Graphical neighborhoods of spatial graphs

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Abstract We give a definition of a graphical neighborhood of a spatial graph which generalizes the tubular neighborhood of a link in S^3. Furthermore we prove existence and uniqueness of graphical tubular neighborhoods.

1 Introduction

In this paper we give a precise definition of the notion of a spatial graph. In our opinion the goals of any good definition in knot theory and related subjects should be two-fold:

1. the definition should be flexible enough to encompass all "physical" objects that one has in mind,
2. the definition should be rigid enough to allow for a reasonable theory.

At least to our taste the definitions of a spatial graph used in the literature are often vague or fall short of (1) or (2). For example, a spatial graph is often defined as "an embedded (topological) graph in S^3". Since a topological graph (i.e. a finite 1-dimensional CW-complex) is in general not a manifold it is not entirely clear what the word "embedded" should really mean in this context.

We start out with a precise definition of a "spatial graph". We hope that the reader will be convinced that it satisfies (1). Afterwards we will attempt to show "spatial graphs" in our sense satisfy (2). More precisely, we will show that spatial graphs admit a graphical neighborhood, which is unique in an appropriate sense, which makes

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it possible to sensibly study spatial graphs. These results had been announced, without proofs, in [2].

Before we turn to the definition of a spatial graph we recall the notion of an abstract graph.

Definition 1. An *abstract graph* G is a triple (V,E,φ) where V is a finite non-empty set, E is a finite set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *the vertices of* G and the elements of E are called the *edges of* G. Furthermore, given $e \in E$ the elements of $\varphi(e) \subset V$ are called the *endpoints of* e.

We turn to topology.

Definition 2. An *arc* in S^3 is a subset E of S^3 for which there exists a map $\varphi\colon [0,1] \to S^3$ with the following properties:

1. the map φ is smooth, i.e. all derivatives are defined on the open interval $(0,1)$ and they extend to continuous maps on the closed interval $[0,1]$ that we also call derivatives,
2. the first derivative $\varphi'(t)$ is non-zero for all $t \in [0,1]$,
3. the restriction of φ to $(0,1)$ is injective,
4. $\varphi((0,1)) \cap \varphi(\{0,1\}) = \varnothing$ and
5. $\varphi((0,1)) = E$.

Given an arc E as above we refer to φ and $\varphi(1)$ as the *endpoints of* E. (Note that the endpoints of E do *not* lie in E.)

Fig. 1 Illustration of arcs with one or two endpoints.

Definition 3. A *spatial graph* G is a pair (V,E) with the following properties:

1. V is a finite non-empty subset of S^3.
2. E is a subset of S^3 with the following properties:
   a. E is disjoint from V,
   b. E has finitely many components,
   c. each component of E is an arc and the endpoints of each arc lie in V.

We refer to the points in V as the *vertices of* G and we refer to the components of E as the *edges of* G. Furthermore, given a spatial graph $G = (V,E)$ we write $|G| = V \cup E \subset S^3$.

Note that for a spatial graph (V,E) as above the corresponding triple

$$(V, \pi_0(E), \varphi(\text{arc}) := \text{endpoints of the arc})$$

is an abstract graph. Also note that a spatial graph admits an obvious CW-structure, i.e. the underlying topological space is indeed a topological graph in the above sense.

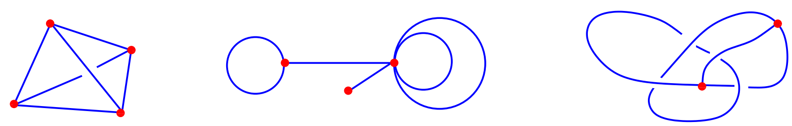

Fig. 2 Examples of spatial graphs.

As mentioned above, a good definition in topology should be flexible enough to capture the examples one has in mind, but it should also be rigid enough to allow for a sensible theory. One can easily convince oneself that the Figure 2 gives three examples of spatial graphs, so our definition seems to be broad enough to capture all "reasonable" examples. On the other hand, one of the key tools in the study of knots and links is the tubular neighborhood. The goal of the remainder of this paper is to show that our notion of a spatial graph is rigid enough to define an analogue of the tubular neighborhood of a knot or link.

More precisely, in Section 3 we introduce the notion of a "graphical neighborhood" of a spatial graph. The following three theorems summarize the key properties of graphical neighborhoods.

Theorem 1. *Every spatial graph admits a graphical neighborhood.*

Theorem 2. *Let $G = (V,E)$ be a spatial graph and let N be a graphical neighborhood for G.*

1. *N contains $|G| = V \cup E$ in the interior $N^o = N \setminus \partial N$ of N,*
2. *$|G|$ is a deformation retract of N,*
3. *∂N is a deformation retract of $N \setminus |G|$,*
4. *the exterior $E_G = S^3 \setminus N^o$ is a compact 3-dimensional manifold that is a deformation retract of $S^3 \setminus |G|$.*

Since every compact 3-dimensional manifold admits a finite CW-structure we obtain the following corollary to Theorem 2 (4).

Corollary 1. *Given a spatial graph G the fundamental group $\pi_1(S^3 \setminus G)$ is finitely presented and all homology groups $H_*(S^3 \setminus G)$ are finitely generated.*

The following theorem concludes our list of three theorems dealing with the key properties of graphical neighborhoods.

Theorem 3. *Any two graphical neighborhoods of a given spatial graph are equivalent (see Section 5 for the precise statement).*

The following corollary is an immediate consequence of Theorem 3.

Corollary 2. *Let G be a spatial graph and let N be a graphical neighborhood for G. The diffeomorphism type of the exterior $E_G := S^3 \setminus N^o$ does not depend on the choice of a graphical neighborhood N.*

Definition 4. We say that two spatial graphs $G = (V,E)$ and $G' = (V',E')$ are *equivalent* if there exists an orientation-preserving homeomorphism $\Psi\colon S^3 \to S^3$ with $\Psi(V) = V'$, $\Psi(E) = E'$ and which restricts to a diffeomorphism $S^3 \setminus V \to S^3 \setminus V'$.

The following lemma relates graphical neighborhoods of spatial graphs.

Lemma 1. *Let $G = (V,E)$ and $G' = (V',E')$ be two spatial graphs.*

1. *Let $\Psi\colon S^3 \to S^3$ be an orientation-preserving homeomorphism with $\Psi(V) = V'$ and $\Psi(E) = E'$ and which restricts to a diffeomorphism $S^3 \setminus V \to S^3 \setminus V'$. If N is a graphical neighborhood for G, then $\Psi(N)$ is a a graphical neighborhood of G'.*
2. *If G and G' are equivalent, then the exteriors of G and G' are diffeomorphic.*

Proof. The lemma follows immediately from the definitions and the following basic fact: if $f\colon \overline{B}^3 \to \mathbb{R}^3$ is a homeomorphism onto its image such that the restriction of f to $\overline{B}^3 \setminus \{0\}$ is a diffeomorphism onto its image, then $f(\overline{B}^3)$ is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to the closed 3-ball, even though $f\colon \overline{B}^3 \to f(\overline{B}^3)$ is not necessarily a diffeomorphism. □

We conclude this introduction with a few remarks:

Remark 1. 1. It is also interesting to consider unions $G \sqcup L$ where G is a spatial graph and L is a link. A graphical neighborhood in this setting is defined as the union $Z \sqcup W$ where Z is a graphical neighborhood for Z and W is a tubular neighborhood for the submanifold L. All of the previous results also hold in that more general context.
2. The theory of graphical neighborhoods works basically the same for spatial graphs in any closed orientable 3-manifold. For simplicity's sake we only deal with the most important case, namely the case of spatial graphs in S^3.
3. The reader might be surprised to note how much effort we spend in our proofs on ensuring that maps are actually smooth and not just continuous. Even though in 3-dimensional topology we have Moise's Theorem which says that any two 3-manifolds that are homeomorphic are also diffeomorphic, this does not imply that analogous statements hold if one wants to keep more control over subsets. For example, consider the two three spatial graphs G, G' and G'' that are shown in Figure 3. They are equivalent in our sense, but there is no self-diffeomorphism h of S^3 that turns any of the spatial graphs into any of the other two spatial graphs.

