A reduced basis method for stochastic wave propagation

M. Ganesh

Colorado School of Mines
Golden
CO 80401

Stuart C. Hawkins

MACQUARIE
University

http://www.maths.mq.edu.au/∼shawkins/research.html

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  - Stochastic wave propagation models
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  - T-matrix
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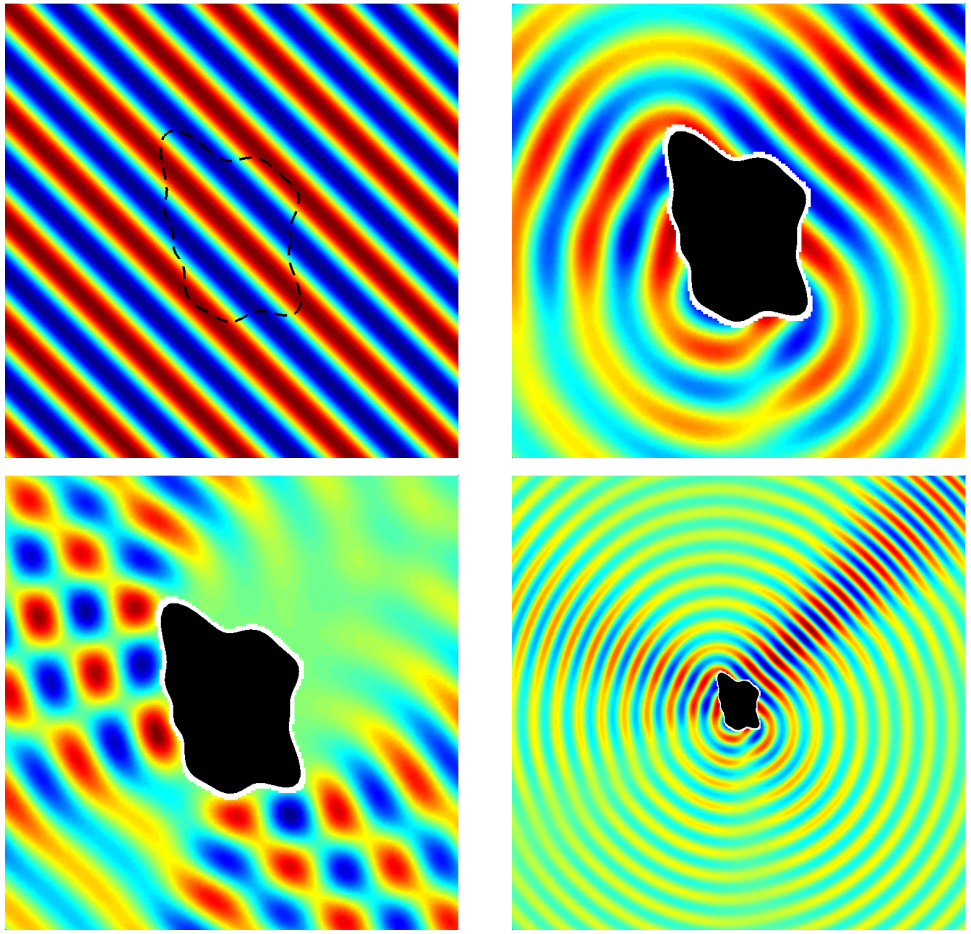
Motivation



- Scattered field

$$u(\boldsymbol{x};\boldsymbol{\sigma}) \approx \frac{e^{ik|\boldsymbol{x}|}}{\sqrt{|\boldsymbol{x}|}}\, u^{\infty}(\hat{\boldsymbol{x}};\boldsymbol{\sigma})$$

- Observation direction $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$
- Parameters describing medium $\boldsymbol{\sigma}$

Motivation

- Inverse problem. . . find $\boldsymbol{\sigma}$ satisfying

$$u^{\infty}(\hat{\boldsymbol{x}};\boldsymbol{\sigma}) = \hat{\boldsymbol{y}}$$

  - Noisy data $\hat{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{\eta}$
  - Noise $\boldsymbol{\eta}$

- Bayesian inversion

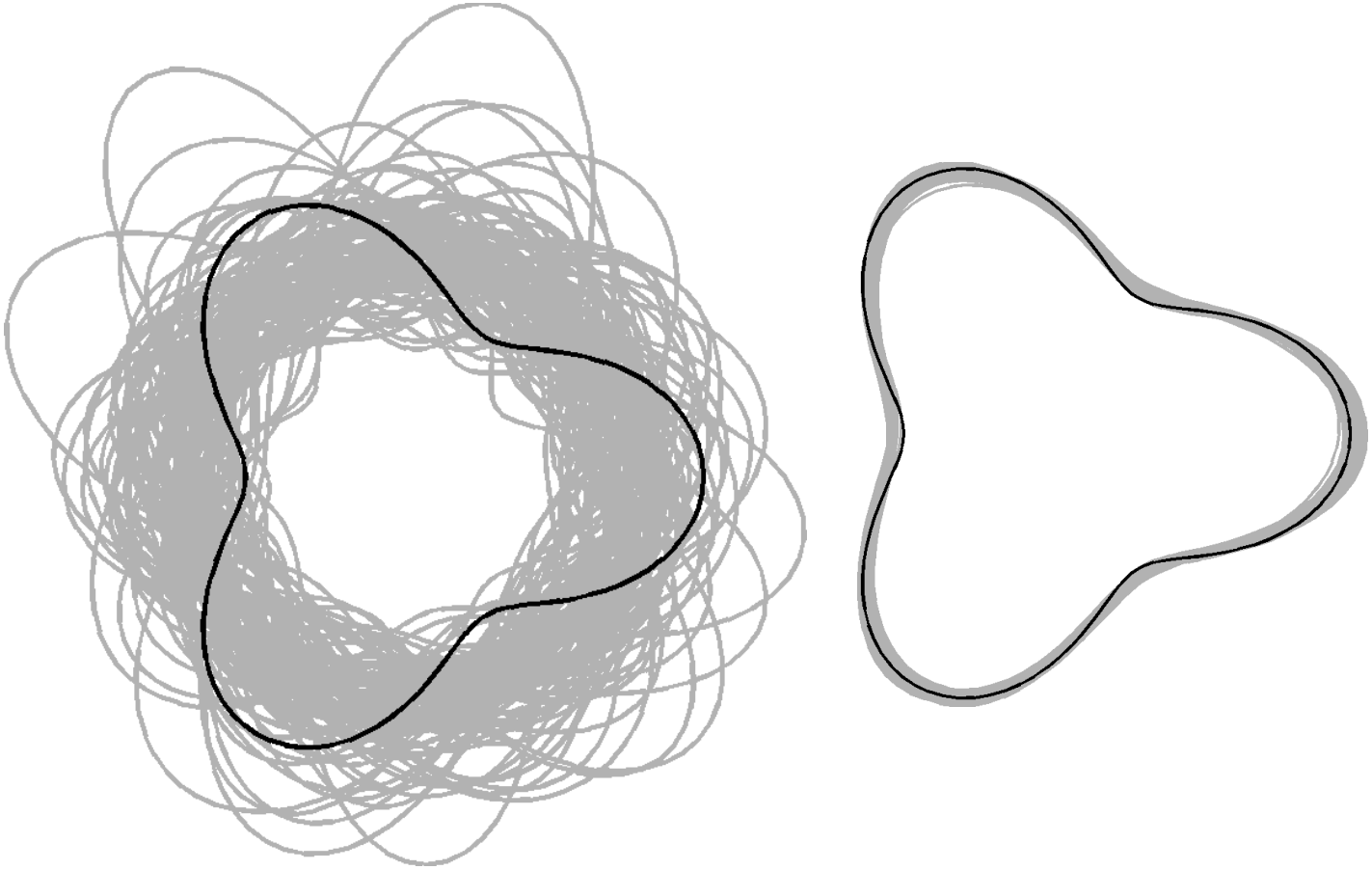
$$\rho_{\hat{\boldsymbol{y}}}(\boldsymbol{\sigma}) \propto \rho(\hat{\boldsymbol{y}} - u^{\infty}(\hat{\boldsymbol{x}};\boldsymbol{\sigma}))\, \rho_0(\boldsymbol{\sigma})$$

