PD_4-complexes and 2-dimensional duality groups

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Abstract This paper is a synthesis and extension of three earlier papers on PD_4-complexes X such that $\pi = \pi_1(X)$ has one end and $c.d.\pi = 2$. The basic notion is that of strongly minimal PD_4-complex, one for which the equivariant intersection pairing λ_X on $\pi_2(X)$ is null. The first main result is that two PD_4-complexes with the same strongly minimal model are homotopy equivalent if and only if their intersection pairings are isometric. If $c.d.\pi \leq 2$ every such complex has a strongly minimal model, and the second half of the paper focuses largely on determining the minimal models. In particular, if π is a surface group or is a semidirect product $F(r) \rtimes \mathbb{Z}$ then the homotopy type of X is determined by π, the Stiefel-Whitney classes and λ_X. Although we expect that the strategy in the surface group case should extend to all π such that $c.d.\pi = 2$ and π has one end, we do not yet have a unified proof that covers the known cases. We conclude with an application to 2-knots and a short list of questions for further research.

1 Introduction

It remains an open problem to give a homotopy classification of closed 4-manifolds or PD_4-complexes, in terms of standard invariants such as the fundamental group, characteristic classes and intersection pairings. Hambleton and Kreck showed that if X is orientable and $H_2(X;\mathbb{Q}) \neq 0$ the homotopy type of X is determined by its Postnikov 2-stage $P_2(X)$ and the image of the fundamental class $[X]$ in $H_4(P_2(X);\mathbb{Z})$, and if $\pi_1(X)$ is finite and of cohomological period dividing 4 this image is in turn determined by the equivariant intersection pairing on $\pi_2(X)$ [27]. Baues and Bleile have extended the first part of this result to all PD_4-complexes: two PD_4-complexes X and Y are homotopy equivalent if and only if there is a homotopy equivalence $h : P_2(X) \to P_2(Y)$ such that $h^*w_1(Y) = w_1(X)$, and which carries the image of $[X]$

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in $H_4(P_2(X);\mathbb{Z}^{w_1(X)})$ to the image of $\pm[Y]$ in $H_4(P_2(Y);\mathbb{Z}^{w_1(Y)})$. (Here $w_1(X)$ and $w_1(Y)$ are the orientation characters and $\mathbb{Z}^{w_1(X)}$ and $\mathbb{Z}^{w_1(X)}$ the associated twisted coefficient modules.) They also give a homotopy classification of PD_4-complexes (up to 2-torsion) in terms of homotopy classes of chain complexes with a homotopy commutative diagonal and an additional quadratic structure [5]. However, there is still the question of how to characterize the classes in $H_4(P_2(X);\mathbb{Z}^{w_1(X)})$ which correspond to PD_4-complexes.

We shall extend the work of [27] to relate such classes to intersection pairings, for certain cases with $\pi = \pi_1(X)$ infinite. The central idea is that of "strongly minimal PD_4-complex", one for which the equivariant intersection pairing is identically 0. (We shall in fact use the equivalent cohomological pairing.) If there is a 2-connected degree-1 map $f : X \to Z$, with Z strongly minimal, and if the orientation character $w = w_1(X) : \pi \to \mathbb{Z}^\times$ does not split then the homotopy type of X is determined by the homotopy type of Z and the equivariant intersection pairing. Every PD_4-complex X with fundamental group π has such a "strongly minimal model" Z if and only if $c.d.\pi \leq 2$. (See Theorem 21 below.) This class of groups is both tractable and of direct interest to low-dimensional geometric topology, as it includes all surface groups, knot groups and the groups of many other bounded 3-manifolds. We expect that if $c.d.\pi \leq 2$ the homotopy type of Z is determined by π, w and the Wu class $v_2(Z)$, and that if $v_2(X)$ is induced from π then the minimal model is unique. (In the latter case, the homotopy type of X is determined by π, w, $v_2(X)$ and the equivariant intersection pairing.) However, this is only known for π a free group, a surface group, a semidirect product $F(r) \rtimes \mathbb{Z}$ or a solvable Baumslag-Solitar group $\mathbb{Z}*_m$.

We shall now outline the paper in more detail. The first two sections are algebraic. In particular, Theorem 1 (in §2) establishes a connection between hermitian pairings and the Whitehead quadratic functor Γ_W. Sections 3–8 consider the homotopy classification of PD_4-complexes, and introduce several notions of minimality. The first main result is Theorem 7 in §7, where it is shown that two PD_4-complexes with the same strongly minimal model and $\pm$isometric intersection pairings are homotopy equivalent, provided $w : \pi \to \mathbb{Z}^\times$ does not split. Sections 9 and 10 determine the strongly minimal PD_4-complexes with $\pi_2 = 0$ and for which π has finitely many ends. Strongly minimal PD_4-complexes with π a semidirect product $\nu \rtimes \mathbb{Z}$ (with ν finitely presentable) are shown to be mapping tori in §11. When ν is a free group the homotopy type of such a mapping torus is determined by π and the Stiefel-Whitney classes, by Theorem 22. The next five sections lead to the second main result, Theorem 27 (in §16), which extends the result of Theorem 22 to the case when π has one end and $c.d.\pi = 2$ *provided* that the image of the symmetric square $\Pi \odot \Pi$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ is 2-torsion free, where $\Pi = \pi_2(X) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$. This theorem is modelled on the much simpler case analyzed in §14, in which π is a PD_2-group. Apart from the notion of minimality, the main technical points are the connection between hermitian pairings and Γ_W, the fact that a certain "cup product" defines an isomorphism, and the 2-torsion condition. In [40], we showed that the cup-product condition held for surface groups, torus knot groups and solvable Baumslag-Solitar groups. Here we show that it holds for all finitely presentable groups π with one end and $c.d.\pi = 2$ (Theorem 26). The 2-torsion condition is only known for π a PD_2-

group or π a solvable Baumslag-Solitar group (Theorem 30), and does not hold for all the cases covered by Theorem 22. The penultimate section considers the classification up to TOP s-cobordism or homeomorphism of closed 4-manifolds with groups as in Theorem 27. In particular, it is shown that a remarkable 2-knot discovered by Fox is determined up to TOP isotopy and reflection by its knot group. In the body of the text we raise a number of questions, some on points of detail, that we have not been able to settle. The most significant of these have been collected in the final section.

