Open Problems in the Topology of Manifolds

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Introduction

The problems in this list were collected at MATRIX, during the workshop on the Topology of Manifolds: Interactions between High and Low Dimensions, January 7th – 18th 2019. Several of the problems below were discussed in the problem sessions during the MATRIX workshop and *the organisers wish to thank all participants for their enthusiasm during the problem sessions and throughout the meeting.* A description of how the problem sessions were run can be found in the preface.

Below, we give a selection of eleven problems that were posed at the workshop. This selection illustrates the range and scope of the discussions at the meeting. We would like to thank all participants who contributed problems and further questions

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that helped shape many of them. An evolving record of these and other problems and questions posed at the workshop can be found at the Manifold Atlas:

http://www.map.mpim-bonn.mpg.de/

We have attributed each problem in this list to the participant(s) who presented the problem at the workshop. The style in which problems were posed varied widely, and our selection reflects this. The first nine problems listed here are succinctly formulated and self-contained: references to the literature are minimal and references for each problem, where they exist, are at the end of the problem. The subject matter of the final two problems necessitated recalling more background and a somewhat more detailed referencing of the literature. The order in which we list the problems is chronological, rather than by subject matter.

Problem 1: A quotient of $\mathbf{S^2 \times S^2}$

presented by Jonathan Hillman

Let $C_4 = \langle \sigma \rangle$ act freely on $S^2 \times S^2$ with that action of σ defined by the equation $\sigma(x,y) = (y,-x)$ and let M be the quotient manifold.

The real projective plane $\mathbb{R}P^2 = S^2/\sim$ embeds in M via $[x] \mapsto [x,x]$ and its disk bundle neighborhood N in M is the tangent disk bundle of $\mathbb{R}P^2$. The complement of the open disk bundle neighborhood is the mapping cylinder of the double cover of lens spaces $L(4,1) \to L(8,1)$. Thus

$$M = N \cup \mathrm{MCyl}(\mathrm{L}(4,1) \to \mathrm{L}(8,1)).$$

This geometric analysis of M was given in [1] where it was shown that there are at most four closed topological manifolds in this homotopy type, half of which are stably smoothable.

The smooth manifold $M' = N \cup \mathrm{MCyl}(\mathrm{L}(4,1) \to \mathrm{L}(8,3))$ is homotopy equivalent to M.

Question. *Are M and M' homeomorphic? diffeomorphic?*

Reference

1. I. Hambleton and J. Hillman, *Quotients of $S^2 \times S^2$*, Preprint 2017. Available at arXiv1712.04572

Problem 2: Connected sum decompositions of high-dimensional manifolds

presented by Stefan Friedl

Let Cat be one of the categories Top, PL or Diff. A Cat-manifold M is called *irreducible* if, whenever we can write M as a connected sum of Cat-manifolds at least one of the summands is a homotopy sphere. The Kneser-Milnor theorem [2] says that every compact Cat 3-manifold admits a connected sum decomposition into irreducible 3-manifolds, and this connected sum decomposition is unique up to permutation of the summands.

Stefan Friedl asked to what degree this statement holds in higher dimensions. During the two weeks of the workshop and during discussions afterwards Imre Bokor, Diarmuid Crowley, Stefan Friedl, Fabian Hebestreit, Daniel Kasprowski, Markus Land and Johnny Nicholson obtained a fairly comprehensive answer which appears in the proceedings [1]. Before we discuss the results, note that exotic spheres do not have a decomposition into irreducible manifolds. Thus it is reasonable to consider all questions "up to homotopy spheres".

In the following we summarize a few of the results.

1. It follows from standard algebraic topology and group theory that every Cat-manifold admits a connected sum decomposition into irreducible manifolds and a homotopy sphere.
2. The uniqueness statement (up to homotopy spheres) fails to hold in any of the dimensions ≥ 4 and any of the categories.
3. If one restricts attention to simply connected manifolds, then it is shown that in any dimension ≥ 17 uniqueness (up to homotopy spheres) fails to hold in any of the categories.
4. In contrast for many even dimensions $2k$, if one restricts attention to the case of $(k-1)$-connected smooth manifolds, uniqueness does hold.

References

1. I. Bokor, D. Crowley, S. Friedl, F. Hebestreit, M. Land, D. Kasprowski and J.Nicholson, *Connected sum decompositions of high-dimensional manifolds*, to appear in the MATRIX Annals (2019). Available at arXiv:1909.02628
2. J. Milnor, *A unique decomposition theorem for 3-manifolds,* Amer. J. Math. 84 (1962), 1–7.