  - Assumed distribution for the noise ρ
  - Prior distribution ρ_0
  - Posterior distribution $\rho_{\hat{\boldsymbol{y}}}$

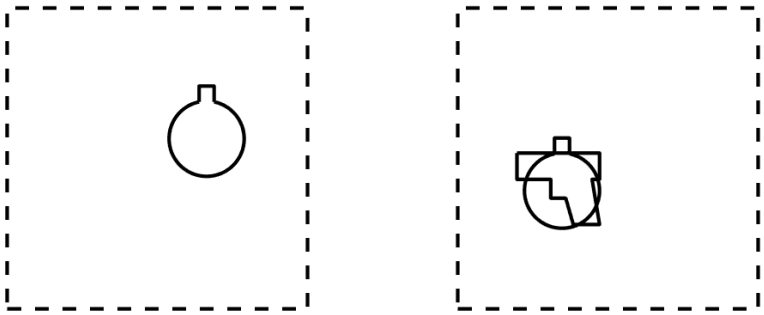
Motivation

prior

posterior

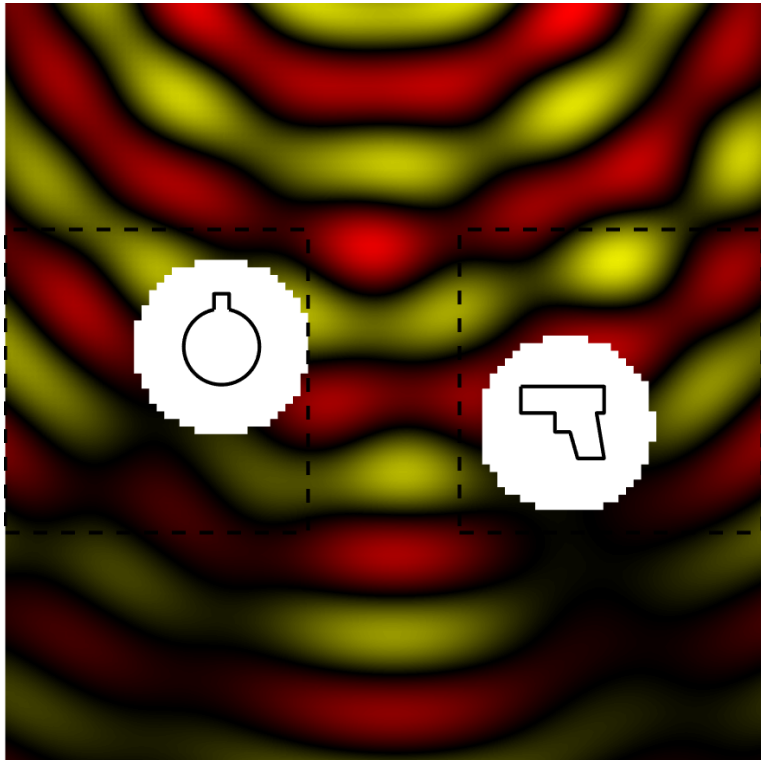


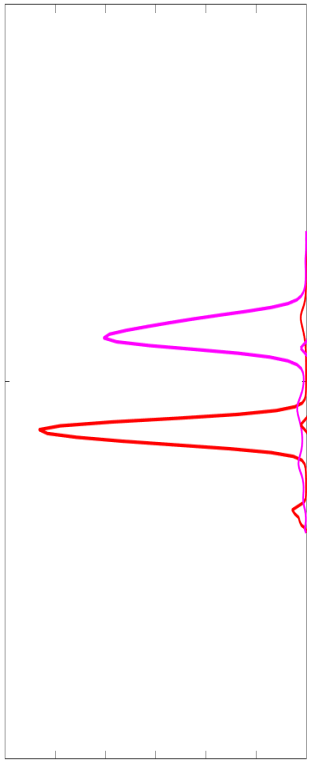
Motivation

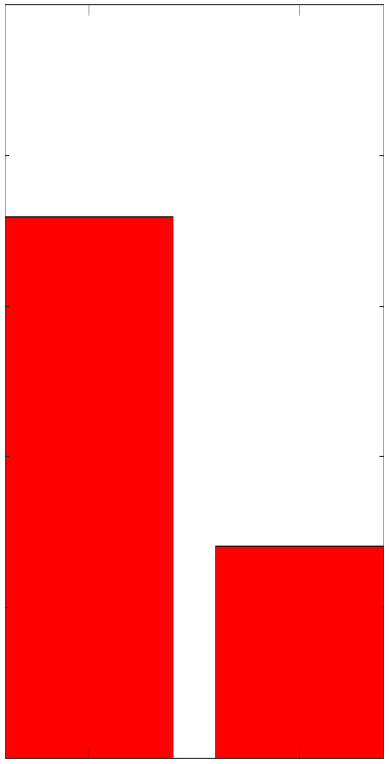


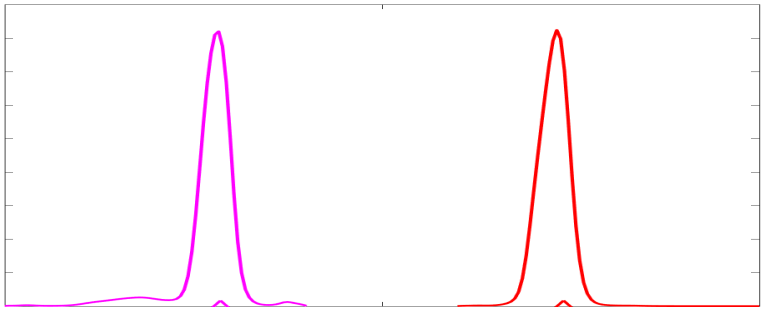