The interactions of cohomology of groups, Poincaré duality and the lower stages of Postnikov towers are central to the arguments. We refer particularly to [9, 51, 54, 62] for more on these topics. Some of the other techniques invoked, such as L^2-homology (used in §5 to compare various notions of minimality) or Farrell cohomology (used in §10 in connection with PD_4-complexes whiose universal covers have two ends) may seem more recondite, but these are mostly used in excursions aside the main theme, and familiarity with such notions is not essential.

The theme of Hambleton, Kreck and Teichner [29] is close to ours, although their methods are very different. They use Kreck's modified surgery theory to classify up to s-cobordism closed orientable 4-manifolds with fundamental groups of geometric dimension 2 (subject to some K- and L-theoretic hypotheses), and they show also that every automorphism of the algebraic 2-type is realized by an s-cobordism, in many cases. (They do not require that π have one end, which is a restriction imposed by our arguments. However, when π is a free group there is a simpler, more homological approach, which also uses the ideas of §2 below [37].)

This paper is a synthesis and extension of three papers [38, 39, 40] which explored the role of minimality in the classification of PD_4-complexes, in particular, those with fundamental group π such that $c.d.\pi = 2$ and π has one end. (Some aspects were considered much earlier [35, 36].) Apart from the benefits of revision, the main novelties are in showing that strongly minimal finite PD_4-complexes have minimal Euler characteristic (Corollary 3), strong minimality is equivalent to order minimality if and only if $c.d.\pi \leq 2$ (Theorem 21), verification that cup product defines an isomorphism for all 2-dimensional duality groups (Theorem 26), clarification of the role of the refined v_2-type, and relaxation of some of the hypotheses.

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2 Modules and group rings

Let π be a finitely presentable group and $w : \pi \to \mathbb{Z}^\times = \{\pm 1\}$ be a homomorphism. (This shall represent the orientation character for a PD_n-complex with fundamental group π.) We shall at times view w as a class in $H^1(\pi;\mathbb{F}_2)$, since this cohomology group may be identified with $Hom(\pi,\mathbb{Z}^\times)$. Define an involution on $\mathbb{Z}[\pi]$ by $\bar{g} = w(g)g^{-1}$, for all $g \in \pi$. Let $\mathbb{Z}$ and $\mathbb{Z}^w$ be the augmentation and w-twisted augmentation rings, and $\varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$ and $\varepsilon_w : \mathbb{Z}[\pi] \to \mathbb{Z}^w$ be the augmentation and the w-twisted augmentation, defined by $\varepsilon(g) = 1$ and $\varepsilon_w(g) = w(g)$, for all $g \in \pi$, respectively. Let $I_w = \mathrm{Ker}(\varepsilon_w)$.

All modules considered here shall be left modules, unless otherwise noted. However, if L is a left $\mathbb{Z}[\pi]$-module the dual $Hom_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])$ and the higher extension groups $Ext^i_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])$ are naturally right modules. If R is a right $\mathbb{Z}[\pi]$-module let $\overline{R}$ be the corresponding left $\mathbb{Z}[\pi]$-module with the conjugate structure given by $g.r = r.\bar{g}$, for all $g \in \mathbb{Z}[\pi]$ and $r \in R$. Let $L^\dagger = \overline{Hom_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])}$ and $E^iL = \overline{Ext^i_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])}$, for $i \geq 0$ be the conjugate dual left modules. If L is free, stably free or projective then so is $E^0L = L^\dagger$. We shall consider $\mathbb{Z}$ and $\mathbb{Z}^w$ to be bimodules, with the same left and right π-structures. (Note that $\overline{\mathbb{Z}} = \mathbb{Z}^w$.)

The modules $E^q\mathbb{Z} = \overline{H^q(\pi;\mathbb{Z}[\pi])}$ with $q \leq 3$ shall recur throughout this paper. In particular, $E^0\mathbb{Z} \cong \mathbb{Z}^w$ if π is finite and is 0 otherwise, while $E^1\mathbb{Z}$ reflects the number of ends of π. It is 0 if π is finite or has one end, infinite cyclic if π has two ends (i.e., is virtually infinite cyclic) and is free abelian of infinite rank otherwise.

Lemma 1. *Let M be a $\mathbb{Z}[\pi]$-module with a finite resolution of length n and such that $E^iM = 0$ for $i < n$. Then $Aut(M) \cong Aut(E^nM)$.*

Proof. Since $E^iM = 0$ for $i < n$ the dual of a resolution of length n for M is a finite resolution for E^nM. Taking duals again recovers the original resolution, and so $E^nE^nM \cong M$. If $f \in Aut(M)$ it extends to an endomorphism of the resolution inducing an automorphism E^nf of E^nM. Taking duals again gives $E^nE^nf = f$. Thus $f \mapsto E^nf$ determines an isomorphism $Aut(M) \cong Aut(E^nM)$.

A group π is an *n-dimensional duality group* over $\mathbb{Z}$ if the augmentation $\mathbb{Z}[\pi]$-module $\mathbb{Z}$ has a finite projection resolution of length n, $H^i(\pi;\mathbb{Z}[\pi]) = 0$ for $i < n$ and the *dualizing module* $\mathscr{D} = H^n(\pi;\mathbb{Z}[\pi])$ is torsion free as an abelian group. (See [9, Theorem VIII.10.1].) We then have $Aut(E^n\mathbb{Z}) = \mathbb{Z}^\times$, by Lemma 1. Finitely generated free groups are duality groups of dimension 1. If π is finitely presentable and $c.d.\pi = 2$ then $H^2(\pi;\mathbb{Z}[\pi]) \neq 0$, and it is torsion free, by [25, Proposition 13.7.1]. Hence π is a 2-dimensional duality group if and only if it has one end.

In general, $H^2(\pi;\mathbb{Z}[\pi])$ is 0, $\mathbb{Z}$ or not finitely generated ([21] – see [25, Proposition 13.7.12]). In the latter case, $H^2(\pi;\mathbb{Z}[\pi])$ must have infinite rank, by the main result of [8]. It remains open whether $H^2(\pi;\mathbb{Z}[\pi])$ must be free as an abelian group.

