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Absolute Value Information

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MATRIX
June 11, 2018

(joint work with P. Siedlecki and H. Woźniakowski)

IBC: worst case setting

Solution operator:

$$S : F \to G$$

F - linear space, G - normed space with $\|\cdot\|$

Approximation:

$$S(f) \sim \mathcal{A}_n(f) = \varphi(\mathbf{y})$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is *information* about f,

$$\begin{aligned} y_i &= L_i(f) && \text{(nonadaptive)} \\ y_i &= L_i(f; y_1, \dots, y_{i-1}) && \text{(adaptive)} \end{aligned}$$

$L_i(\cdot; y_1, \dots, y_{i-1}) \in \Lambda$ - a class of linear functionals on F

$$\Lambda \subseteq \Lambda^{\text{all}}$$

Worst case error

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Algorithm error:
(for a set $\mathcal{F} \subset F$ of problem instances)

$$e^{\mathrm{wor}}(\mathcal{A}_n) = \sup_{f \in \mathcal{F}} \|S(f) - \mathcal{A}_n(f)\|$$

nth minimal error:
(for a class Λ of information functionals)

$$r_n^{\mathrm{wor}}(\Lambda) = \inf_{\mathcal{A}_n} e^{\mathrm{wor}}(\mathcal{A}_n)$$

(information) ε-complexity:

$$\mathrm{comp}^{\mathrm{wor}}(\varepsilon) = \min\{n : \, r_n^{\mathrm{wor}}(\Lambda) \leq \varepsilon\}$$

Absolute Value Information (AVI)

(Motivated by the problem of *phase retrival*, I. Daubechies, S. Mallat, ...)

For a given class Λ of information functionals we define

$$|\Lambda| = \left\{ |L| : \; L \in \Lambda \right\}$$

Then AVI about f is given as $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ where

$$y_i = \left| L_i(f; y_1, \ldots, y_{i-1}) \right|, \qquad L_i(\cdot; y_1, \ldots, y_{i-1}) \in \Lambda$$

We want to compare the powers of Λ and $|\Lambda|$ by

$$\mathrm{r}_n^{\mathrm{wor}}(\Lambda) \quad \text{and} \quad \mathrm{r}_n^{\mathrm{wor}}(|\Lambda|)$$

Approximation of Lipschitz functions

$$F = G = C([0,1]), \qquad S(f) = f, \quad f \in F$$

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R} : \ |f(x) - f(y)| \le |x-y| \quad \forall x, y\}$$

$$\Lambda = \Lambda^{\mathrm{std}} = \{f \mapsto L_t(f) = f(t) : 0 \le t \le 1\}$$

Then

$$\mathrm{r}_n^{\mathrm{wor}}(\Lambda) = \frac{1}{2n} \qquad \text{but} \qquad \mathrm{r}_n^{\mathrm{wor}}(|\Lambda|) = +\infty$$

More generally, if

(i) $\mathcal{F}$ is balanced, and

(ii) for all $f \in \mathcal{F}$ is $S(-f) = -S(f)$

then *for any* Λ

$$\mathrm{r}_n^{\mathrm{wor}}(|\Lambda|) \ = \ \mathrm{r}_0^{\mathrm{wor}}(|\Lambda|) \ = \ \sup_{f \in \mathcal{F}} \|S(f)\|$$

Zero finding

Solve

$$f(x) = 0$$

for the function class

$$\mathcal{F} = \{ f \in C([0,1]) : \quad f(0)f(1) < 0 \quad \& \quad \exists!\, \eta_f \ \text{s.t.}\ f(\eta_f) = 0 \}$$

and $S(f) = \eta_f$. Then

$$\mathrm{r}_n^{\mathrm{wor}}(\Lambda^{\mathrm{std}}) = 2^{-n} \qquad \text{and} \qquad \mathrm{r}_n^{\mathrm{wor}}(|\Lambda^{\mathrm{std}}|) = 1/2$$

But, for

$$\Lambda_+^{\mathrm{std}} = \{ f \mapsto f(x) + f(y) : \ 0 \le x, y \le 1 \}$$

we have

$$\mathrm{r}_n^{\mathrm{wor}}(|\Lambda_+^{\mathrm{std}}|) \ \le \ 2^{-\lfloor n/2 \rfloor}$$

Question

What to do with those negative results for $|\Lambda|$?

Modified worst case error

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Define

$$\mathrm{e}^{\mathrm{mod}}(\mathcal{A}_n) = \sup_{f\in\mathcal{F}} \inf_{|z|=1} \|S(f) - z\,\mathcal{A}_n(f)\|$$

and correspondingly

$$\mathrm{r}_n^{\mathrm{mod}}(\Lambda) \quad \text{and} \quad \mathrm{r}_n^{\mathrm{mod}}(|\Lambda|)$$

(Already used for the problem of *phase retrival*, I. Daubechies, S. Mallat, ...)

Approximation of Lipschitz functions

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We still have

$$\mathrm{r}_n^{\mathrm{mod}}(|\Lambda^{\mathrm{std}}|) = 1$$

But, if

$$\Lambda_+^{\mathrm{std}} = \{ L_1 + L_2 : L_1, L_2 \in \Lambda^{\mathrm{std}} \}$$

then

$$\mathrm{r}_{2n}^{\mathrm{mod}}(|\Lambda_+^{\mathrm{std}}|) = \mathrm{r}_n^{\mathrm{wor}}(\Lambda^{\mathrm{std}}) = \frac{1}{2n}$$

AVI for general IBC problems: upper bound

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Theorem

Suppose that

- *(i)* *S is homogenous,*
- *(ii)* *there exists an algorithm $\mathcal{A}_n$ that uses nonadaptively chosen information functionals from Λ, such that*

$$\mathrm{e}^{\mathrm{wor}}(\mathcal{A}_n) \le C\, \mathrm{r}_n^{\mathrm{wor}}(\Lambda)$$

Then for

$$\Lambda_{\pm} = \{ L_1 \pm L_2 : \ L_1, L_2 \in \Lambda \}$$

we have

$$\mathrm{r}_n^{\mathrm{mod}}(|\Lambda_{\pm}|) \le C\, \mathrm{r}_{\lfloor n/2 \rfloor}^{\mathrm{wor}}(\Lambda)$$

AVI for general IBC problems: lower bound

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Theorem

Let S be linear and $\mathcal{F} \subset F$ be convex and balanced. Then

$$\mathrm{r}_n^{\mathrm{mod}}(|\Lambda|) \;\geq\; \frac{1}{4}\,\mathrm{r}_n^{\mathrm{wor}}(\Lambda)$$

AVI for linear problems

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Corollary

If S is linear, $\mathcal{F}$ is convex and balanced, then

$$\frac{1}{4}\,\mathrm{r}_n^{\mathrm{wor}}(\Lambda^{\mathrm{all}}) \;\le\; \mathrm{r}_n^{\mathrm{mod}}(|\Lambda^{\mathrm{all}}|) \;\le\; 2\,\mathrm{r}_{\lfloor n/2\rfloor}^{\mathrm{wor}}(\Lambda^{\mathrm{all}})$$

Conclusions for modified error

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- Λ^{std} and $|\Lambda^{\mathrm{std}}|$ are generally not related
- $\Lambda^{\mathrm{all}} \sim |\Lambda^{\mathrm{all}}|$ for linear problems