Problem 3: An analogue of Casson-Gordon theory for trisections

presented by Stephan Tillmann

Heegaard splittings have long been used in the study of 3–manifolds. They were introduced in 1898 by Poul Heegaard, and provide a decomposition of each closed 3–manifold into two 1–handlebodies. A key concept introduced in the theory by Casson and Gordon [1] was the notion of *strong irreducibility*, with their main theorem stating that if a closed 3–manifold has a splitting that is not strongly irreducible, then either the splitting is reducible or the manifold contains an incompressible surface of positive genus. That is, one can either simplify the splitting, or one obtains topological information on the 3–manifold. Strongly irreducible Heegaard surfaces turn out to have many useful properties that one usually only associates with incompressible surfaces in 3–manifolds. Casson and Gordon also discovered a local condition, the *rectangle condition*, which guarantees that a Heegaard splitting is irreducible.

The challenge for the analogous theory of trisections of 4–manifold is to determine properties of trisections that have strong topological consequences and that can be determined by local information, for instance, from a trisection diagram.

Reference

1. A. Casson and C.McA. Gordon, *Reducing Heegaard splittings*, Topology Appl. 27 (1987), 275–283.

Problem 4: Aspherical manifolds whose fundamental group has nontrivial centre

presented by Fabian Hebestreit and Markus Land

Given a closed aspherical manifold M whose fundamental group has nontrivial centre, we can ask the following:

Question A. *Does there exist a finite cover of M with a principal S^1-action?*

Question B. *Is such an M null-cobordant?*

Motivation and background for these questions is found in [1, Section 7].

Reference

1. F. Hebestreit, M. Land, W. Lück and Oscar Randal-Williams, *A vanishing theorem for tautological classes of aspherical manifolds*, to appear in Geom. Topol.. Available at arXiv:1705.06232

Problem 5: Is the trisection genus additive under connected sum?

presented by Peter Lambert-Cole

Let M be a closed smooth 4-manifold. The "trisection genus" of M is the minimal genus of the central surface appearing in a trisection of M.

Question. *Is the trisection genus additive under connected sum?*

If so, then the following hold:

1. The trisection genus of M is a homeomorphism invariant.
2. The manifolds S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ have a unique smooth structure.

An affirmative answer to the question is known for the class of all standard simply connected PL 4-manifolds [1].

Reference

1. J. Spreer and S. Tillmann, *The trisection genus of standard simply connected PL 4-manifolds*, 34th International Symposium on Computational Geometry, Art. No. 71, 13 pp., LIPIcs. Leibniz Int. Proc. Inform., 99, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

Problem 6: Compact aspherical 4-manifolds

presented by Jim Davis

Let M_0 and M_1 be a compact aspherical 4-manifolds with boundary. The Borel Conjecture in this setting states that a homotopy equivalence of pairs

$$f\colon (M_0, \partial M_0) \to (M_1, \partial M_1),$$

which is a homeomorphism on the boundary is homotopic, relative to the boundary, to a homeomorphism.

By topological surgery, the Borel Conjecture is valid when the fundamental group $\pi = \pi_1(M_0) \cong \pi_1(M_1)$ is good, for example, if π is elementary amenable. One now proceeds to the following three problems:

1. Decide which good π are the fundamental groups of compact aspherical 4-manifolds.
2. Determine the possible fundamental groups of the boundary components.
3. Determine the homeomorphism types of the boundary components.

These problems could be considered for compact aspherical 4-manifolds even when the fundamental group is not good, also in the smooth case.

Question. *Let M be a closed smooth aspherical 4-manifold. Is every smooth 4-manifold homotopy equivalent to M diffeomorphic to M?*

The question has not been answered for any M, not even the 4-torus.

Problem 7: Embedding integral homology 3-spheres into the 4-sphere

presented by Jonathan Hillman

Question. *Let Σ be an integral homology 3-sphere, not homeomorphic to S^3. Is there a locally flat embedding $\Sigma \hookrightarrow S^4$ such that one or both complementary regions are not simply-connected?*

This problem is motivated by the problem of classifying such embeddings up to isotopy. If a complement has non-trivial fundamental group, then a ‘satellite’ construction yields infinitely many isotopy classes of embeddings of Σ into S^4.