We shall use the "free differential calculus" of Fox and Lyndon to provide partial resolutions of augmentation modules. (See [23] and [46].) Let $F(n)$ be the free group with basis $\{x_1,\dots,x_n\}$. The augmentation ideal of $\mathbb{Z}[F(n)]$ is freely generated by $\{x_1-1,\dots,x_n-1\}$ as a left $\mathbb{Z}[F(n)]$-module and so we may write

$$r-1 = \Sigma_{1\leq i\leq n}\frac{\partial r}{\partial x_i}(x_i-1),$$

for $r \in F(n)$. Since $rs-1 = r-1+r(s-1)$, for all $r,s \in F(\mu)$, the Leibniz conditions

$$\frac{\partial rs}{\partial x_i} = \frac{\partial r}{\partial x_i} + r\frac{\partial s}{\partial x_i}$$

hold for all $r,s \in F(\mu)$ and $1 \le i \le n$. In particular, $\frac{\partial 1}{\partial x_i} = 0$ and $\frac{\partial r^{-1}}{\partial x_i} = -r^{-1}\frac{\partial r}{\partial x_i}$, for $1 \le i \le n$. We may extend these functions linearly to "derivations" of $\mathbb{Z}[F(n)]$.

Now let π be a group with a finite presentation

$$\mathscr{P} = \langle x_1,\dots,x_g | r_1,\dots,r_h\rangle^{\varphi},$$

where $\varphi : F(g) \to \pi$ is an epimorphism with kernel the normal closure of $\{r_1,\dots,r_h\}$. Let $def(\mathscr{P}) = g-h$ be the deficiency and $C(\mathscr{P})$ be the 2-complex corresponding to this presentation. Then $\chi(C(\mathscr{P})) = 1-def(\mathscr{P})$. A choice of lifts of the q-cells of $C(\mathscr{P})$ to the universal cover $\widetilde{C(\mathscr{P})}$ determines a basis for $C_q(\widetilde{C(\mathscr{P})})$ as a free left $\mathbb{Z}[\pi]$-module. We view these as modules of column vectors. The differentials are given by $\partial_1(c_1^{(i)}) = (\varphi(x_i)-1)c_0$ and $\partial_2(c_2^{(j)}) = \Sigma_{1\le i\le g}\varphi(\frac{\partial r_j}{\partial x_i})c_1^{(i)}$. (We extend φ linearly to the group rings.) The module of 0-cycles $Z_0(\widetilde{C(\mathscr{P})})$ is isomorphic to $I(\pi)$, and so $I(\pi)$ has a $g\times h$ presentation matrix with (i,j)th entry $\varphi(\frac{\partial R_j}{\partial x_i})$. (We shall refer to $C_*(\widetilde{C(\mathscr{P})})$ as the *Fox-Lyndon resolution* of $\mathbb{Z}$ associated to $\mathscr{P}$.)

Lemma 2. *Let $\pi = G * F(n)$, where $G = *_{i=1}^m G_i$ is the free product of $m \ge 1$ finitely generated, one-ended groups G_i and $n \ge 0$. Then $E^1\mathbb{Z} \cong \mathbb{Z}[\pi]^{m+n-1}$.*

Proof. If $n=0$ the result follows from the Mayer-Vietoris sequence for the free product, with coefficients $\mathbb{Z}[\pi]$.

In general, let $C_*(G)$ be a resolution of the augmentation module by free $\mathbb{Z}[G]$-modules with $C_0(G) = \mathbb{Z}[G]$. Then there is a corresponding resolution $C_*(\pi)$ with $C_q(\pi) \cong \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[G]} C_q(G)$ if $q \ne 1$ and $C_1(\pi) \cong \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[G]} C_q(G) \oplus \mathbb{Z}[\pi]^n$. Hence there is a short exact sequence of chain complexes

$$0 \to \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[G]} C_*(G) \to C_*(\pi) \to \mathbb{Z}[\pi]^n \to 0,$$

where the third term is concentrated in degree 1. The exact sequence of cohomology with coefficients $\mathbb{Z}[\pi]$ and conjugation give a short exact sequence

$$0 \to \mathbb{Z}[\pi]^s \to \overline{H^1(\pi;\mathbb{Z}[\pi])} \to \overline{H^1(Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi]\otimes_{\mathbb{Z}[G]} C_*(G),\mathbb{Z}[\pi])} \to 0.$$

We may identify the right-hand term with $\mathbb{Z}[\pi]\otimes_{\mathbb{Z}[G]} \overline{H^1(G;\mathbb{Z}[G])} \cong \mathbb{Z}[\pi]^{m-1}$, since G is finitely generated. The middle term is $E^1\mathbb{Z}$, and so the lemma follows easily.

The hypothesis of this lemma holds if π is torsion free but not free. On the other hand, if π is a nontrivial free group then $E^1\mathbb{Z}$ has projective dimension 1 as a $\mathbb{Z}[\pi]$-module, and so the conclusion fails.

If M is a $\mathbb{Z}[\pi]$-module and ν is a subgroup of π then $M|_\nu$ shall denote the $\mathbb{Z}[\nu]$-module obtained by restriction of scalars.

3 The Whitehead functor and hermitian pairings

Let A and B be abelian groups. A function $f : A \to B$ is *quadratic* if $f(-a) = f(a)$ for all $a \in A$ and if $f(a+b) - f(a) - f(b)$ defines a bilinear function from $A \times A$ to B. The *Whitehead quadratic functor* Γ_W assigns to each abelian group A an abelian group $\Gamma_W(A)$ and a quadratic function $\gamma_A : A \to \Gamma_W(A)$ which is universal for quadratic functions with domain A. The natural epimorphism from A onto $A/2A = \mathbb{F}_2 \otimes A$ is quadratic, and so induces a canonical epimorphism q_A from $\Gamma_W(A)$ to $A/2A$. Let $A \odot A = A \otimes A / \langle a \otimes b - b \otimes a | \forall a, b \in A \rangle$ be the *symmetric square* of A. Then the kernel of q_A is the image of $A \odot A$ under the homomorphism s from $A \odot A$ to $\Gamma_W(A)$ given by $s(a \odot b) = \gamma_A(a+b) - \gamma_A(a) - \gamma_A(b)$. Thus there is an exact sequence

$$A \odot A \xrightarrow{s} \Gamma_W(A) \xrightarrow{q_A} /2A \to 0.$$

Moreover, $2\gamma_A(a) = s(a \odot a)$, for all $a \in A$. (Topologically, if $\eta : S^3 \to S^2$ is the Hopf map and $x \in \pi_2(X)$ then $2x \circ \eta = [x, x]$, the Whitehead product in $\pi_3(X)$.) This sequence is short exact if A is torsion free [4, §1.2].