- Far field $u^{\infty}(\hat{\boldsymbol{x}}; x_1, y_1, x_2, y_2, \sigma)$
  - First scatterer position (x_1, y_1)
  - Second scatterer position (x_2, y_2)
  - Second scatterer type σ

Motivation









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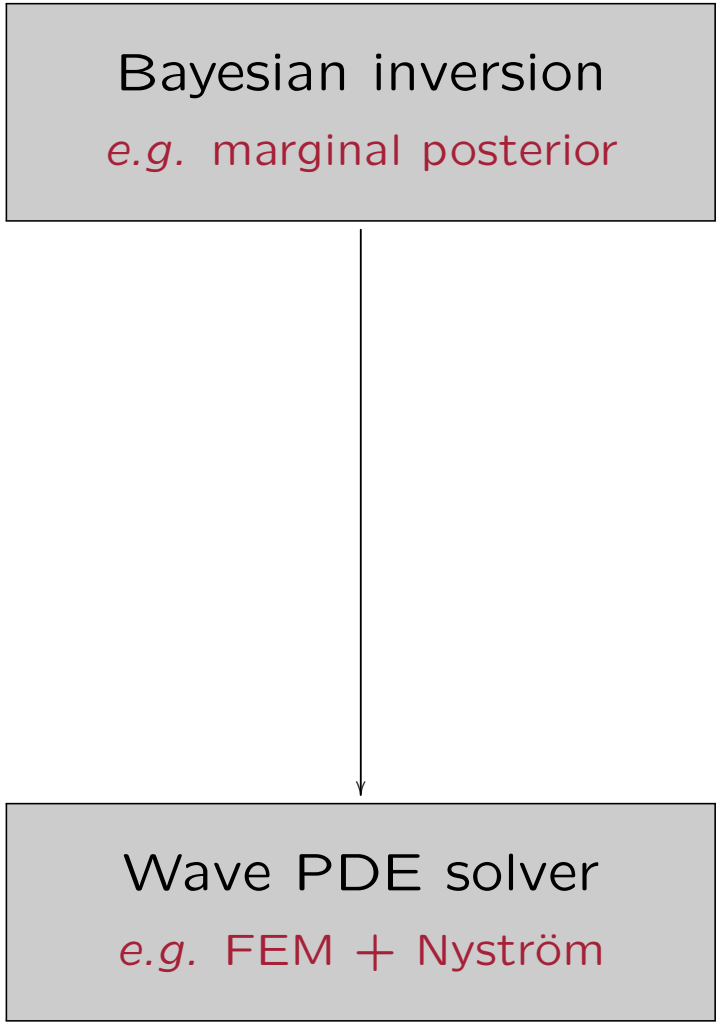
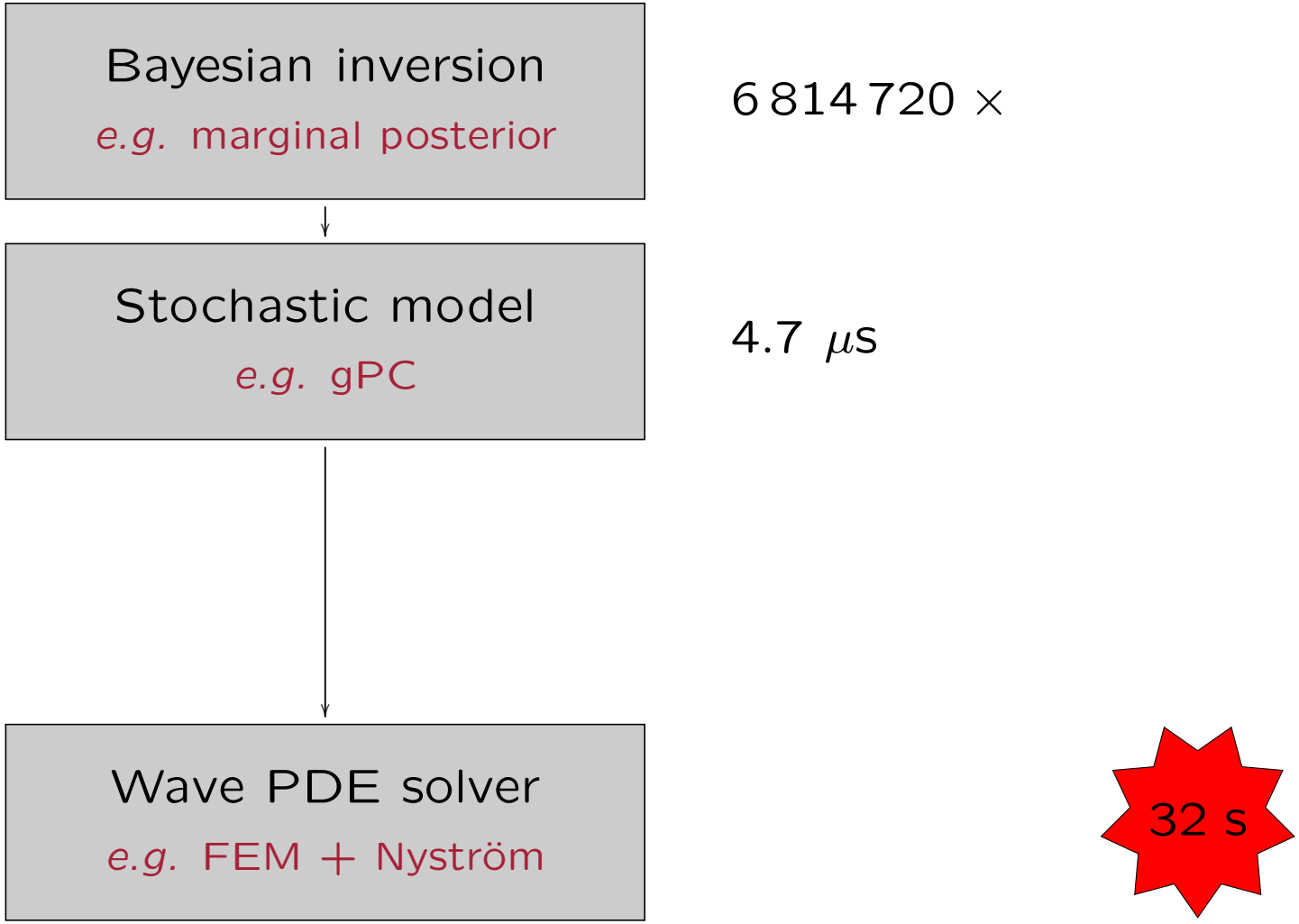