Problem 8: Stabilising number of knots and links

presented by Anthony Conway

Let W be a compact 4–manifold with boundary $\partial W \cong S^3$. We say that a properly embedded disk $(\Delta, \partial\Delta) \subset (W, \partial W)$ is *nullhomologous*, if its fundamental class $[\Delta, \partial\Delta] \in H_2(W, \partial W; \mathbb{Z})$ vanishes.

A link $L \subset S^3$ is *stably slice* if there exists $n \geq 0$ such that the components of L bound a collection of disjoint locally flat nullhomologous discs in the manifold $D^4 \# n(S^2 \times S^2)$. The *stabilising number* $\mathrm{sn}(L)$ of a stably slice link is the minimal such n.

Schneiderman proved that a link L is stably slice if and only if the following invariants vanish: the triple linking numbers $\mu_{ijk}(L)$, the mod 2 Sato-Levine invariants of L, and the Arf invariants of the components of L [2].

Question A. *Does the inequality* $\mathrm{sn}(L) \leq g_4^{\mathrm{top}}(L)$ *hold for stably slice links L of more than one component?*

This question is settled in the knot case: together with Matthias Nagel, we showed that $\mathrm{sn}(K) \leq g_4^{\mathrm{top}}(K)$ holds for stably slice knots [1]. We are currently unable to generalise this proof to links.

Remark. *The definition of the stabilising number also makes sense in the smooth category (one requires that the discs be smoothly embedded). Just as in the topological category, the inequality* $\mathrm{sn}^{\mathrm{smooth}}(K) \leq g_4^{\mathrm{smooth}}(K)$ *holds, and is unknown for links.*

This discussion of categories leads to the following question:

Question B. Is there a difference between the topological and smooth stabilising numbers of a knot? More precisely, is there a non-topological slice, Arf invariant zero knot K such that $0 < \mathrm{sn}^{\mathrm{top}}(K) < \mathrm{sn}^{\mathrm{smooth}}(K)$?

References

1. A. Conway and M. Nagel, *Stably slice disks of links*, J. Topol. 13 (2020), 1261–1301.
2. R. Schneiderman *Stable concordance of knots in 3-manifolds*, Algebr. Geom. Topol. 10 (2010), 373–432.

Problem 9: Unknotted surfaces in the 4-spheres

presented by Jim Davis

Kawauchi has published several accounts of the theorem below; however, none of them are satisfactory. The problem is to give a satisfactory proof.

Theorem. *Any two locally flat topological embeddings of a closed oriented surface in* S^4 *whose complement has infinite cyclic fundamental group are homeomorphic.*

There is a corresponding statement in the nonorientable case. Complex conjugation on $\mathbb{C}P^2$ has fixed set $\mathbb{R}P^2$ and orbit space S^4. Likewise for $\overline{\mathbb{C}P^2}$. The involution on $a\mathbb{C}P^2 \# b\overline{\mathbb{C}P^2}$ thus gives a locally flat embedding of $\#_{a+b}\mathbb{R}P^2$ in S^4.

Conjecture. *Any locally flat topological embedding of a closed nonorientable surface in S^4 whose complement has order 2 fundamental group is homeomorphic to one of the above embeddings.*

See also Massey [1] which determined the possible normal bundles.

Reference

1. W. S. Massey *Proof of a conjecture of Whitney*, Pacific J. Math. 31 (1969), 143–156.

Problem 10: Genus bounds for cancellations

presented by Diarmuid Crowley

This problem and the next are about the classification of compact $2q$-manifolds for $q \geq 2$. For simplicity, we assume that all manifolds are *connected*. We state these problems in the smooth category: there are obvious analogues for *PL*-manifolds and topological manifolds but we only discuss the topological case in dimension 4, which is of course an exceptional dimension and the *PL* case not at all.

For a natural number g, define $W_g := \#_g(S^q \times S^q)$ to be the g-fold connected sum of $S^q \times S^q$ with itself. If M_0 and M_1 are compact smooth $2q$-manifolds of the same Euler characteristic, then a *stable diffeomorphism* from M_0 to M_1 is a diffeomorphism

$$f\colon M_0 \sharp W_g \to M_1 \sharp W_g$$

for some $g \geq 0$. In this case we say write $M_0 \cong_{\mathrm{st}} M_1$ and we say that M_0 and M_1 are *stably diffeomorphic*. Of course, to define the connected sum operation, M_0, M_1 and W_g must be locally oriented. Since W_g admits an orientation reversing diffeomorphism for all g, if for $i = 0, 1$ the manifold M_i is orientable, then the diffeomorphism type of $M_i \sharp W_g$ does not depend on the orientation chosen for M_i.