If A and B are abelian groups the inclusions into $A \oplus B$ induce a canonical splitting $\Gamma_W(A \oplus B) \cong \Gamma_W(A) \oplus \Gamma_W(B) \oplus (A \otimes B)$. Since $\Gamma(\mathbb{Z}) \cong \mathbb{Z}$ it follows by a finite induction that if $A \cong \mathbb{Z}^r$ then $\Gamma_W(\mathbb{Z}^r)$ is finitely generated and free, and that s is injective. If A is any free abelian group, every finitely generated subgroup of such a group lies in a finitely generated direct summand, and so $\Gamma_W(A)$ is again free, and s is injective.

A w-hermitian pairing on a finitely generated $\mathbb{Z}[\pi]$-module M is a function $b : M \times M \to \mathbb{Z}[\pi]$ which is linear in the first variable and such that $b(n,m) = \overline{b(m,n)}$, for all $m, n \in M$. The *adjoint homomorphism* $\widetilde{b} : M \to M^\dagger$ is given by $\widetilde{b}(n)(m) = b(m,n)$, for all $m, n \in M$. The pairing b is *nonsingular* if $\widetilde{b}$ is an isomorphism.

Let $Her_w(M)$ be the group of w-hermitian pairings on M. Let $ev_M(m)(n,n') = \overline{n(m)}n'(m)$ for all $m \in M$ and $n, n' \in M^\dagger$. Then $ev_M(m)(n,n')$ is quadratic in m and w-hermitian in n and n' and $ev_M(gm) = w(g)ev_M(m)$ for all $g \in \pi$ and $m \in M$. Hence ev_M determines a homomorphism

$$B_M : \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M) \to Her_w(M^\dagger).$$

Let $M \odot M$ have the diagonal π-action, given by $g(m \odot n) = gm \odot gn$, for all $g \in \pi$ and $m, n \in M$, and let $M \odot_\pi M = \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (M \odot M)$.

Theorem 1. *Let π be a group, $w : \pi \to \mathbb{Z}^\times$ a homomorphism and M a finitely generated projective $\mathbb{Z}[\pi]$-module. If* $\mathrm{Ker}(w)$ *has no element of order* 2 *then B_M is surjective, while if there is no element $g \in \pi$ of order* 2 *such that $w(g) = -1$ then B_M is injective.*

Proof. Since M is a free abelian group there is a short exact sequence

$$0 \to M \odot M \to \Gamma_W(M) \to M/2M \to 0,$$

and $\Gamma_W(M)$ is free as an abelian group. This is a sequence of $\mathbb{Z}[\pi]$-modules and homomorphisms. Since M is projective, $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} M$ is also free as an abelian group. Hence the sequence

$$0 \to M \odot_\pi M \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M) \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} M/2M = \mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M \to 0$$

is also exact, since $Tor_1^{\mathbb{Z}[\pi]}(\mathbb{Z}^w, M/2M) = \mathrm{Ker}(2.id_{\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} M}) = 0$.

Let $\eta_M : M \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$ be the composite of γ_M with the reduction from $\Gamma_W(M)$ to $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$. Then the composite of η_M with the projection to $\mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M$ is the canonical epimorphism. Let $[m \odot n]$ be the image of $m \odot n$ in $M \odot_\pi M$.

Suppose first that M is a free $\mathbb{Z}[\pi]$-module, with basis $e_1, \dots, e_r$, and let $e_1^*, \dots, e_r^*$ be the dual basis for $M^\dagger$, defined by $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ if $i \neq j$. Since

$$[m \odot gn] = [g(g^{-1}m \odot n)] = [\bar{g}m \circ n] \quad \text{in } M \odot_\pi M,$$

the typical element of $M \odot_\pi M$ may be expressed in the form $\mu = \Sigma_{i\le j}(r_{ij}e_i) \odot e_j$. For such an element

$$B_M(\mu)(e_k^*, e_l^*) = r_{kl}, \quad \text{for } k < l,$$

and

$$B_M(\mu)(e_k^*, e_l^*) = r_{kk} + \bar{r}_{kk}, \quad \text{for } k = l.$$

In particular, $B_M(\mu)$ is *even*: if $\varepsilon_2 : \mathbb{Z}[\pi] \to \mathbb{F}_2$ is the composite of the augmentation with reduction *mod* (2) then $\varepsilon_2(B_M(\mu)(n,n)) = 0$ for all $n \in M^\dagger$. If $m \in M$ has non-trivial image in $\mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M$ then $\varepsilon_2(e_i^*(m)) \neq 0$ for some $i \le r$. Hence $B_M(\eta_M(m))$ is not even, and it follows easily that $\mathrm{Ker}(B_M) \le M \odot_\pi M$. If $B_M(\mu) = 0$, for some $\mu = \Sigma_{i\le j}(r_{ij}e_i) \odot e_j$, then $r_{kl} = 0$, if $k < l$, and $r_{ii} + \bar{r}_{ii} = 0$, for all i.

If π has no orientation reversing element of order 2 and $B_M(\mu) = 0$, where $\mu = \Sigma_{i\le j}(r_{ij}e_i) \odot e_j$, then $r_{ii} = \Sigma_{g\in F(i)} a_{ig}(g - \bar{g})$, where $F(i)$ is a finite subset of π, for $1 \le i \le r$. Since $((g - \bar{g})e_i) \odot e_i = 0$ it follows easily that $\mu = \Sigma(r_{ii}e_i) \odot e_i = 0$. Hence B_M is injective.