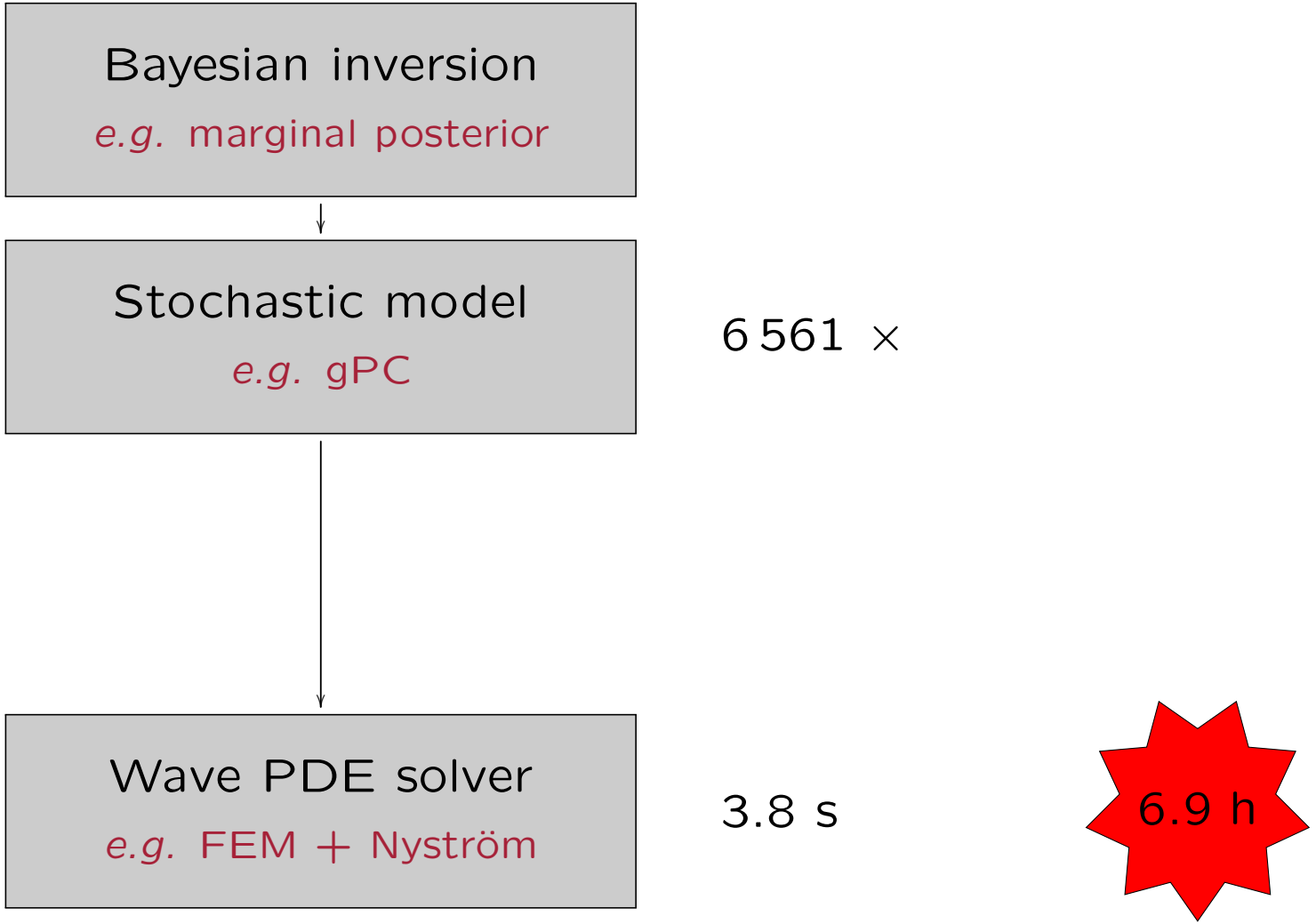
6 814 720 ×

3.8 s

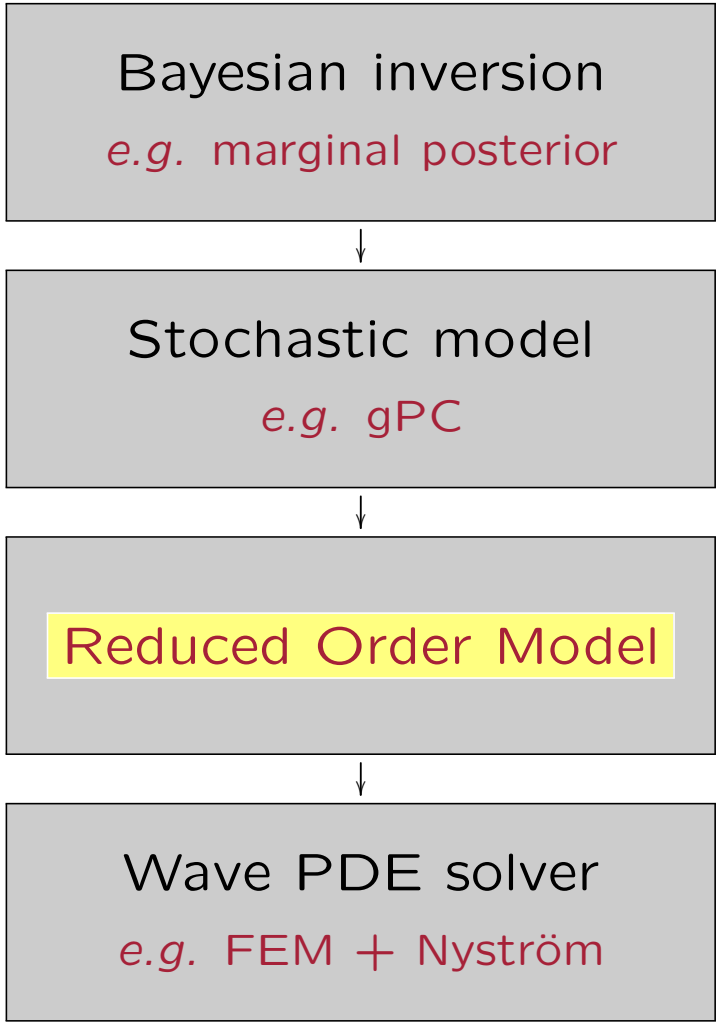
Motivation


Bayesian inversion
e.g. marginal posterior
Stochastic model
e.g. gPC
Wave PDE solver
e.g. FEM + Nyström
6 814 720 ×
4.7 μs
32 s