The *stable class* of a $2q$-manifold M is defined to be the set of diffeomorphism classes of $2q$-manifolds M' with same Euler characteristic as M and which are stably diffeomorphism to M:

$$\mathscr{S}^{\mathrm{st}}(M) = \{M' \mid \chi(M) = \chi(M') \text{ and } M \cong_{\mathrm{st}} M'\}/\text{diffeomorphism}$$

We say that *cancellation holds* for M if every manifold which is stably diffeomorphic to M is diffeomorphic to M; i.e. $|\mathscr{S}^{\mathrm{st}}(M)| = 1$. Our purpose here is to summarise some of what is known about when cancellation holds and to identify two basic problems about cancellation which remain open. For this we require a further definition, notation and discussion.

The *genus* of M, $g(M)$, is defined to be the largest natural number g such that

$$M \cong M' \sharp W_g$$

for some other compact smooth $2q$-manifold M'. Since we have assumed that $q \geq 2$, the fundamental group of M, which we denote by π, is unchanged by stabilisation with W_g. So far, the majority of work on the cancellation problem has been to identify cases where cancellation holds via the fundamental group π, the genus g and the parity of q. For example, the following theorem of Hambleton and Kreck shows the power of cancellation as a classification technique in dimension 4.

Theorem A. (Topological cancellation for $q = 2$ and finite π; [2, Thm. B]) *Let M be a closed oriented topological 4-manifold with finite fundamental group and of genus at least 1. Then cancellation holds for M.*

Recall next that a finitely presented group π is polycyclic-by-finite if it has a finite index subgroup which has a subnormal series where each quotient is cyclic. The minimal number of infinite cyclic quotients is an invariant of π called the *Hirsch length* of π and is denoted $h(\pi)$. The results of the following theorem all use Kreck's theory of modified surgery: the first three are [3, Theorem 5] and the fourth is [2, Theorem 1.1].

Theorem B. (Cancellation results for $q \geq 3$; [3, Thm. 5] and [2, Thm. 1.1]) *Let M be a compact $2q$-manifold of genus g with polycyclic-by-finite fundamental group π and let N be stably diffeomorphic to M with the same Euler characteristic as M.*

1. *If q is odd and π is trivial then M and N are diffeomorphic;*
2. *If π is trivial and $g \geq 1$, then M and N are diffeomorphic;*
3. *If π is finite and $g \geq 2$, then M and N are diffeomorphic;*
4. *If $g \geq h(\pi) + 3$, then M and N are diffeomorphic.*

We now state two problems relating the genus of M to the cancellation problem. The first of these was explained to the author by Ian Hambleton and uses the following further terminology: let π be a finitely presented group and $\varepsilon \in \{\pm 1\}$. We say that g_0 is a *ε-genus cancellation bound* for π if cancellation holds for every $2q$-manifold M with $\varepsilon = (-1)^q$, $\pi_1(M) \cong \pi$ and genus $g(M) \geq g_0$. If such a g_0 exists, the *ε-cancellation genus of π* is defined to be minimum genus cancellation bound

$$\mathrm{cg}_\varepsilon(\pi) := \min\{g_0 \mid g_0 \text{ is an } \varepsilon\text{-genus cancellation bound for } \pi\}.$$

If there is no ε-genus cancellation bound for π, we set $\mathrm{cg}_\varepsilon(\pi) = \infty$.

Problem A. (Genus bounds for general groups) *Is there an example of a finitely presented group π which is not polycyclic-by-finite and for which $\mathrm{cg}_\varepsilon(\pi) < \infty$ for some ε?*

It perhaps remarkable that Problem A is still open, but in fact our knowledge of the cancellation bound for almost all groups π is minimal. By Theorem B(1), we have $\mathrm{cg}_-(\{e\}) = 0$ and there examples which combine with Theorem B(2) to give $\mathrm{cg}_+(\{e\}) = 1$ and indeed $\mathrm{cg}_+(\pi) \geq 1$ for all π. However, there are no known examples where $\mathrm{cg}_\varepsilon(\pi) \geq 2$; i.e. the following problem is still open.