To show that B_M is surjective when $\mathrm{Ker}(w)$ has no element of order 2 it shall suffice to assume that M has rank 1 or 2, since h is determined by the values $h_{ij} = h(e_i^*, e_j^*)$. Let $\varepsilon_w[m, m']$ be the image of $m \odot m'$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$. Then

$$B_M(\varepsilon_w[m, m'])(n, n') = \overline{n(m)}n'(m') + \overline{n(m')}n'(m),$$

for all $m, m' \in M$ and $n, n' \in M^\dagger$. Suppose first that M has rank 1. Since $h_{11} = \bar{h}_{11}$ and $\mathrm{Ker}(w)$ has no element of order 2 we may write $h_{11} = 2b + \delta + \Sigma_{g\in F}(g + \bar{g})$, where $b = \bar{b}$, $\delta = 1$ or 0 and F is a finite subset of π. Let

$$\mu = \varepsilon_w[(b + \delta + \Sigma_{g\in F} g)e_1, e_1] + \delta\eta_M(e_1).$$

Then $B_M(\mu)(e_1^*, e_1^*) = h_{11}$. If M has rank 2 and $h_{11} = h_{22} = 0$ let $\mu = \varepsilon_w[h_{12}e_1, e_2]$. Then $B_M(\mu)(e_i^*, e_j^*) = h_{ij}$. In each case $B_M(\mu) = h$, since each side of the equation is a w-hermitian pairing on $M^\dagger$.

Now suppose that M is projective, and that P is a finitely generated projective complement to M, so that $M\oplus P\cong\mathbb{Z}[\pi]^r$ for some $r\geq 0$. The inclusion of M into the direct sum induces a split monomorphism from $\Gamma_W(M)$ to $\Gamma_W(\mathbb{Z}[\pi]^r)$ which is clearly compatible with B_M and $B_{\mathbb{Z}[\pi]^r}$. We may extend an hermitian pairing h on $M^\dagger$ to a pairing h_1 on $M^\dagger\oplus P^\dagger$ by setting $h_1(n,p)=h_1(p',p)=0$ for all $n\in M^\dagger$ and $p,p'\in P^\dagger$. Clearly $h_1|_{M\times M}=h$ and so this extension determines a split monomorphism from $Her_w(M^\dagger)$ to $Her_w((\mathbb{Z}[\pi]^r)^\dagger)$. If $h_1=B_{\mathbb{Z}[\pi]^r}(\theta)$ then $h=B_M(\theta_M)$, where θ_M is the image of θ under the homomorphism induced by the projection from $M\oplus P$ onto M. Thus if $B_{\mathbb{Z}[\pi]^r}$ is a monomorphism or an epimorphism so is B_M.

In particular, if π has no 2-torsion then B_M is an isomorphism, for any projective $\mathbb{Z}[\pi]$-module M. The restriction on 2-torsion is necessary, as can be seen by considering the group $G=\mathbb{Z}/2\mathbb{Z}=\langle g\mid g^2\rangle$ with w trivial and h the pairing on $M=\mathbb{Z}[G]$ determined by $h(m,n)=mg\bar{n}$.

Let E be another left $\mathbb{Z}[\pi]$-module. Then the summand $M\otimes E$ of $\Gamma_W(M\oplus E)$ has the diagonal left $\mathbb{Z}[\pi]$-module structure. Let $d:M\to M^{\dagger\dagger}$ and $t:\mathbb{Z}\otimes_{\mathbb{Z}[\pi]}(M\otimes E)\to Hom_{\mathbb{Z}[\pi]}(M,E)$ be given by $d(m)(\mu)=\overline{\mu(m)}$ and $t(\mu\otimes e)(m)=\mu(m)e$, for all $m\in M$, $\mu\in M^\dagger$ and $e\in E$. If M is finitely generated and projective these functions are isomorphisms (of left $\mathbb{Z}[\pi]$-modules and abelian groups, respectively). Let $\widetilde{B_M}(\gamma)$ be the adjoint of $B_M(1\otimes\gamma)$, for all $\gamma\in\Gamma_W(M)$.

Lemma 3. *Let M be a finitely generated projective $\mathbb{Z}[\pi]$-module and $\theta:M\to E$ be a $\mathbb{Z}[\pi]$-module homomorphism. Let $d:M\to M^{\dagger\dagger}$ and $t:\mathbb{Z}\otimes_{\mathbb{Z}[\pi]}(M\otimes E)\to Hom_{\mathbb{Z}[\pi]}(M,E)$ be the isomorphisms defined above, and let*

$$\alpha_\theta(m,e)=(m,e+\theta(m)),$$

for all $(m,e)\in M\oplus E$. Then α_θ is an automorphism of $M\oplus E$ and

$$\Gamma_W(\alpha_\theta)(\gamma)-\gamma\equiv(d\otimes 1)^{-1}[(\widetilde{B_M}(\gamma)\otimes 1)(t^{-1}(\theta))]\quad mod\quad \Gamma_W(E),$$

for all $\gamma\in\Gamma_W(M)$.

Proof. The homomorphism α_θ is clearly an automorphism of $M\oplus E$ which restricts to the identity on the summands E and M, and

$$\Gamma_W(\alpha_\theta)(\gamma_{M\oplus E}(m))=\gamma_{M\oplus E}(m)+\gamma_{M\oplus E}(\theta(m))+m\otimes\theta(m),$$

for all $m\in M$ [4, 1.2.7].

Let $\beta_m=B_M(1\otimes\gamma_M(m))$, for $m\in M$. Now the adjoint homomorphism $\widetilde{\beta_m}$ is given by $\widetilde{\beta_m}(\mu)=\overline{\mu(m)}d(m)$. Since t is surjective we have $\theta=t(\Sigma\mu_i\otimes e_i)$, for some $\mu_i\in M^\dagger$ and $e_i\in E$. Then $(\widetilde{\beta_m}\otimes 1)(t^{-1}(\theta))=$

$$\Sigma\widetilde{\beta_m}(\mu_i)\otimes e_i=\Sigma d(m)\otimes\mu_i(m)e_i=d(m)\otimes\theta(m)=(d\otimes 1)(m\otimes\theta(m)).$$

Since

$$\Gamma_W(\alpha_\theta)(\gamma_{M\oplus E}(m))-\gamma_{M\oplus E}(m)\equiv(d\otimes 1)^{-1}[(\widetilde{\beta_m}\otimes 1)(t^{-1}(\theta))]\ \ mod\ \ \Gamma_W(E),$$

for all $m\in M$, and since each side is quadratic in m, we have

$$\Gamma_W(\alpha_\theta)(\gamma)-\gamma\equiv(d\otimes 1)^{-1}[(\widetilde{B_M}(\gamma)\otimes 1)(t^{-1}(\theta))]\quad mod\quad \Gamma_W(E),$$

for all $\gamma\in\Gamma_W(M)$.

