Randomized lattice rules

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On the Frontiers of High Dimensional Computation
MATRIX, Creswick
June 2018

Numerical integration

Approximate d-dimensional integral

$$I(f) := \int_{[0,1]^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

by an n-point cubature rule

$$Q(f; \{(\boldsymbol{x}_k, w_k)\}_{k=1}^n) := \sum_{k=1}^{n} w_k \, f(\boldsymbol{x}_k).$$

Interested in error

$$|I(f) - Q(f; \{(\boldsymbol{x}_k, w_k)\}_{k=1}^n)| \leq ?$$

for f in a certain function class.

$\Rightarrow$ First deterministic, then randomized.

Lattice rule

Given $n \in \mathbb{N}$ and a *generating vector* $\boldsymbol{z} \in \mathbb{Z}_n^d$,

$$\boldsymbol{x}_k = \frac{\boldsymbol{z}\, k \bmod n}{n}, \qquad w_k = \frac{1}{n}, \qquad k \in \mathbb{Z}_n,$$

then for $f \in A(\mathbb{T}^d)$, with $\mathbb{T}^d \simeq [0,1]^d$, and

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^d} \hat{f}(\boldsymbol{h})\, \mathrm{e}^{2\pi \mathrm{i}\, \boldsymbol{h} \cdot \boldsymbol{x}}, \qquad \hat{f}(\boldsymbol{h}) := \int_{[0,1]^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i}\, \boldsymbol{h} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}.$$

we have

$$\int_{[0,1]^d} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \frac{1}{n} \sum_{k \in \mathbb{Z}_n} f(\boldsymbol{x}_k) = I(f) - Q(f; \boldsymbol{z}, n) = \sum_{\substack{\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \pmod{n}}} \hat{f}(\boldsymbol{h}).$$

Hölder inequality

For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q \le \infty$, and $r_{\alpha,\gamma}(\boldsymbol{h}) > 0$

$$|I(f) - Q(f; \boldsymbol{z}, n)| = \left| \sum_{\substack{\boldsymbol{0} \ne \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \pmod{n}}} \hat{f}(\boldsymbol{h}) \, r_{\alpha,\gamma}(\boldsymbol{h}) \, r_{\alpha,\gamma}^{-1}(\boldsymbol{h}) \right|$$

$$\le \left(\sum_{\boldsymbol{h} \in \mathbb{Z}^d} |\hat{f}(\boldsymbol{h})|^p \, r_{\alpha,\gamma}^p(\boldsymbol{h}) \right)^{1/p} \left(\sum_{\substack{\boldsymbol{0} \ne \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \pmod{n}}} r_{\alpha,\gamma}^{-q}(\boldsymbol{h}) \right)^{1/q}$$

$$= \quad \|f\|_{p,\alpha} \quad e_{q,\alpha}(\boldsymbol{z}, n).$$

This is "norm of f" (in some function space determined by $r_{\alpha,\gamma}$ and p) times "worst-case error" of the lattice rule in this function space.

Korobov space of dominating mixed smoothness

Using "product weights" $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots > 0$ (SloanW 1998) and classical Korobov smoothness

$$r_{\alpha,\gamma}(\boldsymbol{h}) = \prod_{\substack{j=1 \\ h_j \neq 0}}^{d} \frac{|h_j|^\alpha}{\gamma_j} = \prod_{j=1}^{d} \max(1, \frac{|h_j|^\alpha}{\gamma_j}).$$

For $p = 2$ and $\alpha \in \mathbb{N}$ using Parseval:

$$\|f\|_{2,\alpha}^2 = \sum_{\boldsymbol{h} \in \mathbb{Z}^d} |\hat{f}(\boldsymbol{h})|^2 r_{\alpha,\gamma}^2(\boldsymbol{h})$$

$$= \sum_{\substack{\boldsymbol{\tau} \in \{0,\alpha\}^d \\ \mathfrak{u} := \{j : \tau_j \neq 0\}}} \prod_{j \in \mathfrak{u}} \frac{\gamma_j^{-2}}{(2\pi)^{2\tau_j}} \left\| \int_{[0,1]^{d-|\mathfrak{u}|}} D^{\boldsymbol{\tau}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_{-\mathfrak{u}} \right\|_{L_2}^2 .$$

Worst-case error / figures of merit

- Take $p = \infty$: classical P_α criterion (SloanJ 1994, N 1992):

$$P_\alpha(\boldsymbol{z}, n) = e_{1,\alpha}(\boldsymbol{z}, n) = \sum_{\substack{\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z} \equiv 0 \pmod{n}}} r_{\alpha,\gamma}^{-1}(\boldsymbol{h}), \qquad \alpha > 1.$$

- Take $p = 1$: Zaremba index:

$$\rho_\alpha(\boldsymbol{z}, n) = \frac{1}{e_{\infty,\alpha}} = \min_{\substack{\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z} \equiv 0 \pmod{n}}} r_{\alpha,\gamma}(\boldsymbol{h}), \qquad \alpha > 0.$$

- Take $p = 2$: Hilbert case worst-case error (DKSloan 2013):

$$e_{2,\alpha}^2(\boldsymbol{z}, n) = \sum_{\substack{\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z} \equiv 0 \pmod{n}}} r_{\alpha,\gamma}^{-2}(\boldsymbol{h}), \qquad \alpha > 1/2.$$

$\| \cdot \|_{\ell_\infty} \leq \| \cdot \|_{\ell_2} \leq \| \cdot \|_{\ell_1}$ thus $\rho_\alpha^{-1}(\boldsymbol{z}, n) \leq e_{2,\alpha}(\boldsymbol{z}, n) \leq P_\alpha(\boldsymbol{z}, n)$

... image of lattice rule point set ...