Problem B. ((s the cancellation genus ever greater than one?) *Is there a finitely presented group π and $\varepsilon \in \{\pm 1\}$ such that $\mathrm{cg}_\varepsilon(\pi) \geq 2$? i.e. is there a pair of $2q$-manifolds M and N with $\pi_1(M) \cong \pi_1(M) \cong \pi$ such that we have $M \sharp W_g \cong N \sharp W_g$ for some g but $M \sharp W_1$ and $N \sharp W_1$ are not diffeomorphic?*

Remark. One source of manifolds in the stable class of M comes from the action of the L-group, $L_{2q+1}(\mathbb{Z}[\pi], w_1)$, by Wall realisation. To be precise about torsions, the torsion requirements correspond to the L-group denoted $L^E_{2q+1}(\pi)$ in [7, 17 D] and this L-group is defined as the group of units in the little-ℓ surgery monoid $l_{2q+1}(\mathbb{Z}[\pi], w)$; see [3, p. 773]. Hence the formations and lagrangians are based, but the formation is *not* required to be simple and there is an exact sequence

$$0 \to L^s_{2q+1}(\mathbb{Z}[\pi], w_1) \to L_{2q+1}(\mathbb{Z}[\pi], w_1) \xrightarrow{\tau} \mathrm{Wh}(\pi),$$

where $\mathrm{Wh}(\pi)$ is the Whitehead group of π and the image of τ is described precisely in [2, Lemma 6.2].

The Wall realisation procedure entails that if $\rho \in L_{2q+1}(\mathbb{Z}[\pi], w_1)$ is represented by a formation on a hyperbolic form of rank $2g_0$ and if $M' = \rho M$, then we have $M \sharp W_{g_0} \cong M' \sharp W_{g_0}$. Moreover, applying [2, First theorem of §1.3], it follows that if $g(M) \geq g_0$, then $M' \cong M$. Hence the cancellation problem is related to the algebraic problem of the determining the minimal rank of a formation representing a given $\rho \in L_{2q+1}(\mathbb{Z}[\pi], w_1)$. For example, if every element of $L_{2q+1}(\mathbb{Z}[\pi], w_1)$ is represented by a formation of rank $2g_0$ or less and $g(M) \geq g_0$, then $L_{2q+1}(\mathbb{Z}[\pi], w_1)$ acts trivially on $\mathscr{S}^{\mathrm{st}}(M)$.

References

1. D. Crowley and J. Sixt, *Stably diffeomorphic manifolds and $l_{2q+1}(\mathbb{Z}[\pi])$*, Forum Math. **23** (2011), 483–538.
2. I. Hambleton and M. Kreck, *Cancellation of hyperbolic forms and topological four-manifolds*, J. Reine Angew. Math. **443** (1993), 21–47.
3. M. Kreck, *Surgery and Duality*, Ann. of Math. **149** (1999), 707–754.
4. C. T. C. Wall, *Surgery on compact manifolds*, Second edition. Edited and with a foreword by A. A. Ranicki. Mathematical Surveys and Monographs, **69**. American Mathematical Society, Providence, RI, 1999.

Problem 11: The Q-form Conjecture

presented by Diarmuid Crowley

This problem follows on from the previous problem on genus bounds for cancellation. We use the same notation but now for simplicity *we assume that all manifolds are closed*, as well as connected. Recall that $\pi = \pi_1(M)$ and $w_1 = w_1(M)$ are the fundamental group and orientation character of M and that the L-group $L_{2q+1}(\mathbb{Z}[\pi], w_1)$ acts on the stable class of M via Wall realisation:

$$\mathscr{S}^{\mathrm{st}}(M) \times L_{2q+1}(\mathbb{Z}[\pi], w_1) \to \mathscr{S}^{\mathrm{st}}(M)$$

Given that the L-groups $L_{2q+1}(\mathbb{Z}[\pi], w_1)$ have been intensively studied, we focus on the quotient of the action above and suggest the following

Problem. *Determine $\mathscr{S}^{\mathrm{st}}(M)/L_{2q+1}(\mathbb{Z}[\pi],w_1)$, the set of orbits of the action of $L_{2q+1}(\mathbb{Z}[\pi],w_1)$ on the stable class.*

Below we present a conjectural solution to this problem, along with some evidence for the conjecture. To do this, we assume the reader is familiar with the setting of modified surgery; the details are found in [3, §1]. Let $\xi\colon B \to BO$ be a fibration over a connected space B. An m-dimensional *normal smoothing* in (B,ξ) is a pair $(M,\bar{\nu})$, where M is a compact m-manifold and $\bar{\nu}\colon M \to B$ is a lift of the stable normal bundle of M, $\bar{\nu}_M$, as in the following diagram:

$$\begin{array}{ccc} & & B_M^{q-1} \\ & \overset{\bar{\nu}}{\nearrow} & \big\downarrow \xi_M^{q-1} \\ M & \xrightarrow{\nu_M} & BO \end{array}$$