4 Postnikov stages

Let X be a based, connected cell complex with fundamental group π, and let $p_X:\widetilde{X}\to X$ be its universal covering projection. Let $E_0(X)$ be the group of based homotopy classes of based self-homotopy equivalences of X, and $E_\pi(X)$ be the subgroup which induces the identity on π. If we fix a basepoint for $\widetilde{X}$ over the basepoint of X then there are well-defined Hurewicz homomorphisms

$$hwz_q:\pi_q(X)=\pi_q(\widetilde{X})\to H_q(\widetilde{X};\mathbb{Z}),\quad \text{for all } q\geq 2.$$

Let $f_{X,k}:X\to P_k(X)$ be the k^{th} stage of the Postnikov tower for X. We may construct $P_k(X)$ by adjoining cells of dimension at least $k+2$ to kill the higher homotopy groups of X. The map $f_{X,k}$ is then given by the inclusion of X into $P_k(X)$, and is a $(k+1)$-connected map. In particular, $P_1(X)\simeq K=K(\pi,1)$ and $c_X=f_{X,1}$ is the classifying map for the fundamental group $\pi=\pi_1(X)$.

If M is a left $\mathbb{Z}[\pi]$-module let $L_\pi(M,n)$ be the *generalized Eilenberg-Mac Lane space* over $K=K(\pi,1)$ realizing the given action of π on M. Thus the classifying map for $L=L_\pi(M,n)$ is a principal $K(M,n)$-fibration with a section $\sigma:K\to L$. The pair (c_L,σ) is an object in the category *ex-K* of spaces over K with sections, and we may view $L_\pi(M,n)$ as the *ex-K* loop space $\overline{\Omega}L_\pi(M,n+1)$ [53], with section σ and projection c_L. Let $\mu:L\times_K L\to L$ be the (fibrewise) loop multiplication. Then $\mu(id_L,\sigma c_L)=\mu(\sigma c_L,id_L)=id_L$ in $[L;L]_K$. Let $\iota_{M,n}\in H^n(L;M)$ be the characteristic element.

Let $[X;Y]_K$ be the set of homotopy classes over $K=K(\pi,1)$ of maps $f:X\to Y$ such that $c_X=c_Yf$. (These may also be considered as π-equivariant homotopy classes of π-equivariant maps from $\widetilde{K}$ to $\widetilde{L}$.) The function $\theta:[X,L]_K\to H^n(X;M)$ given by $\theta(f)=f^*\iota_{M,n}$ is a isomorphism with respect to the addition on $[X,L]_K$ determined by μ. Thus $\theta(id_L)=\iota_{M,n}$, $\theta(\sigma c_X)=0$ and $\theta(\mu(f,f'))=\theta(f)+\theta(f')$ [2, §V.2].

Let $k_1(X)\in H^3(\pi;\pi_2(X))$ be the first k-invariant which may be defined as the primary obstruction to constructing a left inverse to the classifying map c_X. (It may also be identified with the class in $Ext^3_{\mathbb{Z}[\pi]}(\mathbb{Z},\Pi)$ of the iterated extension

$$0\to\pi_2(X)\to C_2/\partial C_3\to C_1\to C_0\to\mathbb{Z}\to 0.$$

This was surely known to Eilenberg, Mac Lane and Whitehead, and appears closely related to the Homotopy Addition Theorem [62, Theorem IV.6.1] or [54, Proposition 7.5.3], but it is difficult to find an accessible published proof. See [10, Theorem 12.2.10] or [49].)

Let $f_X = f_{X,2}$ be the second stage of the Postnikov tower for X. The *algebraic* 2*-type* $[\pi,\pi_2(X),k_1(X)]$ and the Postnikov 2-stage determine each other. More precisely, $P_2(X)\simeq P_2(Y)$ if and only if there are isomorphisms $\alpha:\pi\cong\pi_1(Y)$ and $\beta:\pi_2(X)\cong\pi_2(Y)$ such that β is α-semilinear and $\alpha^*k_1(Y)=\beta_\# k_1(X)$ in $H^3(\pi;\pi_2(Y))$. Moreover,

$$k_1(X)=0 \;\Leftrightarrow\; c_{P_2(X)} \text{ has a section } \;\Leftrightarrow\; P_2(X)\simeq L_\pi(\pi_2(X),2).$$

Let $L=L_\pi(M,2)$. Then $E_\pi(L)$ is the group of units of $[L,L]_K$ with respect to composition. We shall use the following special case of a result of Tsukiyama [56]; we give only the part that we need below.

Lemma 4. *There is an exact sequence*

$$1\to H^2(\pi;M)\to E_\pi(L)\to Aut(M)\to 1.$$

Proof. Let $\theta:[K,L]_K\to H^2(\pi;M)$ be the isomorphism given by $\theta(s)=s^*\iota_{M,2}$, and let $\theta^{-1}(\phi)=s_\phi$ for $\phi\in H^2(\pi;M)$. Then s_ϕ is a homotopy class of sections of c_L, $s_0=\sigma$ and $s_{\phi+\psi}=\mu(s_\phi,s_\psi)$, while $\phi=s_\phi^*\iota_{M,2}$. (Recall that $\mu:L\times_K L\to L$ is the fibrewise loop multiplication.)

Let $h_\phi=\mu(s_\phi c_L,id_L)$. Then $c_L h_\phi=c_L$ and so $h_\phi\in[L;L]_K$. Clearly $h_0=\mu(\sigma c_L,id_L)=id_L$ and $h_\phi^*\iota_{M,2}=\iota_{M,2}+c_L^*\phi\in H^2(L;M)$. We also see that

$$\begin{aligned} h_{\phi+\psi} &= \mu(\mu(s_\phi,s_\psi)c_L,id_L)\\ &= \mu(\mu(s_\phi c_L,s_\psi c_L),id_L)\\ &= \mu(s_\phi c_L,\mu(s_\psi c_L,id_L)) \end{aligned}$$

(by homotopy associativity of μ) and so

$$h_{\phi+\psi}=\mu(s_\phi c_L,h_\psi)=\mu(s_\phi c_L h_\psi,h_\psi)=h_\phi h_\psi.$$

Therefore h_ϕ is a homotopy equivalence for all $\phi\in H^2(\pi;M)$, and $\phi\mapsto h_\phi$ defines a homomorphism from $H^2(\pi;M)$ to $E_\pi(L)$.

