Sparse Isotropic Regularization for Spherical Harmonic Representations of Random Fields on the Sphere

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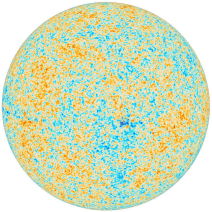
Cosmic Microwave Background

- Cosmic Microwave Background (CMB) is a black-body radiation from the recombination epoch of the early universe, around 13.7 Byr ago.
- CMB is critical to understanding of early universe and evidence of the Big Bang theory. In particular, measurement of CMB fluctuations is key to verification and estimation of the cosmological properties and parameters.
- CMB was discovered by Arno Penzias and Robert Wilson in 1964 (Nobel Prize in Physics 1978), since then NASA and ESA probes have been collecting CMB data.

CMB field on $\mathbb{S}^2$

$$\widetilde{T}(\hat{\boldsymbol{p}}, \boldsymbol{p}_0, t) = T_0(\boldsymbol{p}_0, t)\big(1 + T(\hat{\boldsymbol{p}}, \boldsymbol{p}_0, t)\big)$$

- $\hat{\boldsymbol{p}}$ is the direction the CMB signal comes from.
- $\boldsymbol{p}_0$ is the observation location, near the planet earth.
- t is the time when the CMB signal was received, Planck Space Observatory in 2009.
- $T_0(\boldsymbol{p}_0, t)$ is around $2.73K$ at present.
- $T(\hat{\boldsymbol{p}}) := T(\hat{\boldsymbol{p}}, \boldsymbol{p}_0, t)$ is modelled as a random field on $\mathbb{S}^2$.

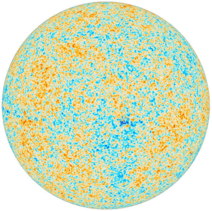


Random fields on $\mathbb{S}^2$

- Probability measure space $(\Omega, \mathcal{F}, \mathrm{P})$

- An $\mathcal{F} \otimes \mathscr{B}(\mathbb{S}^2)$-measurable function $T(\omega, \boldsymbol{x}) : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called a *real-valued random field* on the sphere $\mathbb{S}^2$.

- Space of square integrable functions on product space $L_2(\Omega \times \mathbb{S}^2, \mathrm{P} \otimes \sigma)$



Isotropy of random field

- (Strongly) Isotropy of a random field T means, for any n points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and any rotation $\rho \in SO(3)$,

$$(T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)) \sim (T(\rho\boldsymbol{x}_1), \ldots, T(\rho\boldsymbol{x}_n))$$

  in distribution.

- Isotropy means the field does not change (its distribution) when the coordinate system changes, which is consistent with the physics intuition and is a fundamental hypothesis for the CMB field.

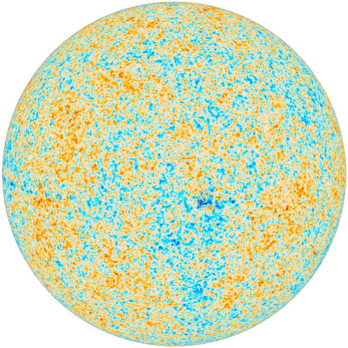
Spherical Harmonic Representation (Karhunen-Loève expansion)

For each realization, we can use spherical harmonics to represent the random field, in $L_2\left(\Omega \times \mathbb{S}^2\right)$ sense,

$$T^o(\omega, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{(T^o)}_{\ell m}(\omega) Y_{\ell m}(\boldsymbol{x}), \quad (\omega, \boldsymbol{x}) \in \Omega \times \mathbb{S}^2.$$

- The $Y_{\ell m}$ is a spherical harmonic orthonormal basis for $L_2(\mathbb{S}^2)$.
- Each $\widehat{(T^o)}_{\ell m}(\omega)$ is a random variable.
- The $L_2\left(\Omega \times \mathbb{S}^2\right)$ convergence holds for the expansion when the field is isotropic.

Big Data of CMB



- Collected by Planck in 2009
- Data stored at $50,331,648$ HEALPix points
- # Fourier coeff. $= (4000+1)^2$ $= 16,008,001$

Sparse representation by ℓ_1 regularization

Cammarota & Marinucci (2015).