Motivation


Bayesian inversion
e.g. marginal posterior
Stochastic model
e.g. gPC
6 561 ×
Wave PDE solver
e.g. FEM + Nyström
3.8 s
6.9 h

Motivation



6 561 $\times$

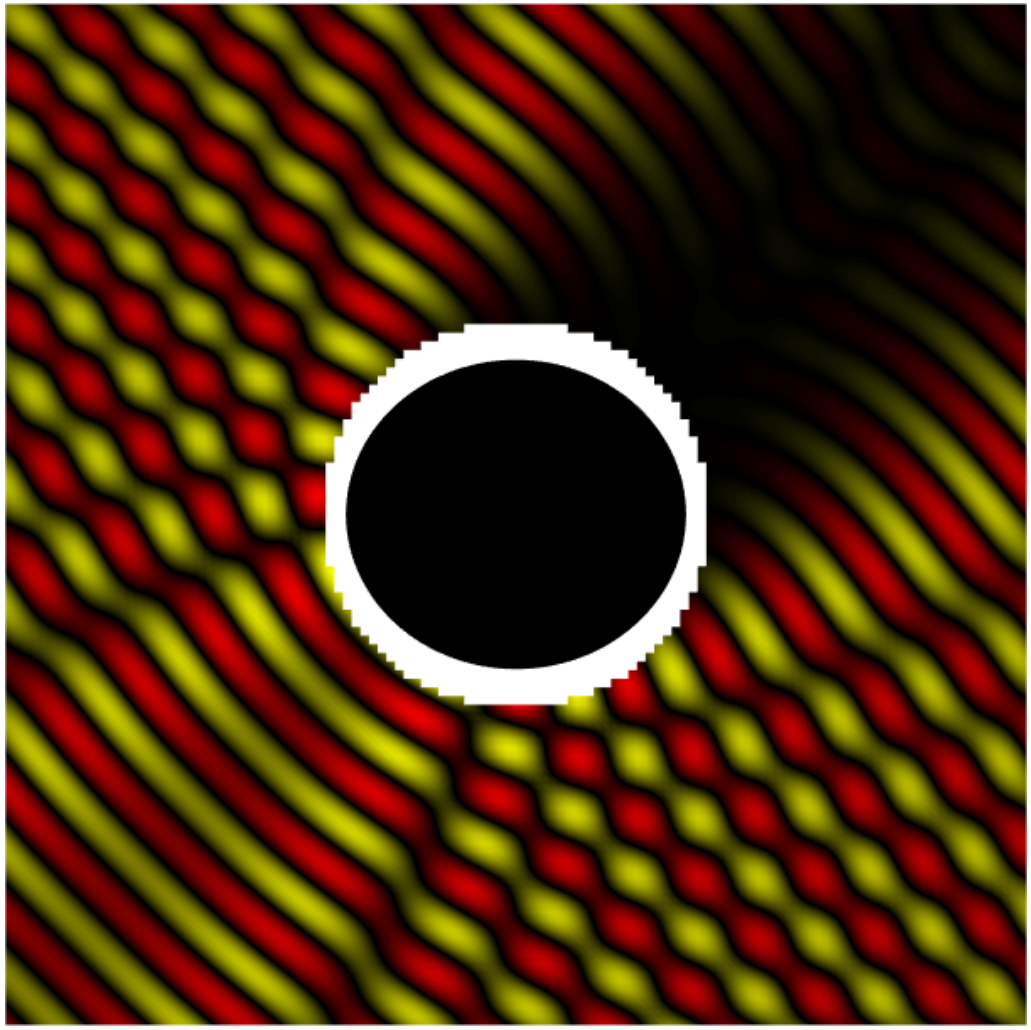
14.4 ms

1.6 min

Motivation

- T-matrix
  - Tool for scattering simulation
  - Ideal for parameter dependent problems
  - Created by Waterman 1965 (EM), 1969 (acoustic)
  - Extensively used
    (meteorology, atmospheric science, oceanography. . .)
- Null Field Method (NFM)
- Numerically unstable