The lift of h_ϕ to the universal cover $\widetilde{L}$ is (non-equivariantly) homotopic to the identity, since the lift of c_L is (non-equivariantly) homotopic to a constant map. Therefore h_ϕ acts as the identity on $M=\pi_2(L)$.

The homomorphism $h:\phi\mapsto h_\phi$ is in fact an isomorphism onto the kernel of the action of $E_\pi(L)$ on M [56], and the extension splits: $E_\pi(L)$ is isomorphic to a semidirect product $H^2(\pi;M)\rtimes Aut(M)$ [3, Corollary 8.2.7]. More generally, if $P=P_2(X)$, $\Pi=\pi_2(X)$ and H is the subgroup of $Aut_\pi(\Pi)\rtimes Aut(\pi)$ which fixes $k_1(X)\in H^3(\pi;\Pi)$ then

$$E_0(P)\cong H^2(\pi;\Pi)\rtimes H$$

(see [53, Part II]). Thus if $P = L_\pi(\Pi)$ every automorphism of π lifts to a self-homotopy equivalence of L, and $E_0(L) \cong E_\pi(L) \rtimes Aut(\pi)$.

Let $X^{[k]}$ be the k-skeleton of X, for all $k \geq 0$, and let $\Pi = \pi_2(X)$. The image of $\pi_3(X^{[2]})$ in $\pi_3(X^{[3]})$ is isomorphic to $\Gamma_W(\Pi)$, and the inclusion of the 3-skeleton induces a homomorphism $\iota_X : \Gamma_W(\Pi) \to \pi_3(X)$. The composite of ι_X with the natural map from $\Pi \odot \Pi$ to $\Gamma_W(\Pi)$ is the Whitehead product $[-,-]$, and there is a natural *Whitehead exact sequence* of abelian groups

$$\pi_4(X) \xrightarrow{hwz_4} H_4(\widetilde{X};\mathbb{Z}) \xrightarrow{b_X} \Gamma_W(\Pi) \xrightarrow{\iota_X} \pi_3(X) \xrightarrow{hwz_3} H_3(\widetilde{X};\mathbb{Z}) \to 0,$$

where b_X is the *secondary boundary homomorphism* [61]. (See [4, 2.1.17].) This is an exact sequence of left $\mathbb{Z}[\pi]$-modules, by naturality. (Note also that the Whitehead sequence for $K(\Pi,2)$ gives $H_4(\Pi,2;\mathbb{Z}) \cong \Gamma_W(\Pi)$.)

The homology spectral sequence for $P_3(\widetilde{X})$ as a fibration over $K(\Pi,2)$ with fibre $K(\pi_3(X),3)$ gives an exact sequence

$$0 \to H_4(P_3(\widetilde{X});\mathbb{Z}) \to H_4(\Pi,2;\mathbb{Z}) \xrightarrow{d^2_{4,0}} H_3(\pi_3(X),3;\mathbb{Z}) \to H_3(P_3(\widetilde{X});\mathbb{Z}) \to 0,$$

in which $d^2_{4,0}$ is the homology transgression. Composing $d^2_{4,0}$ with the inverse of the Hurewicz isomorphism hwz_3 for $K(\pi_3(X),3)$ gives the image of the second k-invariant $k_2(\widetilde{X}) \in H^4(\Pi,2;\pi_3(X))$ in $Hom(H_4(\Pi,2;\mathbb{Z}),\pi_3(X))$ under the evaluation homomorphism, by the interpretation of k-invariants in terms of transgression [47]. In fact $d^2_{4,0} = hwz_3\iota_X$ [4, Theorem 2.5.10].

5 PD_4-complexes and intersection pairings

Let X be a based finitely dominated cell complex, with the natural left $\mathbb{Z}[\pi]$-module structure. The equivariant cellular chain complex $C_* = C_*(X;\mathbb{Z}[\pi])$ of $\widetilde{X}$ is a complex of left $\mathbb{Z}[\pi]$-modules, and is $\mathbb{Z}[\pi]$-chain homotopy equivalent to a finitely generated complex of projective modules. Let $B_q \leq Z_q \leq C_q$ be the submodules of q-boundaries and q-cycles, respectively. Let $C^q = Hom_{\mathbb{Z}[\pi]}(C_q,\mathbb{Z}[\pi])$, for all $q \geq 0$, and let $\Pi = H_2(\widetilde{X};\mathbb{Z}) = H_2(C_*)$. Recall that the choice of a basepoint for $\widetilde{X}$ determines an isomorphism $\pi_2(X) \cong \Pi$.

Let $ev : \overline{H^2(X;\mathbb{Z}[\pi])} \to \Pi^\dagger$ be the evaluation homomorphism, given by

$$ev([c])([z]) = [c] \cap [z] = c(z) \quad \forall\, c \in C^2 \text{ and } z \in C_2.$$

This homomorphism sits in the *evaluation* exact sequence

$$0 \to E^2\mathbb{Z} \to \overline{H^2(X;\mathbb{Z}[\pi])} \xrightarrow{ev} \Pi^\dagger \to E^3\mathbb{Z} \to \overline{H^3(X;\mathbb{Z}[\pi])}.$$

(See [34, Lemma 3.3].) If X is a PD_4-complex then $H^3(X;\mathbb{Z}[\pi]) = H_1(\widetilde{X};\mathbb{Z}) = 0$, and the evaluation sequence is a 4-term exact sequence.

We assume henceforth that X is a PD_4-complex, with orientation character $w = w_1(X)$. Let X^+ be the orientable covering space associated to $\pi^+ = \mathrm{Ker}(w)$. The complex X is finitely dominated and is homotopy equivalent to $X_o \cup_\phi e^4$, where X_o is a complex of dimension at most 3 and $\phi \in \pi_3(X_o)$ [60]. In particular, π is finitely presentable. In [37] and [38] cellular decompositions were used to study the homotopy types of PD_4-complexes. Here we shall rely more consistently on the dual Postnikov approach.

Lemma 5. *If π is infinite the homotopy type of X is determined by $P_3(X)$.*

Proof. If X and Y are two such PD_4-complexes and $h : P_3(X) \to P_3(Y)$ is a homotopy equivalence then $hf_{X,3}$ is homotopic to a map $g : X \to Y$. Since π is infinite $H_4(\widetilde{X};\mathbb{Z}) = H_4(\widetilde{Y};\mathbb{Z}) = 0$, by Poincaré duality. Since $\pi_i(g)$ is is an isomorphism for $i \leq 3$ any lift $\tilde{g} : \widetilde{X} \to \widetilde{Y}$ is a homotopy equivalence, by the Hurewicz and Whitehead theorems, and so g is a homotopy equivalence.