To process signals from random fields, people want to use as small proportion of coefficients to represent the main information of the field.

$$\min_{a_{\ell m}} \frac{1}{2}\|T - T^{\mathrm{o}}\|^2_{L_2(\mathbb{S}^2)} + \lambda \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_{\ell,m}|,$$

$$\text{where } \ T(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^2.$$

where T^{o} is an observed field.

ℓ_1 regularization does not preserve isotropy

$$\frac{1}{2}\|T - T^{\mathrm{o}}\|^2_{L_2(\mathbb{S}^2)} + \lambda \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |a_{\ell,m}|.$$

- The ℓ_1 regularization is an effective strategy of reducing the number of coefficients but is not feasible for representing CMB fields as it does not preserve isotropy of the random field.
- The essential reason the ℓ_1 regularization violates isotropy is because, for a given $\ell \geq 1$, the sum $\sum_{m=-\ell}^{\ell} |a_{\ell,m}(\omega)|$ is not rotationally invariant.

Sparse isotropic regularization

$$\min_{a_{\ell m}} \frac{1}{2}\|T - T^{\mathrm{o}}\|^2_{L_2(\mathbb{S}^2)} + \lambda \sum_{\ell=0}^{\infty} \beta_\ell \left(\sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2 \right)^{1/2}.$$

- For any given $\ell \geq 1$ the sum $\sum_{m=-\ell}^{\ell} |a_{\ell,m}(\omega)|^2$ is rotationally invariant.
- λ is a real number (regularization parameter) and β_ℓ is a real-valued sequence.

Solution

$$\min_{a_{\ell m}} \frac{1}{2}\|T - T^{\text{o}}\|^2_{L_2(\mathbb{S}^2)} + \lambda \sum_{\ell=0}^{\infty} \beta_\ell \left(\sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2 \right)^{1/2}$$

$$= \min_{a_{\ell m}} \frac{1}{2}\|a_{\ell m} - a^{o}_{\ell m}\|^2_{\ell_2} + \lambda \sum_{\ell=0}^{\infty} \beta_\ell \left(\sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2 \right)^{1/2}.$$

Solution (continued)

The regularized field for a given observed field T^{o} takes the form

$$T^{\mathrm{r}}(\boldsymbol{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a^{\mathrm{r}}_{\ell,m} Y_{\ell,m}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^2,$$

- $$a^{\mathrm{r}}_{\ell,m} := \begin{cases} \left(1 - \dfrac{\lambda \beta_\ell}{A^{\mathrm{o}}_\ell}\right) a^{\mathrm{o}}_{\ell,m}, & \text{if } A^{\mathrm{o}}_\ell > \lambda \beta_\ell, \\ 0, & \text{if } A^{\mathrm{o}}_\ell \le \lambda \beta_\ell, \end{cases}$$

- $$A^{\mathrm{o}}_\ell := \left(\sum_{m=-\ell}^{\ell} |a^{\mathrm{o}}_{\ell,m}|^2 \right)^{\frac{1}{2}}, \quad \ell \ge 0.$$

- In any application of the present regularization scheme, the choices of the sequence $(\beta_\ell)_{\ell \ge 0}$ and the parameter λ are crucial.

Sparse isotropic regularization preserves isotropy

- The regularized field T^{r} is strongly isotropic if the observed field T^{o} is isotropic.

Numerical example for CMB map (Mollweide)

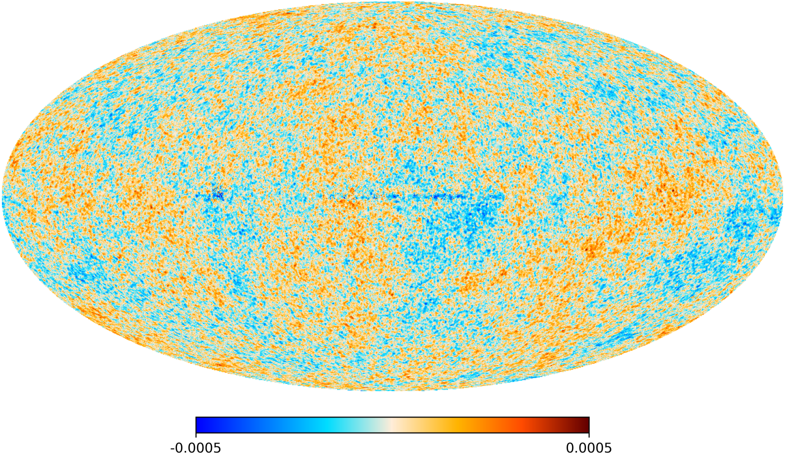


Fig. The CMB map with $N_{\mathrm{side}} = 2048$ as computed by SMICA.