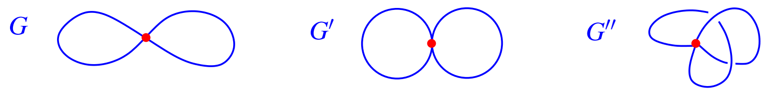

Fig. 3

4. Some authors on spatial graphs write that they work in the PL-category and use the notion of a regular neighborhood, that is for example discussed in [7, Chapter 3], [4, p. 7f] or [3, Chapter III.B]. A regular neighborhood is a much more general concept than a graphical neighborhood. The regular neighborhood of a spatial graph is unique in an appropriate sense (see [7, Theorem 3.24] or alternatively [3, Theorem II.16n]) and by [7, Corollary 3.30] the analogue of Theorem 2 (2) holds. With some effort one can use [7, Corollary 3.18] to show that (3), and thus also (4), are satisfied. But it takes some dedication to understand what a "regular neighborhood" is really supposed to be and at first glance it is not entirely clear that the various definitions given in [7, 3, 4] are actually consistent. At least to the authors it seems like working in the smooth category and working with our graphical neighborhoods is esthetically more pleasing and "closer to reality". On the other hand, for implementing algorithms it seems more reasonable to work in the PL-category.

The paper is organized as follows. In Section 2 we recall the existence and uniqueness of tubular neighborhoods of 1-dimensional submanifolds of 3-dimensional manifolds. In Section 3 we prove Theorem 1 and in Section 4 we provide a proof for Theorem 2. Finally in Section 5 we deal with the hardest part of the paper, namely we prove Theorem 3.

2 Tubular neighborhoods

Before we get started with the technical details we would like to introduce some conventions, definitions and notations:

1. By a manifold we mean a topological manifold equipped with a smooth structure. Every manifold is assumed to be compact and orientable unless we say otherwise. Throughout this paper we try to follow the definitions and conventions of [9].
2. Given a homotopy $F\colon X\times[0,1]\to Y$ and $t\in[0,1]$ we denote by $F_t\colon X\to Y$ the map that is given by $F_t(x)=F(x,t)$.
3. Let M be a manifold. A *diffeotopy of M* is a smooth map $F\colon M\times[0,1]\to M$ such that each $F_t\colon M\to M$ is a diffeomorphism.
4. Given a topological space X and a subset A we denote by A^o its interior and we denote by $\overline{A}$ its closure.
5. As usual we make the identification $S^3=\mathbb{R}^3\cup\{\infty\}$.
6. Given $r\in\mathbb{R}_{\geq 0}$ we denote by $B^3_r\subset\mathbb{R}^3$ the open ball of radius r around the origin. We write $S^2_r=\partial\overline{B^3_r}$.

7. Given $0 < s < t$ we identify $S^2 \times [s,t]$ with $\{z \in \mathbb{R}^3 \,|\, s \leq \|z\| \leq t\} = \overline{B_t^3} \setminus B_s^3$ in the obvious way.

In the next section we will introduce the concept of a graphical neighborhood of a spatial graph. It will build on the notion of a tubular neighborhood of a 1-dimensional submanifold.

Definition 5. 1. Let M be a 3-manifold. A *proper submanifold* of M is a compact submanifold C of M with $\partial C = C \cap \partial M$ and that meets ∂M transversally.
2. A *tubular neighborhood* for a proper 1-dimensional submanifold C of M is an embedding $F\colon C \times \overline{B}^2 \to M$ with the following properties:
   a. we have $F(\partial C \times \overline{B}^2) = F(C \times \overline{B}^2) \cap \partial M$,
   b. the image $F(C \times \overline{B}^2)$ is a submanifold of M with corners (see [9, p. 30] and [1, Chapter 86] for the definition of a submanifold with corners),
   c. there exists a collar neighborhood $\partial M \times [0,1]$ such that the tubular neighborhood of $C \cap (\partial M \times [0,1])$ is a product, i.e. we have

$$(\partial M \times [0,1]) \cap \Phi(C \times \overline{B}^2) \;=\; \Phi((\partial M \cap C) \times \overline{B}^2) \times [0,1].$$

Remark 2. Note that a tubular neighborhood N of a 1-dimensional submanifold C with non-empty boundary is a submanifold of M with non-empty corners, in particular N is strictly speaking not a smooth submanifold. Fortunately in practice this is not a problem. For example we are mostly interested in considering the exterior $E_C := M \setminus N^o$ where N^o is the interior of N. The exterior E_C is a smooth manifold with corner, but by "straightening of corners", see [9, Proposition 2.6.2] we can view $E_C = M \setminus N^o$ as a smooth manifold in a canonical way.

The following two theorems show the existence and uniqueness of tubular neighborhoods in our setting.

Theorem 4. *Every proper 1-dimensional submanifold C of every 3-manifold M admits a tubular neighborhood $F\colon C \times \overline{B}^2 \to M$.*

Proof. This theorem is basically a consequence of [9, Theorem 2.3.3]. We have the extra condition (2c) in the definition of a tubular neighborhood, which is not explicitly mentioned in [9], but using [9, Proposition 1.5.6] one can see that this condition can also be arranged. □

Theorem 5. *[9, Chapter 2.5] Let M be a 3-manifold and let C be a proper 1-dimensional submanifold of M. If $F, G\colon C \times \overline{B}^2 \to M$ are two tubular neighborhoods of C, then there exists a diffeotopy $\Phi\colon M \times [0,1] \to M$ rel C with the following properties:*

1. *$\Phi_t = \mathrm{id}_M$ for small t,*
2. *the restriction of Φ_1 to $F(C \times \overline{B}^2)$ defines a fiber-preserving diffeomorphism from $F(C \times \overline{B}^2)$ to $G(C \times \overline{B}^2)$.*

3 Definition and existence of graphical neighborhoods

Definition 6. Let $G = (V,E)$ be a spatial graph.

1. We say that an orientation-preserving map $\Theta\colon \overline{B^3_R} \to S^3$ is *transverse at* $v \in V$ if Φ is an embedding, if $\Phi = v$ and if for each $r \in (0,R]$ the image $\Theta(S^2_r)$ is a submanifold of $S^3 \setminus V$ that is transverse to the submanifold E of $S^3 \setminus V$.
2. A *small neighborhood of* V is a compact 3-dimensional submanifold X of S^3 with components $\{X_v\}_{v\in V}$ such that for each $v \in V$ there exists a map $\Theta\colon \overline{B^3_R} \to S^3$ that is transverse to v with $\Theta(\overline{B^3_r}) = X_v$.

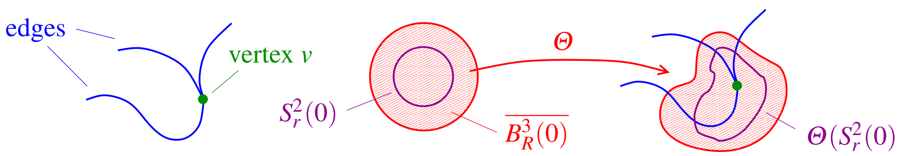

Fig. 4 Illustration of a small neighborhood.

We point out that $E \cap (S^3 \setminus X^o)$ is a proper submanifold of $S^3 \setminus X^o$. Now we can define a graphical neighborhood of a spatial graph.

Definition 7. Let $G = (V,E)$ be a spatial graph. A *graphical neighborhood of* (V,E) is a subset N of S^3 that can be written as a union $N = X \cup Y$ where X is a small neighborhood of V and Y is a tubular neighborhood of $E \cap (S^3 \setminus X^o)$ in $S^3 \setminus X^o$.