If $\bar{\nu}$ is k-connected then $(M,\bar{\nu})$ is called a *normal* $(k-1)$-*smoothing* over (B,ξ) and if in addition ξ is k-coconnected, then the fibration ξ represents the normal $(k-1)$-type of M, which we denote by $\xi_M^{k-1}\colon B_M^{k-1} \to BO$. There is a well-defined notion of (B,ξ)-diffeomorphism, that is diffeomorphism preserving (B,ξ)-structures up to equivalence and also (B,ξ)-bordism of closed (B,ξ)-manifolds; the corresponding bordism group is denoted $\Omega_m(B;\xi)$. For an m-dimensional normal k-smoothing $(M,\bar{\nu})$ over (B,ξ), we let $[M,\bar{\nu}] \in \Omega_m(B;\xi)$ denote its bordism class and define

$$\begin{aligned} &\mathrm{NS}_{\xi}(M,\bar{\nu}) \\ :=&\{(M',\bar{\nu}') \mid \chi(M') = \chi(M),\ [M'\bar{\nu}'] = [M,\bar{\nu}]\}/(B,\xi)\text{-diffeomorphism}, \end{aligned}$$

to be the set of (B,ξ)-diffeomorphism classes of m-dimensional normal k-smoothings which are bordant to $(M,\bar{\nu})$ and have the same Euler characteristic as M.

For $m = 2q$ and $k = q-1$, a foundational result of Kreck [3, Corollary 3] states that if $(M_0,\bar{\nu}_0)$ and $(M_1,\bar{\nu}_1) \in \mathrm{NS}_{\xi}(M,\bar{\nu})$ then M_0 and M_1 are stably diffeomorphic. Combined with [2, Lemma 2.3], we obtain for $(B,\xi) = (B_M^{q-1},\xi_M^{q-1})$ that the forgetful map

$$\mathrm{F}\colon \mathrm{NS}_{\xi_M^{q-1}}(M,\bar{\nu}) \to \mathscr{S}^{\mathrm{st}}(M), \quad (M',\bar{\nu}') \mapsto M',$$

is onto. Moreover $\mathrm{aut}(\xi_M^{q-1})$, the group of fibre homotopy classes of fibre homotopy automorphisms of ξ_M^{q-1}, acts by post-composition $\mathrm{NS}_{\xi_M^{q-1}}(M,\bar{\nu})$ and by [4, Theorem 7.5], the universal properties of the Moore-Postnikov factorisation $\nu_M = \xi_M^{q-1} \circ \bar{\nu}$ ensure that the induced map

$$\mathrm{F}_{\mathrm{aut}(\xi_M^{q-1})}\colon \mathrm{NS}_{\xi_M^{q-1}}(M,\bar{\nu})/\mathrm{aut}(\xi_M^{q-1}) \to \mathscr{S}^{\mathrm{st}}(M) \tag{1}$$

is a bijection. Hence it makes sense to study $\mathrm{NS}_{\xi_M^{q-1}}(M,\bar{\nu})$ together with the action of $\mathrm{aut}(\xi_M^{q-1})$, in order to learn about $\mathscr{S}^{\mathrm{st}}(M)$.

We next define the key new invariant we shall use to formulate our conjectures and this is the *extended quadratic form* of $(M, \bar{\nu})$. Given $\xi\colon B \to BO$, we let $\pi = \pi_1(B)$ be the fundamental group of B and w_1 the orientation character of ξ. We fix a base-point in B and a local orientation of ξ at the base-point. We all assume that all normal smoothings $(M, \bar{\nu})$ over (B, ξ) are base-point preserving and that $\bar{\nu}_*\colon \pi_1(M) \to \pi_1(B)$ an isomorphism, which we use to identify $\pi_1(M) = \pi$. The local orientation of ξ gives M a local orientation and hence defines a fundamental class $[M] \in H_{2q}(M; \mathbb{Z}_{w_1})$ and also the equivariant intersection form $\lambda_{(M,\bar{\nu})}\colon H_q(M; \mathbb{Z}[\pi]) \times H_q(M; \mathbb{Z}[\pi]) \to \mathbb{Z}[\pi]$.