Parameter selection

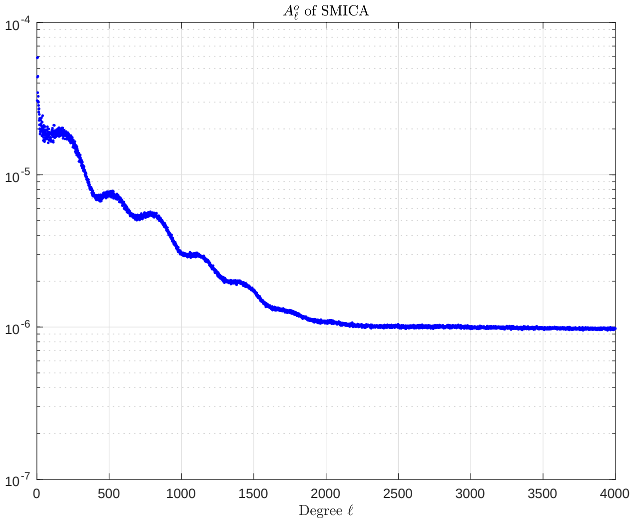


Fig. The observed field $A_\ell^{\rm o}$, empirical angular power spectrum $C_\ell = (A_\ell^{\rm o})^2/(2\ell+1)$.

Parameter selection

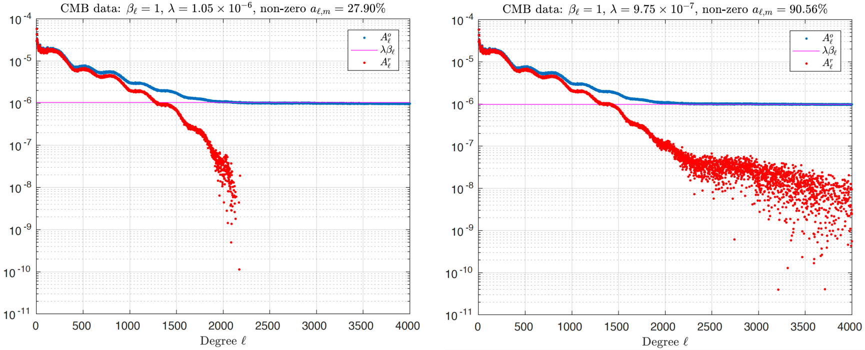
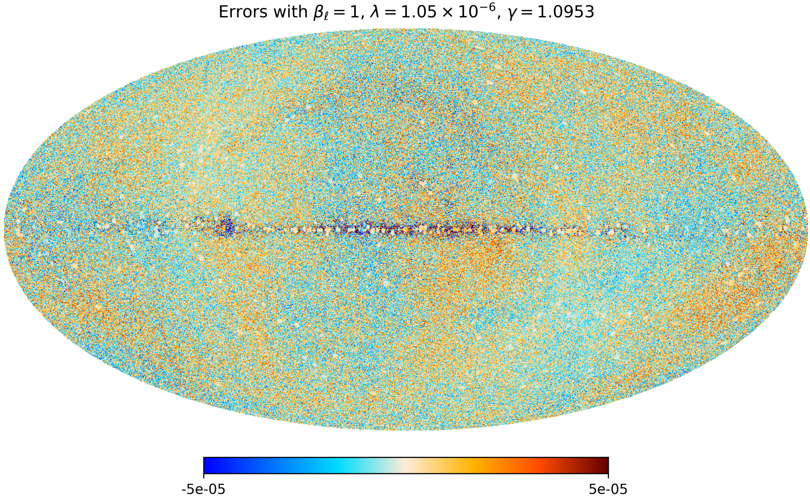


Fig. The regularized field A^{r}_ℓ with $\beta_\ell = 1$, and with $\lambda = 1.05 \times 10^{-6}$ (left graph) and $\lambda = 9.75 \times 10^{-7}$ (right graph).

Errors of regularized field, Sparsity 72.1%, L_2 error 5.82×10^{-5}



Errors of regularized field, Sparsity 9.44%, L_2 error 5.54×10^{-5}