Remark 3. As remarked above, after "straightening of corners", we can view $S^3 \setminus N^o$ as a smooth manifold in a canonical way.

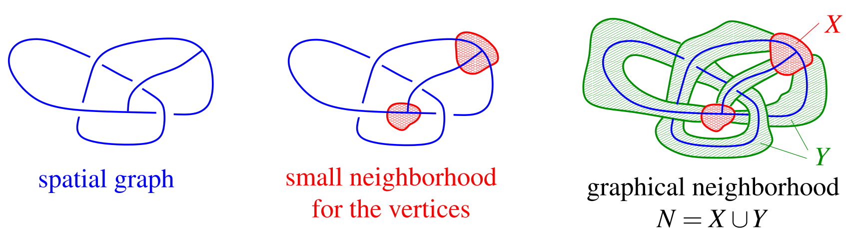

Fig. 5

In the following we will see that every spatial graph admits a graphical neighborhood and that graphical neighborhoods have properties that are very similar to the properties of tubular neighborhoods.

We now prove that every spatial graph admits a graphical neighborhood. This statement should be viewed as the analogue of Theorem 4.

Theorem 1. *Every spatial graph admits a graphical neighborhood.*

Proof. Let $G=(V,E)$ be a spatial graph. First we show that V admits a small neighborhood. Since we can always shrink small neighborhoods it suffices to show that each vertex admits a small neighborhood. Let v be a vertex. To simplify the notation we might as well assume that $v=0\in\mathbb{R}^3\cup\{\infty\}$. By definition of a spatial graph there exist smooth injective maps $\varphi_i\colon [0,\frac{1}{4}]\to S^3$, $i=1,\dots,k$, with the following properties:

1. For each $i\in\{1,\dots,k\}$ we have $\varphi_i=0$ and $\varphi_i(\frac{1}{4})\neq 0$,
2. for each $i\in\{1,\dots,k\}$ we have $\varphi_i'(t)\neq 0$ for all $t\in[0,\frac{1}{4}]$,
3. there exists an $s>0$ such that $\overline{B^3_s}\cap E=\overline{B^3_s}\cap\Big(\bigcup\limits_{i=1}^{k}\varphi_i([0,\frac{1}{4}])\Big)$.

Let $i\in\{1,\dots,k\}$. Since the φ_i are smooth and since $\varphi_i=0$ we see that we can write $\varphi_i(t)=t\cdot\varphi_i'+\nu(t)$ where $\lim_{t\to 0}\frac{\|\nu(t)\|}{t}=0$. It follows from this observation and the fact that $\varphi_i'\neq 0$ that there exists an $s_i\in(0,\frac{1}{4})$ such that $\varphi_i(t)\cdot\varphi_i'(t)>0$ for all $t\in(0,s_i)$. We set $R:=\frac{1}{2}\cdot\min\{s,s_1,\dots,s_k\}$. It is now straightforward to verify that $\Phi=\mathrm{id}_{\overline{B^3_R}}$ has the desired properties.

Now let X be a small neighborhood for V. By the Tubular Neighborhood Theorem 4 there exists a tubular neighborhood Y of the proper submanifold $E\cap(S^3\setminus X^o)$ of $S^3\setminus X^o$. Then $X\cup Y$ is a graphical neighborhood of G. □

4 Properties of Graphical Neighborhoods

In this section we want to provide the proof for Theorem 2. Whereas statements (1) and (4) of Theorem 2 are basically obvious, the proof of the remaining two statements requires a little effort. We will need the following proposition.

Proposition 1. *Let $G=(V,E)$ be a spatial graph and let X be a small neighborhood for V. Let $v\in V$. We denote by X_v the corresponding component of X.*

1. *There exist points $P_1,\dots,P_k\in S^2$ and a homeomorphism $\Theta\colon\overline{B^3}\to X_v$ such that*

$$\Theta\Big(\bigcup_{i=1}^{k}\{r\cdot P_i\in\overline{B^3}\,|\,r\in[0,1]\}\Big)\;=\;E\cap X_v$$

and such that each $\Theta(S^2_t)$ is transverse to E.

2. *Given any neighborhood U of v there exist points $P_1,\dots,P_k\in S^2$, some $\eta>0$ with $\Theta(\overline{B^3_\eta})\subset U$ and an orientation-preserving diffeomorphism $\Theta\colon\overline{B^3}\to X_v$ such that*

$$\Theta^{-1}(E)\cap(S^2\times[\eta,1])\;=\;\bigcup_{i=1}^{k}\{P_k\}\times[\eta,1]$$

and such that each $\Theta(S_t^2)$ is transverse to E.

Definition 8. Let $a < b$ be real numbers.

1. A *string* in $S^2 \times [a,b]$ is a connected 1-dimensional submanifold with one boundary point on $S^2 \times \{a\}$ and one boundary point on $S^2 \times \{b\}$.
2. A string is called *linear* if it is of the form $\{x\} \times [a,b]$ for some $x \in S^2$.
3. A *collection of strings* is defined as a finite set of disjoint strings.
4. We call a collection of strings E in $S^2 \times [a,b]$ *unknotted* if there exists a diffeomorphism $\Phi\colon S^2 \times [a,b] \to S^2 \times [a,b]$ with $\Phi_{S^2\times\{a\}} = \mathrm{id}$ such that for every $t \in [a,b]$ the submanifold $\Phi(E)$ is transverse to $S^2 \times \{t\}$.

Lemma 2. *Let E be a collection of strings in $S^2 \times [a,d]$. We suppose that E is transverse to $S^2 \times \{t\}$ for all $t \in [a,d]$. Let $c \in [a,d]$. We write $b = \frac{a+c}{2}$. There exists a level-preserving diffeomorphism $\phi\colon S^2 \times [a,d] \to S^2 \times [a,d]$ with $\phi|_{S^2\times[a,b]} = \mathrm{id}$ such that every component of $\phi(E) \cap (S^2 \times [c,d])$ is a linear string.*

Proof. We enumerate the components of E by $E_1,\dots,E_n$ and write $v_i = E_i \cap S^2 \times \{a\}$. For notational simplicity we assume the case that $[a,d] = [0,2]$. We pick parametrizations $\alpha_i\colon [0,2] \to E_i$ for each i. Since E_i intersects each $S^2 \times \{t\}$ transversally we can reparametrize α_i to a smooth map $\tilde{\alpha}_i\colon [0,2] \to S^2 \times [0,2]$ which is level preserving, i.e. such that for any $t \in [0,2]$ we have $\tilde{\alpha}_i(t) \in S^2 \times \{t\}$. In other words, we can write $\tilde{\alpha}_i(t) = (\beta_i(t),t)$ for some smooth map $\beta_i\colon [0,2] \to S^2$. Thus we obtain a map:

$$\begin{aligned} h\colon V \times [0,2] &\to S^2 \times [0,2] \\ (v_i,t) &\mapsto (\beta_i(t),t). \end{aligned}$$

By the diffeotopy extension theorem [9, Theorem 2.4.2], we obtain a level preserving diffeomorphism $\phi\colon S^2 \times [0,2] \to S^2 \times [0,2]$ extending the map h. A quick look at the proof of the diffeotopy extension theorem [9, Theorem 2.4.2] shows that ϕ can be chosen to be the identity on $S^2 \times \{0\}$. If $c = 0$, then ϕ^{-1} is the desired diffeomorphism. If $c \neq 0$, then again by notational convenience we assume $c = 1$. We take a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ which is f is monotonously increasing, with $f(t) = 0$ for $t \le \frac{1}{2}$ and with $f(t) = 1$ for $t \ge 1$. Since ϕ is level-preserving, there is a smooth map $\varphi\colon S^2 \times [0,2] \to S^2$ such that $\phi(x,t) = (\varphi(x,t),t)$. Note that for every $t \in [0,2]$ the map $\varphi(\,\cdot\,,t)$ is a diffeomorphism. The map $\widetilde{\phi}(x,t) := (\varphi(x,t\cdot f(t)),t)$ is a diffeomorphism with inverse $(\varphi(x,t\cdot f(t))^{-1},t)$. Moreover, for $t \in [0,\frac{1}{2}]$ we have $f(t) = 0$ and hence $\widetilde{\phi} = \mathrm{id}$ and for $t \in [1,2]$ we have $f(t) = 1$ and hence $\widetilde{\phi}^{-1}(\,\cdot\,,t) = \phi^{-1}(\,\cdot\,,t)$. Therefore in $S^2 \times [1,2]$ the map $\widetilde{\phi}$ maps linear strings to E. This shows that $\widetilde{\phi}^{-1}$ is the desired diffeomorphism. □