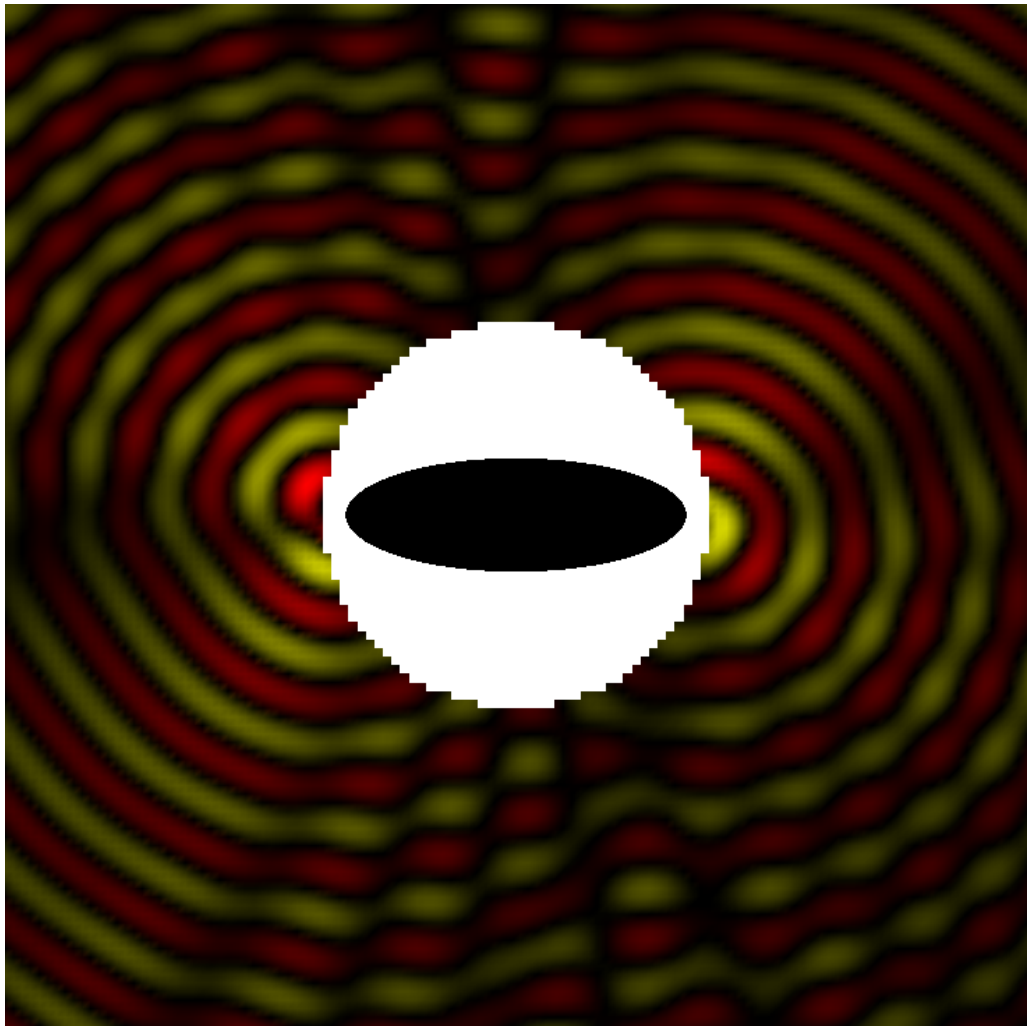
Motivation



Null Field Method

aspect ratio = 1.1

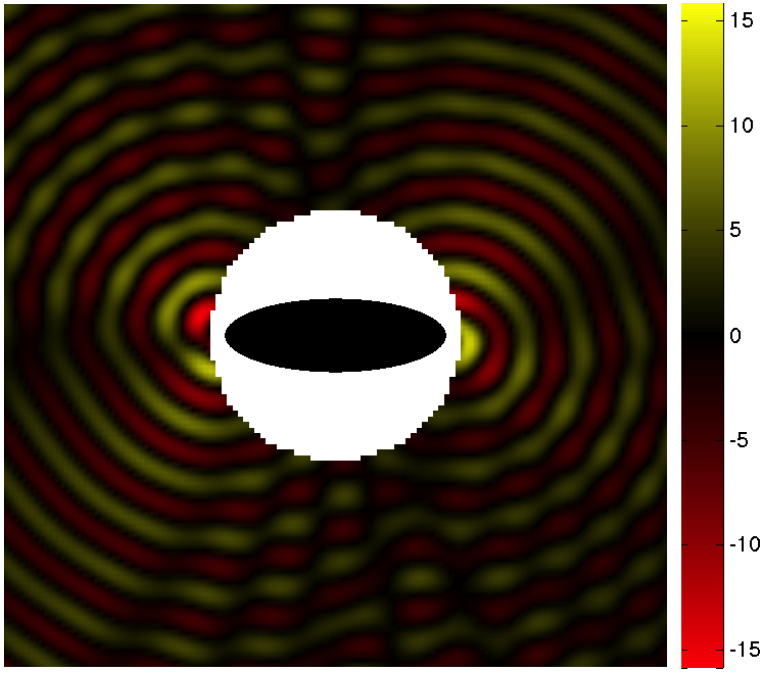
Motivation



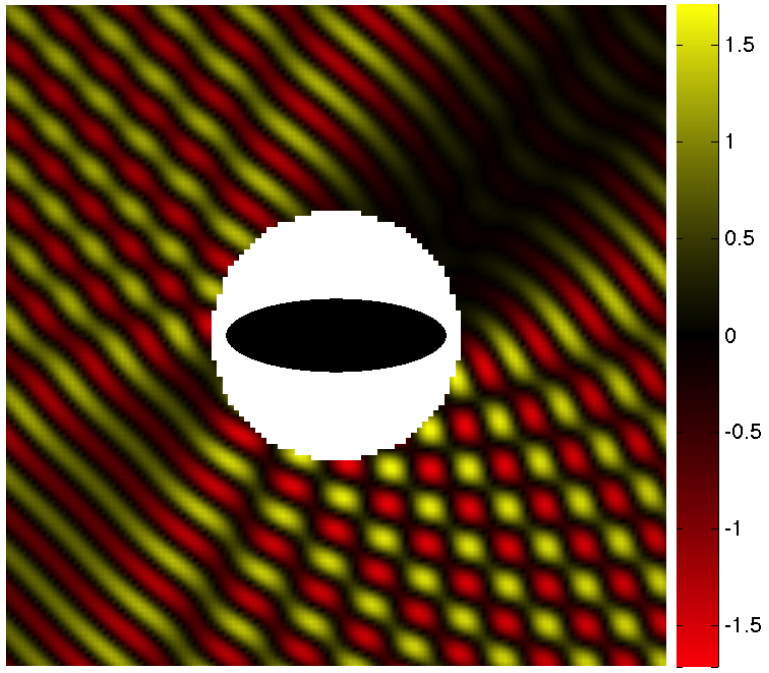
Null Field Method

aspect ratio = 3

Motivation



NFM



true solution

Motivation

Null Field Method

Relative error in T-matrix for ellipses

aspect ratio ρ

size	$\rho = 1.1$	$\rho = 1.5$	$\rho = 3.0$	$\rho = 6.0$
1	1.9e-15	1.0e-15	5.6e-02	6.1e+01
2	1.4e-15	1.3e-14	8.1e-01	2.0e+03
4	2.3e-15	3.9e-14	3.8e+01	1.5e+04
8	2.2e-15	1.3e-11	1.9e+02	1.1e+03
16	6.3e-14	7.4e-08	7.3e+04	5.0e+02

Motivation

- Numerical instability in Null Field Method
  - Somerville et al. 2012, 2013
  - Khlebtsov 2010
  - Mishchenko and Travis 1994
  - Mishchenko et al. 1996
  - Petrov and Shkuratov 2007
  - Waterman 1999, 2007, 2009
  - Havemann and Baran 2004
  - Sarkissian et al. 1993
- No method that is numerically stable for all shapes!