For every positive integer n, Ranicki [6, §10], defines a *quadratic form parameter* over the twisted group ring $(\mathbb{Z}[\pi], w_1)$, $Q_n(\xi)$, which is associated to the stable spherical fibration underlying the stable bundle ξ. In general, if we fix a ring with involution Λ, then a quadratic form parameter over Λ is a triple $Q = (Q, \mathrm{h}, \mathrm{p})$, written

$$Q = (Q \xrightarrow{\mathrm{h}} \Lambda \xrightarrow{\mathrm{p}} Q).$$

Here Q is an abelian group together with a *quadratic* action of Λ and h and p are equivariant homomorphisms with respect to the conjugation of $\mathbb{Z}[\pi]$ on itself, which satisfy certain equations. We refer the reader to [6, §10] for the details and point out that a similar but more general notion of quadratic form parameter can be found in the work of Baues [1]. We also mention that there is an exact sequence of abelian groups (see [6, p. 37])

$$Q_{(-1)^n}(\mathbb{Z}[\pi]) \to Q_\xi(n) \to H_n(B; \mathbb{Z}[\pi]) \to 0,$$

where $Q_{(-1)^n}(\mathbb{Z}[\pi])$ is the classical Q-group appearing in Wall's quadratic form [7, Theorem 5.2] and where the homomorphism $Q_\xi(n) \to H_n(B; \mathbb{Z}[\pi])$ is equal to the quotient map $Q_\xi(n) \to Q_\xi(n)/\mathrm{Im}(\mathrm{p})$.

An *extended quadratic form* over a form parameter Q, briefly a Q-form, is a triple

$$(H, \lambda, \mu),$$

where H is a Λ-module, $\lambda\colon H \times H \to \Lambda$ is a sesqui-linear form and $\mu\colon H \to Q$ is a quadratic refinement of λ which means in part that for all $x, y \in H$ we have

$$\mu(x+y) = \mu(x) + \mu(y) + \mathrm{p}(\lambda(x,y)) \quad \text{and} \quad \lambda(x,x) = \mathrm{h}(\mu(x)).$$

The *linearisation* of (H, λ, μ) is the Λ-module homomorphism

$$S(\mu)\colon H \to Q/\mathrm{Im}(\mathrm{p}), \quad x \mapsto [\mu(x)].$$

If (H, λ, μ) and (H', λ', μ') are Q-forms then an isometry between them is an λ-module isomorphism preserving the sesquilinear forms and their quadratic refinements and we write

$$\mathrm{Hom}_\Lambda(Q)$$

for the set of isometry classes of Q-forms on finitely generated Λ-modules.

The theory of [6, §10] ensures that a normal $(q-1)$-smoothing $\bar{\nu}\colon M \to B$ over a stable bundle $\xi\colon B \to BO$ defines a $Q_\xi(q)$-form

$$\mu(M,\bar{\nu}) := \big(H_q(M;\mathbb{Z}[\pi]), \lambda_{(M,\bar{\nu})}, \mu(\bar{\nu})\big),$$

where $(H_q(M;\mathbb{Z}[\pi]), \lambda_{(M,\bar{\nu})})$ is the equivariant intersection form of $(M,\bar{\nu})$ and the map $\mu(\bar{\nu})\colon H_q(M;\mathbb{Z}[\pi]) \to Q_\xi(q)$ is a quadratic refinement of $\lambda_{(M,\bar{\nu})}$, which has linearisation

$$S(\mu(\bar{\nu})) = \bar{\nu}_*\colon H_q(M;\mathbb{Z}[\pi]) \to H_q(B;\mathbb{Z}[\pi]).$$

It follows from the definitions that if $f\colon M_0 \to M_1$ is a (B,ξ)-diffeomorphism between $2q$-dimensional $(q-1)$-smoothings $(M_0,\bar{\nu}_0)$ and $(M_1,\bar{\nu}_1)$ over (B,ξ), then the induced homomorphism $f_*\colon H_q(M_0;\mathbb{Z}[\pi]) \to H_q(M_1;\mathbb{Z}[\pi])$ is an isometry of $Q_\xi(q)$-forms. It follows that there is a well-defined map

$$\mathrm{NS}_\xi(M,\bar{\nu}) \to \mathrm{Hom}_{(\mathbb{Z}[\pi],w_1)}(Q_\xi(q)), \quad (M,\bar{\nu}) \mapsto \mu(M,\bar{\nu}).$$

Now Wall realisation also defines an action of $L_{2q+1}(\mathbb{Z}[\pi],w_1)$ on $\mathrm{NS}(M,\bar{\nu})$ and it is elementary to check that the isometry class of the extended quadratic forms is invariant under this action. Hence the map above descends to define the map

$$\mu\colon \mathrm{NS}_\xi(M,\bar{\nu})/L_{2q+1}(\mathbb{Z}[\pi],w_1) \to \mathrm{Hom}_{(\mathbb{Z}[\pi],w_1)}(Q_\xi(q)). \tag{2}$$

At last, we can state the first version of the Q-form Conjecture.