Proof (Proof of Proposition 1). Let $G = (V,E)$ be a spatial graph and let X be a small neighborhood for V. Given $v \in V$ and $X_v \subset X$ we can and will pick an orientation-preserving diffeomorphism $\Theta\colon \overline{B_R^3} \to X_v$ with $\Theta = v$ and such that for each $r \in (0,R]$ the image $\Theta_i(S_r^2)$ is a submanifold of $S^3 \setminus V$ that is transverse to the submanifold E of $S^3 \setminus V$.

1. We pick a strictly decreasing sequence $R = a_1, a_2, a_3, \ldots$ of real numbers with $\lim_{i\to\infty} a_i = 0$ and we iteratively apply Lemma 2 (with $c = a$) to corresponding strings in $S^2 \times [a_{i+1}, a_i] = \overline{B^3_{a_i}} \setminus B^3_{a_{i+1}}$. We combine the resulting diffeomorphisms to obtain the desired homeomorphism.
2. Let U be neighborhood U of v. We pick $a < c < R$ such that $\Theta(S^2 \times [a,c]) \subset U$. We apply Lemma 2 and obtain a map $\phi\colon S^2 \times [a,R] \to S^2 \times [a,R]$. Since ϕ is the identity in a neighborhood of $S^2 \times \{a\}$ we see that ϕ extends to a smooth map $\phi\colon \overline{B^3_R} \to \overline{B^3_R}$ that is the identity on $\overline{B^3_a}$. The map $\Theta \circ \phi\colon \overline{B^3_R} \to X_v$ has the desired properties. □

Now we can finally prove Theorem 2.

Theorem 2. *Let G be a spatial graph and let N be a graphical neighborhood for G.*

1. *N contains $|G|$ in the interior $N^o = N \setminus \partial N$ of N,*
2. *$|G|$ is a deformation retract of N,*
3. *∂N is a deformation retract of $N \setminus |G|$,*
4. *the exterior $E_G = S^3 \setminus N^o$ is a compact 3-dimensional manifold that is a deformation retract of $S^3 \setminus |G|$.*

Proof. Let $N = X \cup Y$ be a graphical neighborhood. Statement (1) is immediate. Statement (4) is a consequence of of statement (3). Statement (2) and (3) can be proved easily using Proposition 1 (1), using the fact that Y is a product and using the following elementary claim.

Claim.

1. a. There exists a deformation retraction from $[0,1] \times \overline{B^2}$ to $([0,1] \times \{0\}) \cup (\{0,1\} \times \overline{B^2})$,
   b. there exists a deformation retraction from $[0,1] \times (\overline{B^2} \setminus \{0\})$ to $[0,1] \times S^1$.
2. Let $P_1, \ldots, P_k \in S^2$ with $k \geq 1$. We write $Y := \bigcup_{i=1}^{k} \{r \cdot P_i \in \overline{B^3} \,|\, r \in [0,1]\}$.
   a. There exists a deformation retraction from $\overline{B^3}$ to Y.
   b. There exists a deformation retraction from $\overline{B^3} \setminus Y$ to $S^2 \setminus \{P_1, \ldots, P_k\}$.

The proof of the claim is left to the reader. □

5 Uniqueness of graphical neighborhoods

Finally we define what is means for two graphical neighborhoods of a given spatial graph to be equivalent.

Definition 9. Let G be a spatial graph and let N and N' be two graphical neighborhoods of G. We say N and N' are *equivalent* if there exists a map $\Phi\colon S^3 \times [0,1] \to S^3$ with the following properties:

1. each $\Phi_t\colon S^3 \to S^3$ is a diffeomorphism,
2. each Φ_t is the identity on the vertex set and it preserves each edge setwise,
3. $\Phi_0 = \mathrm{id}$,
4. we have $\Phi_1(N) = N'$.

The goal of this section is to prove Theorem 3 from the introduction, i.e. we want to show that any two graphical neighborhoods of a given spatial graph are equivalent.

We turn to our first technical lemma of this section.

Lemma 3. *We consider the manifold $S^2 \times [a,b]$. Suppose we are given points $P_1,\dots,P_n$ in S^2 with $n \geq 1$. Let C be a proper submanifold of $S^2 \times (a,b)$ which is diffeomorphic to S^2, which is transverse to all the strings $P_i \times [a,b]$ and which meets each string $P_i \times [a,b]$ exactly once. Then there exist a diffeomorphism*

$$\Psi\colon S^2 \times [a,b] \ \to\ S^2 \times [a,b]$$

and some $c \in (a,b)$ with the following properties:

1. *Ψ is the identity near the boundary,*
2. *Ψ preserves each string $\{P_i\} \times [a,b]$ setwise,*
3. *$\Psi(S^2 \times \{c\}) = C$,*
4. *for each $t \in [a,b]$ the image $\Psi(S^2 \times \{t\})$ is transverse to each string $\{P_i\} \times [a,b]$.*

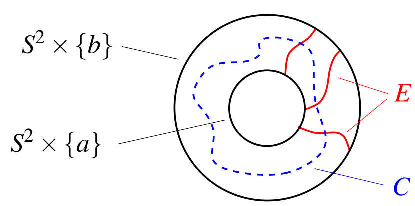

Fig. 6

In the proof of Lemma 3 we need the following well-known purely group theoretic statement.

Lemma 4. *Let $\varphi\colon A \to B$ be a homomorphism between free groups of the same finite rank. If $\varphi_*\colon H_1(A) \to H_1(B)$ is an epimorphism, then φ is a monomorphism.*

For the reader's convenience we include the proof of the lemma.

Proof. Let k be the rank of A and B. Note that $\Gamma := \varphi(A)$ is a subgroup of the free group B, thus it is also free group. The rank of Γ is evidently $\leq k$. The epimorphism $\varphi_*\colon H_1(A) \to H_1(B)$ factors through the inclusion induced map $H_1(\varphi(A)) \to H_1(B)$. Therefore we see that the rank of $\varphi(A)$ is $\geq k$. Thus $\varphi(A)$ is in fact a free group of rank k. The map $A \to \varphi(A)$ is evidently an epimorphism. Since free groups are Hopfian [6, p. 109] we see that $A \to \varphi(A)$ is also a monomorphism. But this implies that φ itself is a monomorphism. □

We will also make use of the following theorem.

Theorem 6. *Let F be a surface (possibly with boundary) that is not diffeomorphic to S^2. Let $D \subset F \times (0,1)$ be a properly embedded surface. If D is incompressible, i.e. if the inclusion induced map $\pi_1(D) \to \pi_1(F \times [0,1])$ is a monomorphism, then there exists an orientation-preserving diffeomorphism $\Psi\colon F \times [0,1] \to F \times [0,1]$ with $\Psi(F \times \{\frac{1}{2}\}) = D$, Ψ is the identity in a neighborhood of $F \times \{0,1\}$ and the restriction of Ψ to $\partial F \times [0,1]$ is diffeotopic to the identity.*

Proof. This follows, with minor effort, from [8, Proposition 3.1 and Corollary 3.2]. □

Proof (Proof of Lemma 3). For notational convenience we suppose that $a = -1$ and $b = 1$. Thus we consider the manifold $S^2 \times [-1,1]$. We denote by $p\colon S^2 \times [-1,1] \to S^2$ the obvious projection. Suppose we are given points $P_1, \dots, P_n \in S^2$ with $n \geq 1$. Let C be a submanifold of $S^2 \times (-1,1)$ which is diffeomorphic to S^2, which is transverse to each string $\{P_i\} \times [-1,1]$ and which meets each string $\{P_i\} \times [-1,1]$ exactly once.