Motivation

- Numerical instability in Null Field Method
  - Somerville et al. 2012, 2013
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  - Waterman 1999, 2007, 2009
  - Havemann and Baran 2004
  - Sarkissian et al. 1993
- No method that is numerically stable for all shapes!
- MG and SCH 2008, 2009, 2010, 2012, 2017 (TMATROM)

Wave scattering model

- Helmholtz equation

$$\nabla^2 u(\boldsymbol{x}) + k^2 u(\boldsymbol{x}) = 0$$

for $\boldsymbol{x} \in \mathbb{R}^2 \setminus \overline{D}$

- Radiation condition

$$\lim_{|x| \to \infty} \sqrt{|\boldsymbol{x}|} \left(\frac{\partial u}{\partial \boldsymbol{x}}(\boldsymbol{x}) - iku(\boldsymbol{x}) \right) = 0$$

- Boundary condition *e.g.*

$$u(\boldsymbol{x}) = -u^{\text{inc}}(\boldsymbol{x})$$

on boundary ∂D

The T-matrix

- Incident field

$$u^{\text{inc}}(r,\theta) = \sum_n b_n \, J_{|n|}(kr) e^{in\theta}$$

- Scattered field

$$u(r,\theta) = \sum_n a_n \, H^{(1)}_{|n|}(kr) e^{in\theta}$$

- T-matrix

$$\mathsf{a} = \mathsf{T}\mathsf{b}$$

- Coefficient vectors $\mathsf{a} = (a_n)$ and $\mathsf{b} = (b_n)$

The Null Field Method

- Null Field Method (Waterman 1965)

$$\mathsf{T} = -(\mathsf{Rg}\,\mathsf{Q})\,\mathsf{Q}^{-1}.$$

- Regular matrix

$$\mathsf{Rg}\,\mathsf{Q}_{mn} = \frac{i}{4}\int_{\partial D} \frac{\partial}{\partial \boldsymbol{n}}\Big(J_{|n|}(kr)e^{in\theta}\Big)\ \overline{J_{|m|}(kr)e^{im\theta}}\ ds$$

- Radiating matrix

$$\mathsf{Q}_{mn} = \frac{i}{4}\int_{\partial D} \frac{\partial}{\partial \boldsymbol{n}}\Big(J_{|n|}(kr)e^{in\theta}\Big)\ \overline{H^{(1)}_{|m|}(kr)\ e^{im\theta}}\ ds$$

The Null Field Method

- Null Field Method (Waterman 1965)

$$\mathsf{T} = -(\mathsf{Rg}\,\mathsf{Q})\,\mathsf{Q}^{-1}.$$

- Regular matrix

$$\mathsf{Rg}\,\mathsf{Q}_{mn} = \frac{i}{4}\int_{\partial D} \frac{\partial}{\partial \boldsymbol{n}}\Big(J_{|n|}(kr)e^{in\theta}\Big)\,\overline{J_{|m|}(kr)e^{im\theta}}\,ds$$

- Radiating matrix

$$\mathsf{Q}_{mn} = \frac{i}{4}\int_{\partial D} \frac{\partial}{\partial \boldsymbol{n}}\Big(J_{|n|}(kr)e^{in\theta}\Big)\,\overline{H^{(1)}_{|m|}(kr)\,e^{im\theta}}\,ds$$

- Hankel function $H^{(1)}_{|m|}(kr) \to \infty$ as $|m| \to \infty$, $r \to 0$

MG and SCH method

- Ganesh and SCH 2010 (EM), 2008/2009 (acoustic)

$$\mathsf{T}_{mn} = \frac{1}{4}\sqrt{\frac{k}{\pi}} i^{|m|}(1+i)\int_{S^1} u_n^\infty(\theta)\; e^{-im\theta}\, ds$$

- Hankel function $H^{(1)}_{|m|}(kr) \to \mathcal{O}(k^{-1/2})$ as $r \to \infty$
- u_n^∞ is far field induced by incident field

$$u^{\text{inc}}(r,\theta) = J_{|n|}(kr)e^{in\theta}$$

- Scattering solver to compute u_n^∞
  - FEM + Nyström method,...

MG and SCH method

Theorem (Ganesh, SCH and Hiptmair, 2012)

Plane wave $u^{\text{inc}}(\boldsymbol{x}) = e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}}$ scattering

$$\|u^{\infty} - u_M^{\infty}\|_{L^2}^2 \leq CM^2 \left(\frac{Rke}{2M}\right)^{2M} + C'\epsilon^2$$

- T-matrix truncation parameter M
- Solver error parameter ϵ

TMATROM

- Setup T-matrix

```
T = tmatrix(M,kwave,solver,origin);
```

- Setup incident field coefficients

```
inc = plane_wave(dirn,kwave);
b = regularwavefunctionexpansion(M,origin,inc);
```