Conjecture A. (The Q-form Conjecture for normal smoothings) If $q \geq 3$, then the map μ of (2) is injective; i.e. if $q \geq 3$ and $(M_0,\bar{\nu}_0)$ and $(M_1,\bar{\nu}_1)$ are $2q$-dimensional (B,ξ)-bordant normal $(q-1)$-smoothings with equal Euler characteristic and isometric $Q_\xi(q)$-forms, then $(M_0,\bar{\nu}_0)$ and $(M_1,\bar{\nu}_1)$ differ by the action of $L_{2q+1}(\mathbb{Z}[\pi],w_1)$.

Given the bijection of (1), Conjecture A allows us to formulate a conjectural determination of the stable class of M, at least for $q \geq 3$. For this, note that $\mathrm{aut}(\xi_M^{q-1})$ acts on $Q_\xi(q)$ by automorphisms and hence on $\mathrm{Hom}_{(\mathbb{Z}[\pi],w_1)}(Q_\xi(q))$ by post-composition. Thus we obtain the map

$$\mu_{/\mathrm{aut}(\xi)}\colon \mathrm{NS}_\xi(M,\bar{\nu})/\big(L_{2q+1}(\mathbb{Z}[\pi],w_1) \times \mathrm{aut}(\xi)\big) \to \mathrm{Hom}_{(\mathbb{Z}[\pi],w_1)}(Q_\xi(q))/\mathrm{aut}(\xi),$$

which is a bijection if Conjecture A holds. Since the bijection of (1) is equivariant with respect to the action of $L_{2q+1}(\mathbb{Z}[\pi],w_1)$, when $\xi = \xi_M^{q-1}$ is a representative of the normal $(q-1)$-type of M, the map $\mu_{/\mathrm{aut}(\xi)}$ induces another map, also denoted $\mu_{/\mathrm{aut}(\xi)}$,

$$\mu_{/\mathrm{aut}(\xi)}\colon \mathscr{S}^{\mathrm{st}}(M)/L_{2q+1}(\mathbb{Z}[\pi],w_1) \to \mathrm{Hom}_{(\mathbb{Z}[\pi],w_1)}(Q_{\xi_M^{q-1}}(q))/\mathrm{aut}(\xi_M^{q-1}). \tag{3}$$

Conjecture B. (The Q-form Conjecture for the stable class) If $q \geq 3$, then the map $\mu_{/\mathrm{aut}(\xi)}$ of (3) is injective; i.e. if $q \geq 3$ and we have $M_0, M_1 \in \mathscr{S}^{\mathrm{st}}(M)$ then

$M_0 \cong \rho M_1$ for some $\rho \in L_{2q+1}(\mathbb{Z}[\pi], w_1)$ if and only if for $i = 0, 1$, there are normal $(q-1)$-smoothings $\bar{\nu}_i \colon M_i \to B_M^{q-1}$ such that $\mu(M_0, \bar{\nu}_0)$ and $\mu(M_1, \bar{\nu}_1)$ are isometric $Q_{\xi_M^{q-1}}(q)$-forms.

We conclude by briefly discussing Conjectures A and B. Notice that since the map μ of Conjecture A is aut(ξ_M^{q-1})-equivariant, Conjecture A implies Conjecture B. Both conjectures are inspired by the classification of the ℓ-monoids in [2] and to the best of our knowledge, both conjectures are consistent with the extensive literature on classifying $2q$-manifolds for $q \geq 3$. In addition, Conjecture A (hence Conjecture B) has been proven by Nagy in the case where q is even, $\pi = \{e\}$ and $H_q(B;\mathbb{Z})$ is torsion free [5].

At times, it has been tempting to propose Conjectures A and B as *hypotheses*; i.e. as sign posts for organising work on the classification of the stable class, as opposed to statements believed to be true. However, the resilience of these statements to date encourages their proposal as conjectures in the usual sense. This is also consistent with history of the exploration of the stable class, where the "unreasonable effectiveness" of the (equivariant) intersection form has often been observed.

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