By picking small enough closed disks D_i around the P_i we obtain tubular neighborhoods $D_i \times [-1,1]$ for the strings such that for each i the intersection $C \cap (D_i \times [-1,1])$ is a single disk and such that the projection $p\colon S^2 \times [-1,1] \to S^2$ restricts to a diffeomorphism $C \cap (D_i \times [-1,1]) \to D_i$. We write $\Sigma = S^2 \setminus \bigcup_{i=1}^{n} D_i^o$. Since p also restricts to a diffeomorphism $C \cap (\partial D_i \times [-1,1]) \to \partial D_i$ we see that the curve $C \cap (\partial D_i \times [-1,1])$ represents a generator for $H_1(\partial D_i \times [-1,1])$.

Claim. The surface $C' = C \cap (\Sigma \times [-1,1])$ is incompressible in $\Sigma \times [-1,1]$, i.e. the inclusion induced map $\pi_1(C') \to \pi_1(\Sigma \times [-1,1])$ is a monomorphism.

As we noted above, the intersection of the sphere C with each cylinder $D_i \times [-1,1]$ is a single disk. Thus we see that C' is a sphere with n open disks removed, i.e. C' is diffeomorphic to Σ. Thus we see that $\pi_1(C')$ and $\pi_1(\Sigma \times [-1,1])$ are free groups of the same rank. By Lemma 4 it suffices to show that $H_1(C') \to H_1(\Sigma \times [-1,1])$ is an epimorphism.

We consider the following commutative diagram of inclusion induced maps:

$$\begin{array}{ccccc}
H_1(\partial\Sigma) & \xrightarrow{\cong} & H_1((\partial\Sigma) \times [-1,1]) & \twoheadleftarrow & H_1(C' \cap ((\partial\Sigma) \times [-1,1])) \\
\downarrow & & \downarrow & & \downarrow \\
H_1(\Sigma) & \xrightarrow{\cong} & H_1(\Sigma \times [-1,1]) & \longleftarrow & H_1(C').
\end{array}$$

The two horizontal maps on the left are evidently isomorphisms. Furthermore, since Σ is a sphere minus some open disks we see that the left vertical map is an epimorphism. Thus the middle vertical map is an epimorphism. Furthermore, since

$C \cap (\partial D_i \times [-1,1])$ represents a generator for $H_1(\partial D_i \times [-1,1])$ we see that the top right horizontal map is an epimorphism. Thus it follows that the horizontal map on the bottom right is also an epimorphism. This concludes the proof of the claim.

It follows from the fact that C' is properly embedded in $\Sigma \times (-1,1))$ and Theorem 6 that there exists a self-diffeomorphism Ψ of $\Sigma \times [-1,1]$, that is the identity in a neighborhood of $\Sigma \times \{\pm 1\}$, that sends $\Sigma \times \{0\}$ to C' and which has the property that the restriction to each annulus $\partial D_i \times [-1,1]$ is diffeotopic to the identity.

It remains to extend Ψ over the cylinders $D_i \times [-1,1]$ in a suitable way. Recall that the projection $p\colon S^2 \times [-1,1] \to S^2$ restricts for each i to a diffeomorphism $C \cap (D_i \times [-1,1]) \to D_i$. The existence of the desired extensions is thus an immediate consequence of Lemma 5 below. $\square$

The following technical lemma concludes the previous proof.

Lemma 5. *Let $\varepsilon > 0$. We denote by $p\colon \overline{B^2_{1+\varepsilon}} \times [-1,1] \to \overline{B^2_{1+\varepsilon}}$ the obvious projection. Let C be a properly embedded disk in $\overline{B^2_{1+\varepsilon}} \times (-1,1)$ such the restriction of p to $C \cap (\overline{B^2_1} \times [-1,1]) \to \overline{B^2_1}$ is a diffeomorphism. Furthermore suppose we are given an orientation-preserving self-diffeomorphism Ψ of $(S^1 \times [1,1+\varepsilon]) \times [-1,1]$ with the following properties:*

1. *Ψ is the identity near $(S^1 \times [1,1+\varepsilon]) \times \{\pm 1\}$,*
2. *the restriction of Ψ to $S^1 \times \{1\} \times [-1,1]$ is diffeotopic to the identity,*
3. *$\Psi((S^1 \times [1,1+\varepsilon]) \times \{0\}) = C \cap (S^1 \times [1,1+\varepsilon]) \times [-1,1]$.*

Then there exists a self-diffeomorphism Φ of $\overline{B^2_{1+\varepsilon}} \times [-1,1]$ with the following properties:

(0) it equals Ψ on $S^1 \times [1,1+\varepsilon] \times [-1,1]$,
(1) Φ is the identity in a neighborhood of $\overline{B^2_{1+\varepsilon}} \times \{\pm 1\}$,
(2) Φ preserves $\{0\} \times [-1,1]$ setwise,
(3) we have $\Phi\big(\overline{B^2_{1+\varepsilon}} \times \{0\}\big) = C$.

Proof. We write $\Theta = \Psi|_{S^1 \times [-1,1]}$. This is an orientation-preserving self-diffeomorphism of the annulus $S^1 \times [-1,1]$ that is the identity near $S^1 \times \{\pm 1\}$. By hypothesis there exists a diffeotopy $S^1 \times [-1,1] \times [0,1]$ from Θ to the identity and the diffeotopy can be chosen to be the identity near $S^1 \times \{\pm 1\} \times [0,1]$. We use this diffeotopy to extend Ψ to $S^1 \times [1,\frac{1}{2}] \times [-1,1]$. Finally we extend Ψ via the identity to $\overline{B^2_1} \times [-1,1]$. Note that the restriction of the projection p to $\Psi'(C) \to \overline{B^2_{1+\varepsilon}}$ is still a diffeomorphism. We can postcompose Ψ' with a suitable self-diffeomorphism of $\overline{B^2_{1+\varepsilon}} \times [-1,1]$ of the form $(P,z) \mapsto (P, f(P,z))$ to obtain the desired self-diffeomorphism Φ. $\square$

5.1 Proof of Theorem 3

The theorem will be proved in several steps. To formulate the first step we need to introduce a new definition.

Definition 10. Let $G=(V,E)$ be a spatial graph. Let X and X' be small neighborhoods of V.

1. Given $v\in V$ we say that X'_v is *covered* by X_v if the following properties hold:
   a. $X'_v\subset X_v$,
   b. there exists a diffeomorphism $\Theta\colon \overline{B^3_R}\to X_v$ that is transverse at v and an $R'<R$ such that $\Theta(\overline{B^3_{R'}})=X'_v$,
   c. there exist $P_1,\dots,P_n\in S^2$ such that $\Theta^{-1}(E)\cap S^2\times[R',R]=\bigcup_{i=1}^n\{P_i\}\times[R',R]$.
2. We say X' is *covered* by X if each X'_v is covered by X_v.

Remark 4. By rescaling we can always arrange that $R'=1$ and $R=2$.

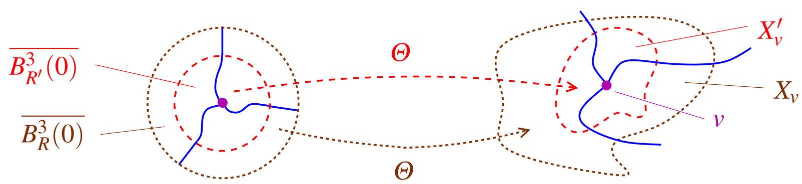

Fig. 7 The small neighborhood Θ'_i is covered by Θ_i.

