Notes on Tractability Conditions For Linear Multivariate Problems

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Preliminaries

$$\mathcal{S} = \{S_d : \mathcal{H}_d \to \mathcal{G}_d\}_{d\in\mathbb{N}}.$$

Here, $\mathcal{H}_d$, $\mathcal{G}_d$ Hilbert spaces, S_d compact linear

$$S_d(f) \approx A_{d,n}(f) = \phi_{d,n}(L_1(f), L_2(f), \dots, L_n(f))$$

with $\phi_n : \mathbb{C}^n \to \mathcal{G}_d$, $L_j \in \mathcal{H}_d^*$ can be adaptive.

The worst case setting:

$$e(A_{d,n}) = \sup_{\substack{f\in\mathcal{H}_d \\ \|f\|_{\mathcal{H}_d}\le 1}} \|S_d(f) - A_{d,n}(f)\|_{\mathcal{G}_d}.$$

The nth minimal error

$$e(n, S_d) = \inf_{A_{d,n}} e(A_{d,n}).$$

Information Complexity

The Information complexity:

$$n(\varepsilon, S_d) = n_{\mathrm{ABS}}(\varepsilon, S_d) \quad = \quad \min\{n : \ e(n, S_d) \le \varepsilon\},$$
$$n(\varepsilon, S_d) = n_{\mathrm{NOR}}(\varepsilon, S_d) \quad = \quad \min\{n : \ e(n, S_d) \le \varepsilon \, \|S_d\|\}.$$

Known:

$$W_d = S_d^* S_d : \mathcal{H}_d \to \mathcal{H}_d.$$
$$W_d \, \eta_{d,j} = \lambda_{d,j} \, \eta_{d,j}$$
$$\lambda_{d,1} \ge \lambda_{d,2}, \dots$$

Then

$$n_{\mathrm{ABS}}(\varepsilon, S_d) \quad = \quad \min\{n : \ \lambda_{d,n+1} \le \varepsilon^2\}, \qquad (1)$$
$$n_{\mathrm{NOR}}(\varepsilon, S_d) \quad = \quad \min\{n : \ \lambda_{d,n+1} \le \varepsilon^2 \lambda_{d,1}\}. \qquad (2)$$

Tractability

We study how $n(\varepsilon, S_d)$ depends on ε and d. We compare two types of tractability:

- Tractability with respect to $(d, 1+\varepsilon^{-1})$ which is called algebraic tractability and abbreviated by ALG.
- Tractability with respect to $(d, 1+\ln(1+\varepsilon^{-1}))$ which is called exponential tractability and abbreviated by EXP.

ALG : Novak and W. [2008,2010,2012], Werschulz and W. [2017]

EXP : conditions given here

Strong Polynomial Tractability

$\mathcal{S}$ is ALG-SPT-ABS/NOR iff $\exists\ C, p \geq 0$ such that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq C(1+\varepsilon^{-1})^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

$\mathcal{S}$ is EXP-SPT-ABS/NOR iff $\exists\ C, p \geq 0$ such that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq C(1+\ln(1+\varepsilon^{-1}))^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

Table 1: **SPT-ABS**

$\mathcal{S}$ is ALG-SPT-ABS iff

$\exists\ \tau > 0$ and $\widetilde{C} \in \mathbb{N}$: $\sup_{d\in\mathbb{N}} \sum_{j=\widetilde{C}}^{\infty} \lambda_{d,j}^{\tau} < \infty.$

$\mathcal{S}$ is EXP-SPT-ABS iff

$\exists\ \tau > 0$ and $\widetilde{C} \in \mathbb{N}$: $\sup_{d\in\mathbb{N}} \sum_{j=\widetilde{C}}^{\infty} \lambda_{d,j}^{j^{-\tau}} < \infty.$

Table 2: **SPT-NOR**

$\mathcal{S}$ **ALG-SPT-NOR iff** $\exists\ \tau \geq 0$: $\sup_{d\in\mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{\tau} < \infty.$
$\mathcal{S}$ **is EXP-SPT-NOR iff** $\exists\ \tau \geq 0$: $\sup_{d\in\mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau}} < \infty.$

Polynomial Tractability

$\mathcal{S}$ is ALG-PT-ABS/NOR iff $\exists\, C, p, q \ge 0$ such that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\, d^{\,q}(1 + \varepsilon^{-1})^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

$\mathcal{S}$ is EXP-PT-ABS/NOR iff $\exists\, C, p, q \ge 0$ such that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\, d^{\,q}(1 + \ln(1 + \varepsilon^{-1}))^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