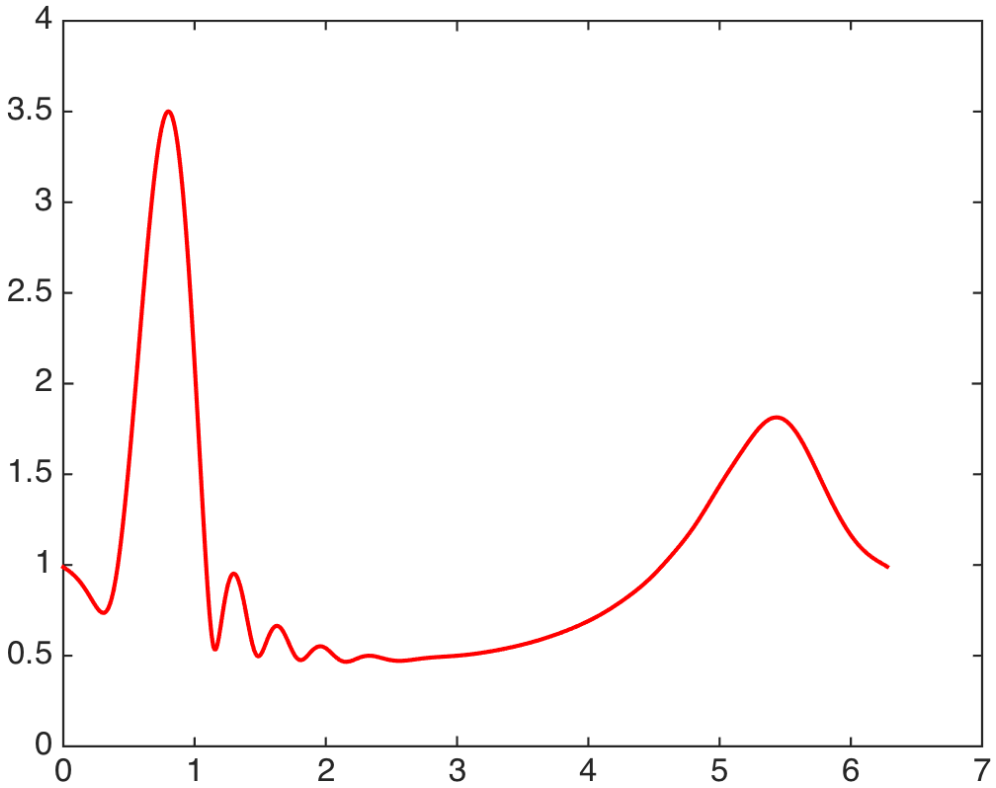
- Apply T-matrix

```
a = T * b;
```

TMATROM

- Visualize

```
a.visualizeFarField();
```



Numerical Results

Theorem

The T-matrix of a sound soft or sound hard particle satisfies

$$\mathsf{T} + \mathsf{T}^* + 2\mathsf{T}^*\mathsf{T} = 0$$

- Error measure for truncated T-matrix

$$\max_{n,m=-M,\ldots,M} |(\mathsf{T} + \mathsf{T}^* + 2\mathsf{T}^*\mathsf{T})_{mn}|$$

- Truncation parameter

$$M \approx ka + 6 + 4(ka)^{1/3}$$

- Radius of scatterer a

Numerical Results

Null Field Method

Relative error in T-matrix for ellipses

aspect ratio ρ

size	$\rho = 1.1$	$\rho = 1.5$	$\rho = 3.0$	$\rho = 6.0$
1	1.9e-15	1.0e-15	5.6e-02	6.1e+01
2	1.4e-15	1.3e-14	8.1e-01	2.0e+03
4	2.3e-15	3.9e-14	3.8e+01	1.5e+04
8	2.2e-15	1.3e-11	1.9e+02	1.1e+03
16	6.3e-14	7.4e-08	7.3e+04	5.0e+02

Numerical Results

TMATROM*

Relative error in T-matrix for ellipses

aspect ratio ρ

size	$\rho = 1.1$	$\rho = 1.5$	$\rho = 3.0$	$\rho = 6.0$
1	3.1e-14	6.5e-14	2.6e-09	3.5e-09
2	1.3e-10	6.0e-11	3.0e-11	4.8e-09
4	1.9e-11	8.9e-12	5.0e-11	7.7e-10
8	4.0e-10	1.6e-10	7.0e-10	7.4e-10
16	3.0e-09	1.0e-09	6.2e-09	6.9e-09

*Solver: Nyström method with error $\epsilon <$ 1.0e-08

Numerical Results

TMATROM*

Relative error in T-matrix for ellipses

aspect ratio ρ

size	$\rho = 1.1$	$\rho = 1.5$	$\rho = 3.0$	$\rho = 6.0$
20	4.4e-09	1.4e-09	3.9e-10	1.6e-10
40	3.0e-10	3.6e-09	8.9e-10	1.0e-09
60	8.4e-10	4.9e-09	2.5e-09	2.8e-09
80	1.4e-09	5.8e-09	5.7e-09	6.3e-09
100	2.1e-09	6.3e-09	8.5e-09	9.6e-09

*Solver: Nyström method with error $\epsilon <$ 1.0e-08

Numerical Results

NFM and **TMATROM***

Relative error in T-matrix for ellipse

aspect ratio $\rho = 6$

size	NFM	NFM ($2 \times$ quad)	NFM ($4 \times$ quad)	TMATROM
1	6.1e+01	1.6e+01	1.9e-06	3.5e-09
2	2.0e+03	4.5e+01	1.3e-02	4.8e-09
4	1.5e+04	8.7e+03	8.3e+02	7.7e-10
8	1.1e+03	3.6e+05	2.0e+04	7.4e-10
16	5.0e+02	6.2e+03	2.4e+03	6.9e-09

*Solver: Nyström method with error $\epsilon < 1.0\text{e-}08$

Numerical Results

NFM and **TMATROM***

Relative error in T-matrix for kite

size	NFM	NFM ($2 \times$ quad)	NFM ($4 \times$ quad)	TMATROM
1	4.8e-01	4.8e-01	4.8e-01	3.9e-10
2	3.0e+01	3.0e+01	3.0e+01	2.9e-10
4	1.4e+01	1.4e+01	1.4e+01	6.3e-11
8	5.2e+01	5.2e+01	5.2e+01	2.4e-09
16	1.4e+03	1.1e+03	1.2e+03	5.5e-09

*Solver: Nyström method with error $\epsilon < 1.0\text{e-}08$

Numerical Results

NFM and **TMATROM***

Relative error in T-matrix for hexagon

size	NFM	NFM (2 × quad)	NFM (4 × quad)	TMATROM
1	2.9e-03	2.9e-03	2.9e-03	4.2e-09
2	8.7e-04	8.7e-04	8.7e-04	1.6e-09
4	9.6e+00	9.6e+00	9.6e+00	3.0e-09
8	1.0e+01	1.0e+01	1.0e+01	8.7e-09
16	8.0e+00	8.0e+00	8.0e+00	9.7e-09

*Solver: MPSPACK with error $\epsilon <$ 1.0e-08

Numerical Results

NFM and **TMATROM***

Relative error in T-matrix for capped cylinder

aspect ratio $\rho = 6$

size	NFM	NFM ($2 \times$ quad)	NFM ($4 \times$ quad)	TMATROM
1	2.2e+02	1.8e+02	1.9e+01	9.4e-05
2	2.0e+03	3.9e+03	5.2e+02	9.0e-05
4	9.5e+03	6.1e+03	3.2e+03	9.9e-05
8	2.1e+04	3.5e+04	4.8e+03	9.9e-05
16	1.6e+05	6.3e+04	3.1e+05	9.7e-05

*Solver: Nyström method with error $\epsilon <$ 1.0e-04

Numerical Results

NFM and **TMATROM***

CPU time (seconds) on quad core 2.6GHz CPU for ellipse aspect ratio $\rho = 6$

size	NFM	NFM ($2 \times$ quad)	NFM ($4 \times$ quad)	TMATROM
1	0.83	0.81	0.97	0.02
2	1.62	1.53	1.92	0.03
4	3.05	3.80	5.29	0.05
8	8.08	10.79	16.47	0.11
16	27.35	40.70	66.73	0.24

*Solver: Nyström method with error $\epsilon < 1.0\text{e-}08$

Numerical Results

References

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