Lemma 6 (Small neighborhood shrinking lemma). *Let $G=(V,E)$ be a spatial graph. Let X be X' two small neighborhoods of V. If $X'\subset X^o$, then X' is covered by X.*

Proof. Let X and X' be small neighborhoods of V with $X'\subset X^o$. Let $v\in V$. We pick a corresponding diffeomorphism $\Theta'_v\colon \overline{B^3_{R'}}\to X'_v$ that is transverse at v. By Proposition 1 (2) we can pick a diffeomorphism $\Theta_v\colon \overline{B^3_R}\to X_v$ that is transverse at v and which admits an $\eta>0$ such that $\Theta_v(\overline{B^3_\eta})\subset (X'_v)^o$ and such that $F:=\Theta_v^{-1}(E)\cap S^2\times[\eta,R]$ is linear. (Here we make the usual identification $\overline{B^3_R}\setminus B^3_\eta=S^2\times[\eta,R]$.)

We write $C:=\Theta_v^{-1}(\Theta'_v(S^2_{R'}))\subset \overline{B^3_R}$. Note that by the choice of η we have $C\subset S^2\times[\eta,R]$. By Lemma 3 there exists a diffeomorphism

$$\Psi\colon S^2\times[\eta,R]\ \to\ S^2\times[\eta,R]$$

and some $c\in(\eta,R)$ with the following properties:

1. Ψ is the identity near the boundary,
2. Ψ preserves the linear strings $\Theta_v^{-1}(E) \cap (S^2 \times [\eta, R]$,
3. $\Psi(S^2 \times \{c\}) = C$,
4. for each $t \in [\eta, R]$ the image $\Psi(S^2 \times \{t\})$ is transverse to F.

We now consider the map $\Xi : \overline{B^3_R} \to \overline{B^3_R}$ that is given by the identity on $\overline{B^3_\eta}$ and that is given by Ψ on $S^2 \times [\eta, R] = \overline{B^3_R} \setminus B^3_\eta$. (Note that Ξ is smooth by the first property of Ψ.) Note that $\Theta_v \circ \Xi : \overline{B^3_R} \to X_v$ is transverse at v and that the image of $\overline{B^3_c}$ under this map is precisely X'_v. We have thus shown that X'_v is covered by X_v. $\square$

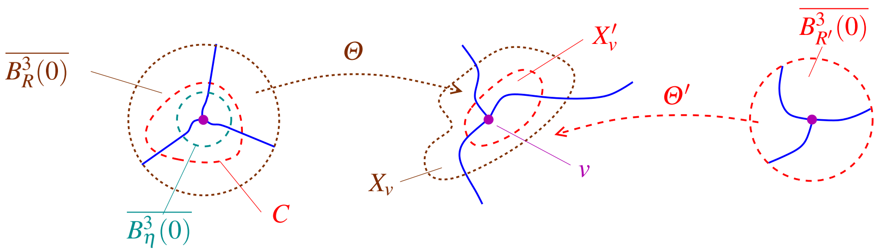

Fig. 8 Illustration of the proof of Lemma 6.

Lemma 7 (Graphical shrinking lemma). *Let $G = (V, E)$ be a spatial graph. Let $Z = X \cup Y$ be a graphical neighborhood for G. Suppose that X' is a small neighborhood for V. If X' is covered by X, then there is a graphical neighborhood Z' with decomposition $Z' = X' \cup Y'$ and which is equivalent to Z.*

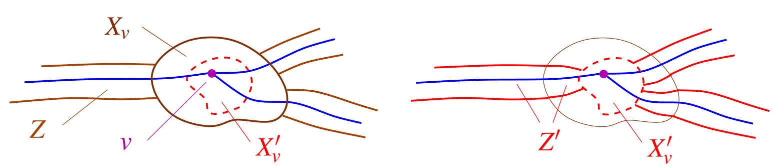

Fig. 9 Illustration of Lemma 7.

We will need the following rather technical lemma.

Lemma 8. *Let $P_1, \dots, P_m$ be points in S^2. Let $\varepsilon \in (0,1)$ and let $f : S^2 \times [1, 1+\varepsilon] \to S^2 \times [1,2]$ be an embedding such that $f|_{S^2 \times \{1\}} = \mathrm{id}$ and such that f preserves $\{P_i\} \times [1,2]$ setwise. Then there exists a diffeomorphism $\Psi : S^2 \times [1,2] \to S^2 \times [1,2]$ with the following properties:*

1. Ψ equals f in a neighborhood of $S^2 \times \{1\}$,

2. *Ψ is the identity in a neighborhood of $S^2 \times \{2\}$,*
3. *for each $t \in [1,2]$ and for each $Q \in S^2$ the submanifolds $\Psi(S^2 \times \{t\})$ and $\{Q\} \times [1,2]$ are transverse,*
4. *for each i the map Ψ preserves $\{P_i\} \times [1,2]$ setwise.*

Proof. We denote by $p\colon S^2 \times [1,2] \to S^2$ the obvious projection. Since $f|_{S^2\times\{1\}} = \mathrm{id}$ we can pick an $\mu > 0$ such that for all $Q \in S^2$ and $t \in [1,1+\mu]$ the differential $Dp_{(Q,t)}$ is an isomorphism and such that for each i we have $f(\{P_i\} \times [1,1+\mu]) \subset D_i \times [1,2]$. Note that for each $t \in [1,1+\mu]$ the map $g_t := p \circ f_t\colon S^2 \to S^2$ is a diffeomorphism that is the identity on $\{P_1,\dots,P_m\}$.

Let $\sigma\colon [1,1+\mu] \to [1,1+\mu]$ be a smooth function which is equal to 1 on some interval $[1,\eta]$ with $\eta > 0$ and which has the property that there exists a $\nu > 0$ such that $\sigma(t) = t$ for all $t \in [1+\mu-\nu, 1+\mu]$. We consider the map

$$\begin{aligned} \alpha\colon S^2 \times [1,1+\mu] &\to S^2 \times [1,1+\mu] \\ (Q,t) &\mapsto f(g^{-1}_{\sigma(t)}(Q),t). \end{aligned}$$

Note that the map α has the property that $p(\alpha(Q,t)) = Q$ for all $Q \in S^2$ and all $t \in [1+\mu-\nu, 1+\mu]$. This means that there exists a smooth function $d\colon S^2 \times [1+\mu-\nu, 1+\mu] \to [1,1+\mu]$ such that $\alpha(Q,t) = (Q,d(Q,t))$ for all $Q \in S^2$ and all $t \in [1+\mu-\nu,1+\mu]$ and for some smooth function $d\colon S^2 \times [1+\mu-\nu,1+\mu] \to [1,1+\mu]$. Note that d has the property that for each $Q \in S^2$ the function $t \mapsto d(Q,t)$ has positive derivative. We pick an extension of d to a smooth function $d\colon S^2 \times [1+\mu-\nu,2] \to [1,2]$ with the following properties:

1. for each $Q \in S^2$ the function $t \mapsto d(Q,t)$ has positive derivative,
2. there exists a neighborhood of $S^2 \times \{2\}$ such that the map d is just the projection.

Now we consider the map

$$\begin{aligned} \Psi\colon S^2 \times [1,2] &\to S^2 \times [1,2] \\ (Q,t) &\mapsto \begin{cases} \alpha(Q,t), & \text{if } t \in [1,1+\mu], \\ (Q,d(Q,t)), & \text{if } t \in [1+\mu,2]. \end{cases} \end{aligned}$$

One easily verifies that the above map Ψ has all the desired properties. □

Proof (Proof of the Graphical shrinking Lemma 7). Let $v \in V$.