Table 3: **PT-ABS**

$\mathcal{S}$ is ALG-PT-ABS iff

$\exists\ \tau_1, \tau_2, \tau_3, \widetilde{C} \ge 0$: $\quad \sup_{d\in\mathbb{N}} d^{-\tau_1} \sum_{j=\lceil \widetilde{C}\, d^{\,\tau_3} \rceil}^{\infty} \lambda_{d,j}^{\tau_2} < \infty.$

$\mathcal{S}$ is EXP-PT-ABS iff

$\exists\ \tau_1, \tau_2, \tau_3, \widetilde{C} \ge 0$: $\quad \sup_{d\in\mathbb{N}} d^{-\tau_1} \sum_{j=\widetilde{C}\, d^{\,\tau_3}}^{\infty} \lambda_{d,j}^{j^{-\tau_2}} < \infty.$

Table 4: **PT-NOR**

$\mathcal{S}$ **ALG-PT-NOR iff** $\exists\ \tau_1, \tau_2 \ge 0: \quad \sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{\tau_2} < \infty.$
$\mathcal{S}$ **is EXP-PT-NOR iff** $\exists\ \tau_1, \tau_2 \ge 0: \quad \sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau}} < \infty.$

(s,t)-Weak Tractability

$\mathcal{S}$ is ALG-(s,t)-WT-ABS/NOR iff

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\ \max\left(1, n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)\right)}{d^{\,t} + (1+\varepsilon^{-1})^s} = 0.$$

$\mathcal{S}$ is EXP-(s,t)-WT-ABS/NOR iff

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\ \max\left(1, n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)\right)}{d^{\,t} + (1+\ln(1+\varepsilon^{-1}))^s} = 0.$$

Table 5: (s,t)-**WT-ABS**

$\mathcal{S}$ is ALG-(s,t)-WT-ABS iff

$$\sup_{d\in\mathbb{N}} \exp(-c\,d^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\,\lambda_{d,j}^{-s/2}\right) < \infty \quad \forall\, c > 0.$$

$\mathcal{S}$ is EXP-(s,t)-WT-ABS iff

$$\sup_{d\in\mathbb{N}} \exp(-c\,d^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left[1 + \ln(2\,\max\left(1, \lambda_{d,j}^{-1})\right)\right]^{s}\right) < \infty \quad \forall\, c > 0.$$

Table 6: (s,t)-WT-NOR

$\mathcal{S}$ ALG-(s,t)-WT-NOR iff

$$\sup_{d\in\mathbb{N}} \exp(-c\,d^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{s/2}\right) < \infty \quad \forall\, c > 0.$$

$\mathcal{S}$ is EXP-SPT-NOR iff

$$\sup_{d\in\mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left[1+\ln\frac{2\lambda_{d,j}}{\lambda_{d,1}}\right]^{s}\right) < \infty \quad \forall\, c > 0.$$

Uniform Weak Tractability

$\mathcal{S}$ is ALG-UWT-ABS/NOR iff
$\mathcal{S}$ is ALG-(s,t)-WT-ABS/NOR for all positive s and t.

$\mathcal{S}$ is EXP-UWT-ABS/NOR iff
$\mathcal{S}$ is EXP-(s,t)-WT-ABS/NOR for all positive s and t.

Siedlecki [2013]

Table 7: **UWT-ABS**

$\mathcal{S}$ is ALG-UWT-ABS iff

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln n]^k} \frac{\ln \frac{1}{\lambda_{d,n}}}{\ln \ln n} = \infty \quad \text{for all } k \in \mathbb{N},$$

$\mathcal{S}$ is EXP-UWT-ABS iff

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln n]^k} \frac{\ln \ln \frac{1}{\lambda_{d,n}}}{\ln \ln n} = \infty \quad \text{for all } k \in \mathbb{N}.$$

Table 8: **UWT-NOR**

$\mathcal{S}$ is ALG-UWT-NOR iff

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln n]^k} \frac{\ln \frac{\lambda_{d,1}}{\lambda_{d,n}}}{\ln \ln n} = \infty \quad \text{for all } k \in \mathbb{N},$$

$\mathcal{S}$ is EXP-UWT-NOR iff

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln n]^k} \frac{\ln \ln \frac{\lambda_{d,1}}{\lambda_{d,n}}}{\ln \ln n} = \infty \quad \text{for all } k \in \mathbb{N}.$$