2+a^{-2},\mathbf{z}\,|\,z_i\to 0)=(a+a^{-1})\,\overline{\Phi}_{\rm n}(a^2+a^{-2},\mathbf{z}\setminus i)F_i(\mathbf{z}), \qquad (20)$$

where the function $F_i(\mathbf{z})$ is a symmetric function of $(\mathbf{z}\setminus i)$, which we do not need, as it does not depend on a.

Second we consider a system of even size L, and send one of its rapidities to zero. Now the larger system has a defect line, and it has weight $(a+1^{-1})$ in all configurations. We have seen above that the zero-rapidity line effectively creates a new path. The defect path in the system of size $(L-1)$ joins up

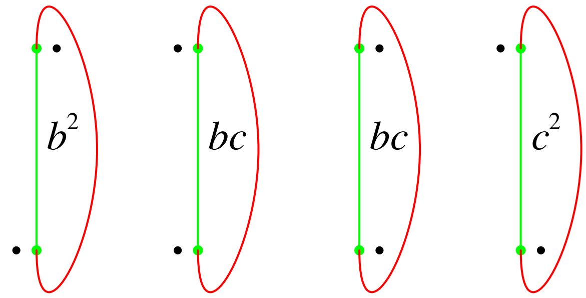


Fig. 4 The possible configurations of the puncture relative to the closed path in the width-(L+1) system made up of the unmatched path in the width-L system and the zero-rapidity line, with repective weights (top) and current factor (bottom).

with this new path to form a loop. While it does this the puncture for either end of the cylinder is placed to the right or left of the juncture with weight b or c, respectively, as illustrated in Figure 4. The defect line and the zero-rapidity line form a loop. If this loop separates the two punctures, it must wind the cylinder. This happens with weight $c^2 + b^2 = -q - q^{-1} = 1$. The loop surrounds the operator insertion if it separates this from both punctures. Irrespective of where the insertion is, it is surrounded with weight 1 and not surrounded with weight 1. Thus, in total the insertion is surrounded with weight 1, and not surrounded with weight 2. Thus we see that the weight $(a+a^{-1})$ for the defect line in the size-$(L-1)$ system is replaced by (a^2+a^{-2}) with weight 1, and by 1 with weight 2. In total the weight $(a+a^{-1})$ in the small system is replaced by $(a^2+a^{-2}+2) = (a+a^{-1})^2$ in the big system. Thus also for even L we find

$$\overline{\Phi}_{\rm n}(a^2+a^{-2}, \mathbf{z}\,|\,z_i \to 0) = (a+a^{-1})\,\overline{\Phi}_{\rm n}(a^2+a^{-2}, \mathbf{z}\setminus i)F_i(\mathbf{z}) \tag{21}$$

just as for odd L. With $\overline{\Phi}_{\rm d}(\mathbf{z}) = \overline{\Phi}_{\rm n}(1,\mathbf{z})$ and $\overline{\Phi}(w,\mathbf{z}) = \overline{\Phi}_{\rm n}(w,\mathbf{z})/\overline{\Phi}_{\rm d}(\mathbf{z})$ this completes the braid recursion relation for the LWCO.

5 The inhomogeneous expression for LWCO

Now that we have found recursion relations that should be satisfied by the LWCO of the inhomogeneous TL model, we have the means to prove an expression if we have it. However, until now only the homogeneous limit was known. In the Matrix workshop on Statistical Mechanics, Combinatorics and Conformal Field Theory in 2017, Christian Hagendorf pointed us to a his publication [8] in which an expression appears equivalent to (1). And it also gives an more general expression of which (1) is the homogeneous limit. Since we had calculated explicit expressions for the inhomogeneous LWCO for small systems ($L < 9$), it was not difficult to verify that indeed his expression is what we need. To make this explicit, we introduce the shorthand $[x] \equiv (x - x^{-1})$, and we propose that

$$\overline{\Phi}_{\mathrm{n}}(a^2 + a^{-2}, \mathbf{z}) = \prod_{1 \le i < j \le L} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{\left[z_i^{1/2} z_j^{-1/2}\right] \left[z_j^{1/2} z_i^{-1/2}\right]} \times$$

$$\prod_{i,j=1}^{L} \left[qz_i^{1/2} z_j^{-1/2}\right] \det_{i,j=1}^{L} \left(\frac{a^{-1}}{\left[qz_i^{1/2} z_j^{-1/2}\right]} + \frac{a}{\left[qz_j^{1/2} z_i^{-1/2}\right]} \right) \tag{22}$$

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