1. Since X' is covered by X we can find for each $v \in V$ an orientation-preserving diffeomorphism $\Theta_v\colon \overline{B^3_2} \to X_v$ such that Θ_v is transverse at v and such that Θ_v restricts to a diffeomorphism $\overline{B^3_1} \to X'_v$ and such that there exists a finite subset $P_v \subset S^2$ with $\Theta_v(S^2 \times [\frac{1}{4},2]) \cap E = \Theta_v(P_v \times [\frac{1}{4},2])$.
2. By condition (3) on a tubular neighborhood we can find orientation-preserving embeddings $\Omega_v\colon S^2 \times [2,3] \to S^3 \setminus X^o$, $v \in V$ with the following properties:
   a. the images are disjoint,
   b. for each v we have $\Omega_v(S^2 \times \{2\}) = \partial X_v$,

c. for each v there exists a finite subset $Q_v \subset S^2$ with $E \cap \Omega_v(S^2 \times [2,3]) = \Omega_v(Q_v \times [2,3])$ and there exist disjoint closed disks $\{D_i\}_{i \in Q_v}$ in S^2 with $Z \cap \Omega_v(S^2 \times [2,3]) = \Omega_v\Big(\bigcup_{i \in Q_v} \{D_i\} \times [2,3]\Big)$.

Note that after possibly postcomposing Ω_v with the map

$$\begin{aligned} S^2 \times [2,3] &\mapsto S^2 \times [2,3] \\ (P,t) &\mapsto ((\Theta_v|_{S^2_2} \circ (\Omega_v|_{S^2_2})^{-1})(P),t) \end{aligned}$$

we can arrange that $\Theta_v|_{S^2_2} = \Omega_v|_{S^2_2}$. In particular we have $P_v = Q_v$.

3. Using the Whitney Approximation Theorem, as formulated in [5, Theorem 6.26], we can extend the embedding $\Omega_v \colon S^2 \times [2,3] \to S^3 \setminus X^o$ to a smooth map $\Omega_v \colon S^2 \times [2-\varepsilon,3] \to S^3$ for a suitably small $\varepsilon > 0$. Furthermore we can arrange that $E \cap \Omega_v(S^2 \times [2-\varepsilon,2]) = \Omega_v(Q_v \times [2-\varepsilon,2])$. After possibly reducing the ε these maps are in fact embeddings. Note that Ω_v restricts to an embedding $S^2 \times [2-\varepsilon,2] \to X_v$, in particular we obtain an embedding $f := (\Theta_v)^{-1} \circ \Omega_v \colon S^2 \times [2-\varepsilon,2] \to S^2 \times [1,2]$ that is the identity on $S^2 \times \{2\}$ and that preserves $Q_v \times [2-\varepsilon,2]$ setwise.

We pick $\Psi_v \colon S^2 \times [1,2] \to S^2 \times [1,2]$ as in Lemma 8. (Here we replace the endpoints $\{1,2\}$ by $\{2,1\}$ and we consider the map $f := (\Theta_v)^{-1} \circ \Omega_v \colon S^2 \times [2-\varepsilon,2] \to S^2 \times [1,2]$.) We define

$$\begin{aligned} \Xi_v \colon \overline{B^3_3} &\to S^3 \\ (P,t) &\mapsto \begin{cases} \Omega_v(P,t), & \text{if } t \in [2,3], \\ \Theta_v(\Psi_v(P,t)), & \text{if } t \in [1,2], \\ \Theta_v(P,t), & \text{if } t \in [0,1]. \end{cases} \end{aligned}$$

We set

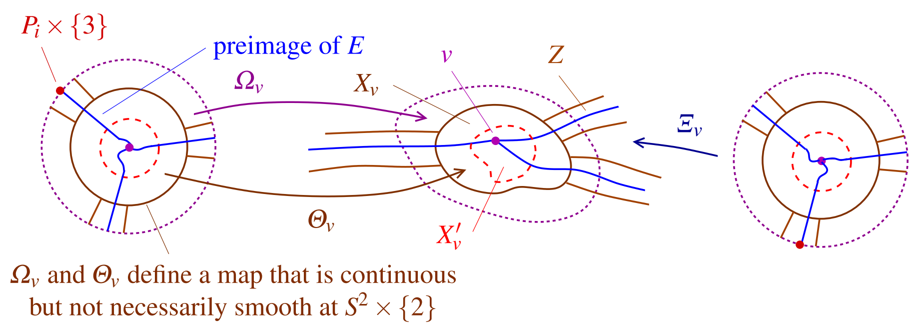

Fig. 10

$$Z' := Z \cup \bigcup_{v \in V} \bigcup_{i \in P_v} \Xi_v\big(D_i \times [1,2]\big).$$

It is fairly straightforward to show that Z' is indeed a graphical neighborhood for G.

It remains to show that the graphical neighborhoods Z and Z' are equivalent. We pick a smooth strictly monotonously function $f\colon [0,3] \to [0,3]$ with $f(t) = t$ for $t \in [0, \frac{1}{2}]$ and $t \in [\frac{5}{2}, 3]$ and with $f(2) = 1$. The map

$$\begin{aligned} \Phi\colon S^3 \times [0,1] &\to S^3 \\ (P,t) &\mapsto \begin{cases} \Xi_v(Q, s\cdot(1-t) + f(s)\cdot t), & \text{if } P = \Xi_v(Q,s) \text{ where } Q \in S^2, t \in [0,3] \\ P, & \text{otherwise} \end{cases} \end{aligned}$$

is a diffeotopy that fixes V pointwise and that fixes E setwise with $\Phi_0 = \mathrm{id}$ and with $\Phi_1(Z) = Z'$. □

Lemma 9. *Let $P_1, \dots, P_m \in S^2$, let $\varepsilon > 0$ and let $\Psi\colon (S^2 \times [1, 1+\varepsilon)) \times [0,1] \to \mathbb{R}^3 \setminus B^3_1$ be a map with the following properties:*

1. *$\Psi_t = \mathrm{id}$ for small t,*
2. *each Ψ_t is an embedding,*
3. *each Ψ_t is the identity on each $\{P_i\} \times [1, 1+\varepsilon)$,*
4. *each Ψ_t preserves $S^2 \times \{1\}$ setwise.*

Then we can extend Ψ to a smooth map $(S^2 \times [\frac{1}{2}, 1+\varepsilon)) \times [0,1] \to \mathbb{R}^3 \setminus B^3_{\frac{1}{2}}$ with the following properties:

1. *$\Psi_0 = \mathrm{id}$,*
2. *each Ψ_t is an embedding,*
3. *for each i we have $\Psi_i(\{P_i\} \times [\frac{1}{2}, 1+\varepsilon)) \subset \{P_i\} \times [\frac{1}{2}, 1+\varepsilon)$,*
4. *there exists a $\nu > 0$ such that each Ψ_t is the identity on $S^2 \times [\frac{1}{2}, \frac{1}{2} + \nu]$.*

Proof. We start out with the following claim.

Claim. There exists a smooth map $\Psi'\colon S^2 \times [\frac{1}{2}, 1+\varepsilon) \times [0,1] \to \overline{B^3_{\frac{1}{2}}}$ with the following properties:

1. The map agrees with Ψ on $S^2 \times [1, 1+\varepsilon) \times [0,1]$,
2. for each i and each $t \in [0,1]$ we have $\Psi'_t(\{P_i\} \times [\frac{1}{2}, 1]) \subset \{P_i\} \times [\frac{1}{2}, 1+\varepsilon]$,
3. on $(S^2 \times [\frac{1}{2}, 1]) \times \{0\}$ the map Ψ' is the identity. (Strictly speaking it is the projection onto the factor in the parenthesis.)