Known results

Lemma

The average over all choices for prime n and $\alpha > 1$ achieves the bound

$$\frac{1}{(n-1)^d} \sum_{\boldsymbol{z} \in \{1,\ldots,n-1\}^d} P_\alpha(\boldsymbol{z}, n) \leq \frac{3}{n} \prod_{j=1}^{d} (1 + 2\gamma_j \zeta(\alpha))$$
$$\leq \frac{3}{n} \exp(2\zeta(\alpha) \sum_{j \geq 1} \gamma_j).$$

$\Rightarrow$ If $\sum_{j\geq1} \gamma_j < \infty$ then this bound is independent of d.

$\Rightarrow$ Can find lattice rule component-by-component (SloanR 2002).

Known results II

Lemma

For n prime and $\alpha > 0$ there are at least $(n-1)^d/2$ choices for $\boldsymbol{z} \in \{1, \ldots, n-1\}^d$ such that

$$\rho_\alpha(\boldsymbol{z}, n) \geq \left(\frac{n/2}{3 \prod_{j=1}^{d} (1 + 2\gamma_j^{1/\lambda} \zeta(\alpha/\lambda))} \right)^\lambda$$

for all $\lambda \in (0, \alpha)$.

Possible improvement for randomized worst-case error

For a deterministic algorithm Q the worst-case error is

$$e(Q,\mathcal{F}) := \sup_{\substack{f\in\mathcal{F}\\ \|f\|\le 1}} |I(f) - Q(f)| \le \frac{C_{d,\alpha}}{n^\alpha}.$$

For a randomized algorithm Q^ω the randomized error is

$$e^{\mathrm{ran}}(Q,\mathcal{F}) := \sup_{\substack{f\in\mathcal{F}\\ \|f\|\le 1}} \mathbb{E}\big[|I(f) - Q^\omega(f)|\big] \le \frac{C_{d,\alpha}}{n^{\alpha+1/2}}.$$

See Bakhvalov...

Typical randomization: randomly shifted lattice rule:

$$\boldsymbol{x}_k = \left(\frac{\boldsymbol{z}k}{n} + \boldsymbol{\Delta}\right) \bmod 1, \qquad k \in \mathbb{Z}_n,$$

does not give this improvement.

Our algorithm

Define the set of "good" generating vectors for a given prime n

$$\mathcal{Z}_n = \mathcal{Z}_{n,\alpha,\gamma} = \Bigg\{ \boldsymbol{z} \in \{1, \ldots, n-1\}^d : \rho_\alpha(\boldsymbol{z}, n) \geq \left(\frac{n/2}{3 \prod_{j=1}^d (1 + 2\gamma_j^{1/\lambda} \zeta(\alpha/\lambda))} \right)^\lambda \quad \text{for all } \lambda \in (0, \alpha) \Bigg\}.$$

We have $|\mathcal{Z}_n| > \lceil (n-1)^d / 2 \rceil$.

Lemma (The randomized algorithm M_m)

1. *Uniformly choose a prime n between $\lceil m/2 \rceil + 1$ and m.*
2. *Uniformly choose a $\boldsymbol{z}$ from $\mathcal{Z}_n$.*
3. *Use this lattice rule.*

… one-dimensional intuition …

Upper bound

Theorem (KKNU201*)

For $\alpha > 1/2$, $\lambda \in (1/2, \alpha)$, $\delta \in (0, \lambda - 1/2)$, *and*

$$m \geq 12 \prod_{j=1}^{d} (1 + 2\gamma_j^{1/\lambda} \zeta(\alpha/\lambda))$$

then for a randomly chosen prime n between $\lceil m/2 \rceil + 1$ *and m and then a randomly chosen* $\boldsymbol{z} \in \mathcal{Z}_n$ *we have*

$$e^{ran}(M_m) \leq \frac{C_{\lambda,\delta}}{m^{\lambda+1/2-\delta}} \left(3 \prod_{j=1}^{d} (1 + 2\gamma_j^{1/\lambda} \zeta(\alpha/\lambda)) \right)^{\lambda},$$

which is independent of d if $\sum_{j \geq 1} \gamma_j^{1/\lambda} < \infty$.

Lower bound

Theorem (KKNU201*)

For $\alpha > 0$, then for a randomly chosen prime n between $\lceil m/2 \rceil + 1$ and m and then a randomly chosen $\boldsymbol{z} \in \mathcal{Z}_n$ we have

$$e^{ran}(M_m) \geq \frac{\gamma_1}{2} \frac{\sqrt{\log n}}{n^{\alpha+1/2}}.$$

Conclusions

- Roughly: Algorithm M_m: randomly select number of points in $m/2$ and m and then randomly select a good generating vector delivers $m^{-\alpha-1/2}$, independent of d if $\sum_{j\geq 1} \gamma_j^{1/\alpha} < \infty$.
- Holds for $\alpha > 0$ if also doing random shift.
- Original Bakhvalov (1961) constructive proof for randomized error in classical Sobolev space was based on embedding in Korobov space and then using a Korobov type lattice rule: select uniformly over all good lattice rules for all prime n in the range $m/2$ and m at the same time.
- But there the bound strongly depends on d.

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