First we extend Ψ to a map Ψ_1 on the following closed subset:

$$\begin{aligned} \Psi_1 = (\Psi_1^x, \Psi_1^y)\colon (S^2 \times [1, 1+\varepsilon)) \times [0,1] \cup (S^2 \times [\tfrac{1}{2}, 1]) \times \{0\} &\to (S^2 \times [\tfrac{1}{2}, 1]) \times [0,1] \\ P &\mapsto \Psi_1(P) = (\Psi_1^x(P), \Psi_1^y(P)) \end{aligned}$$

of $(S^2 \times [\frac{1}{2}, 1+\varepsilon]) \times [0,1]$ by defining it to be the identity on $(S^2 \times [\frac{1}{2}, 1]) \times \{0\}$. Since $\Psi_t = \mathrm{id}$ for small t we see that Ψ_1 is smooth on this closed subset. (Recall that by definition, [5, p. 45], a map $f\colon A \to N$ on an arbitrary subset A of a manifold is smooth if given any point $P \in A$ there exists an open neighborhood U of P and a smooth map on U that agrees with f on $A\cap U$.) It follows from the Whitney Approximation Theorem, as formulated in [5, Theorem 6.26], that Ψ_1 can be extended to a smooth map on $(S^2\times[\frac{1}{2},1+\varepsilon))\times[0,1]$. We denote this extension again by Ψ_1. Next we pick disjoint open neighborhoods $U_1,\dots,U_m$ around $P_1,\dots,P_m$. In the following we make the identification $S^2=\mathbb{R}^2\cup\{\infty\}$ in such a way that $U_1,\dots,U_m\subset\mathbb{R}^2$. Thus we can consider the map

$$\Psi_2\colon (S^2\times[1,1+\varepsilon))\times[0,1]\,\cup\, S^2\times[\tfrac{1}{2},1+\varepsilon)\times\{0\}\,\cup\,\bigcup_{i=1}^{m}U_i\times[\tfrac{1}{2},1+\varepsilon)\times[0,1]\;\to\; S^2\times[\tfrac{1}{2},1+\varepsilon)$$

that is given by

$$(Q,s,t)\;\mapsto\begin{cases}(\Psi_1^x(Q,s,t)+P_i-\Psi_1^x(P_i,s,t),\Psi_1^y(Q,s,t)), & \text{if } Q\in U_i \text{ and } s\in[\frac{1}{2},1],\\ \Psi_1(Q,s,t), & \text{otherwise.}\end{cases}$$

One can easily see that Ψ_2 is smooth on the open subsets $U_i\times[\frac{1}{2},1+\varepsilon)\times[0,1]$. It follows in particular that the restriction of Ψ_2 to the closed subset

$$(S^2\times[1,1+\varepsilon))\times[0,1]\,\cup\,\bigcup_{i=1}^{m}(P_i\times[\tfrac{1}{2},1])\times[0,1]\,\cup\,(S^2\times[\tfrac{1}{2},1])\times\{0\}$$

is smooth. Thus, once again by the Whitney Approximation Theorem, as formulated in [5, Theorem 6.26], we can extend Ψ_2 to a smooth map $\Psi'\colon S^2\times[\frac{1}{2},1+\varepsilon)\times[0,1]$ which now has all the desired properties. This concludes the proof of the claim.

It follows from the claim and the Whitney Approximation Theorem, as formulated in [5, Theorem 6.26], that Ψ' can be extended to a smooth map on $(S^2\times[\frac{1}{2},1+\varepsilon))\times[0,1]$. We denote this extension again by Ψ'.

Note that for a sufficiently small $\eta\in(0,\frac{1}{4})$ each map $\Psi'_t\colon S^2\times[1-\eta,1]\to\overline{B^3_1}$ is an embedding. Thus we can apply Lemma 8 (with the endpoints $\{1,2\}$ replaced by $\{\frac{1}{2},1\}$, with the interval $[1,1+\varepsilon]$ replaced by $[1-\eta,1]$ and with f replaced by Ψ') to obtain an extension of $\Psi\colon (S^2\times[1-\eta,1])\times[0,1]\to\overline{B^3_1}$ to a map on $(S^2\times[\frac{1}{2},1])\times[0,1]\to S^2\times[\frac{1}{2},1]$ that has all the properties we expect on that domain. Together with our original map it defines the desired map $(S^2\times[\frac{1}{2},1+\varepsilon))\times[0,1]\to\mathbb{R}^3\setminus B^3_{\frac{1}{2}}$.
□

Now we can finally give the proof of Theorem 3.

Proof. Let G be a spatial graph and suppose that $Z=X\cup Y$ and $Z'=X'\cup Y'$ are two graphical neighborhoods of G. It follows easily from the proof of the existence of graphical neighborhoods that there exists a graphical neighborhood $Z''=X''\cup Y''$ with the following two properties:

1. we have $X''\subset X^o$ and $X''\subset(X')^o$,
2. for each $v\in V$ there exists a map $\Theta_v\colon\overline{B^3_2}\to X_v$ that is transverse at v and such that $\Theta_v(\overline{B^3_1})=X''_v$ and such that the images $\Theta_v(\overline{B^3_2})$ are disjoint.

Since for graphical neighborhoods being equivalent is indeed an equivalence relation it suffices to prove the desired statement for X'' and X.

By the Shrinking Lemma 6 the small neighborhood X'' is covered by X. We apply the Graphical Shrinking Lemma 7 to X and X'' and we obtain a new graphical neighborhood $\widetilde{Z} = X'' \cup \widetilde{Y}$ which is equivalent to the graphical neighborhood $Z = X \cup Y$. Thus it remains to show that $Z'' = X'' \cup Y''$ is equivalent to $\widetilde{Z} = X'' \cup \widetilde{Y}'$. Now Y'' and $\widetilde{Y}$ are tubular neighborhoods of the proper submanifold $E \cap S^3 \setminus (X'')^o$ in $S^3 \setminus (X'')^o$. By uniqueness of tubular neighborhood, see Theorem 5, we obtain a diffeotopy Ψ of $S^3 \setminus (X'')^o$ rel $E \cap S^3 \setminus (X'')^o$ with $\Psi_0 = \mathrm{id}$ and with $\Psi(Y'') = \widetilde{Y}$.

We continue with the above maps $\Theta_v \colon \overline{B_2^3} \to X_v$. We apply Lemma 9 to the maps

$$\begin{aligned} S^2 \times [1, 1+\varepsilon] \times [0,1] &\to \mathbb{R}^3 \\ (P, s, t) &\mapsto \Theta_v^{-1}(\Psi_t(\Theta_v(P, s, t))) \end{aligned}$$

for a conveniently chosen $\varepsilon > 0$. We can use the resulting extensions given by Lemma 9 to extend Ψ over all of S^3. $\square$

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References

1. Friedl, S.: Algebraic Topology I-V. Lecture notes, University of Regensburg (2020) https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/2020_friedl-algebraic-topology.pdf
2. Friedl, S. and Herrmann, G.: Spatial graphs, to appear in the Encyclopedia of Knot Theory to be published by CRC Press (2020).
3. Glaser L.: Geometrical combinatorial topology. Vol. I. Van Nostrand Reinhold Mathematics Studies, 27. Van Nostrand Reinhold Co. (1970).
4. Hempel J.: 3-manifolds. Annals of Mathematics Studies 86. Princeton, New Jersey: Princeton University Press and University of Tokyo Press. XII (1976).
5. Lee J.: Introduction to smooth manifold, Graduate Texts in Mathematics 218, second edition, Springer Verlag (2002).
6. Magnus W., Karrass A. and Solitar D.: Combinatorial group theory. Presentations of groups in terms of generators and relations. Reprint of the 1976 second edition. Dover Publications (2004).
7. Rourke C. P. and Sanderson B. J.: Introduction to piecewise-linear topology. Ergebnisse der Mathematik und ihrer Grenzgebiete 69 (1972).
8. Waldhausen F.: On irreducible 3-manifolds which are sufficiently large. Ann. of Math. (2) **87**, 56–88 (1968)
9. Wall C. T. C.: Differential topology. Cambridge Studies in Advanced Mathematics 156. Cambridge University Press (2016).