In particular, if π is torsion free but not free then $H_3(X;\mathbb{Z}[\pi]) \cong E^1\mathbb{Z}$ is a free $\mathbb{Z}[\pi]$-module, by Lemma 2, and so $\pi_3(X) \cong \Gamma_W(\Pi) \oplus E^1\mathbb{Z}$. Hence the homotopy type of X is determined by π, w, Π and the first two k-invariants.

Let $H = \overline{H^2(X;\mathbb{Z}[\pi])}$. A choice of generator $[X]$ for $H_4(X;\mathbb{Z}^w) \cong \mathbb{Z}$ determines a Poincaré duality isomorphism $D : H \to \Pi$ by $D(u) = u \cap [X]$, for all $u \in H$. Moreover $H^3(X;\mathbb{Z}[\pi]) = 0$. The *cohomology intersection pairing* $\lambda : H \times H \to \mathbb{Z}[\pi]$ is defined by

$$\lambda(u,v) = ev(v)(D(u)) \quad \text{for all } \ u,v \in H.$$

This pairing is w-hermitian: $\lambda(gu,hv) = g\lambda(u,v)\bar{h}$ and $\lambda(v,u) = \overline{\lambda(u,v)}$ for all $u,v \in H$ and $g,h \in \pi$. If X is a closed 4-manifold this pairing is equivalent under Poincaré duality to the equivariant intersection pairing on Π. (See [51, page 82].) Replacing $[X]$ by $-[X]$ changes the sign of the pairing. Since $\lambda(u,e) = 0$ for all $u \in H$ and $e \in E = E^2\mathbb{Z}$ the pairing λ induces a pairing

$$\lambda_X : H/E \times H/E \to \mathbb{Z}[\pi].$$

The adjoint $\widetilde{\lambda_X}$ is a monomorphism, since $\mathrm{Ker}(ev) = E$. The PD_4-complex X is *strongly minimal* if $\lambda_X = 0$.

The next lemma relates nonsingularity of λ_X, projectivity of Π and H/E and conditions on $E^2\mathbb{Z}$ and $E^3\mathbb{Z}$.

Lemma 6. *Let X be a PD_4-complex with fundamental group π, and let $E = E^2\mathbb{Z}$, $H = \overline{H^2(X;\mathbb{Z}[\pi])}$ and $\Pi = \pi_2(X)$. Then*

1. *$\lambda_X = 0$ if and only if $H = E$, and then $E^3\mathbb{Z} \cong E^\dagger$;*
2. *if λ_X is nonsingular and H/E is a projective $\mathbb{Z}[\pi]$-module then $E^3\mathbb{Z} \cong E^\dagger$;*
3. *if λ_X is nonsingular and $E^\dagger = 0$ then $E^3\mathbb{Z} = 0$;*
4. *if $E^3\mathbb{Z} = 0$ then λ_X is nonsingular;*
5. *if $E^3\mathbb{Z} = 0$ and Π is a projective $\mathbb{Z}[\pi]$-module then $E = 0$;*
6. *if $\pi = G * F(n)$, where $G = *_{i=1}^m G_i$ is the free product of $m \geq 1$ one-ended groups and Π is a projective $\mathbb{Z}[\pi]$-module then $c.d.\pi \leq 4$, with equality if π has one end.*

Proof. Let $p : \Pi \to \Pi/D(E)$ and $q : H \to H/E$ be the canonical epimorphisms. Poincaré duality induces an isomorphism $\gamma : H/E \cong \Pi/D(E)$. It is straightforward to verify that $p^\dagger(\gamma^\dagger)^{-1}\widetilde{\lambda_X}q = ev$, and (1) is clear.

If λ_X is nonsingular then $\widetilde{\lambda_X}$ is an isomorphism, and so $\mathrm{Coker}(p^\dagger) = \mathrm{Coker}(ev)$. If moreover $\Pi/D(E) \cong H/E$ is projective then Π is a direct sum: $\Pi \cong (\Pi/D(E)) \oplus D(E)$. Hence $\Pi^\dagger \cong (\Pi/D(E))^\dagger \oplus E^\dagger$, and so $E^\dagger \cong \mathrm{Coker}(p^\dagger) = E^3\mathbb{Z}$.

If λ_X is nonsingular and $E^\dagger = 0$ then $\widetilde{\lambda_X}$ and $p^\dagger$ are isomorphisms, and so $ev = p^\dagger(\gamma^\dagger)^{-1}\widetilde{\lambda_X}q$ is an epimorphism. Hence $E^3\mathbb{Z} = 0$.

If $E^3\mathbb{Z} = 0$ then $H/E = \Pi^\dagger$ and $ev = q$. Since q is an epimorphism it follows that $p^\dagger(\gamma^\dagger)^{-1}\widetilde{\lambda_X} = id_{\Pi^\dagger}$, and so $p^\dagger$ is an epimorphism. Since $p^\dagger$ is also a monomorphism it is an isomorphism. Therefore $\widetilde{\lambda_X} = \gamma^\dagger(p^\dagger)^{-1}$ is also an isomorphism.

If Π is projective then so is $\Pi^\dagger$. If, moreover, $E^3\mathbb{Z} = 0$ then $H \cong E \oplus \Pi^\dagger$. Hence E is projective, since it is a direct summand of $H \cong \Pi$, and so $E \cong E^{\dagger\dagger} = 0$.

If π is a free product of $m \geq 1$ one-ended groups and n copies of $\mathbb{Z}$ then $E^1\mathbb{Z} \cong \mathbb{Z}[\pi]^{m+n-1}$, by Lemma 2. If, moreover, Π is projective then so are $C_3' = C_3 \oplus \Pi$ and $C_4' = C_4 \oplus E^1\mathbb{Z}$. We may easily extend the differentials of C_* to obtain a projective resolution C_*' of length 4 for $\mathbb{Z}$. Hence $c.d.\pi \leq 4$. If π has one end and Π is projective then $H^4(\pi;\mathbb{Z}[\pi]) = E^4\mathbb{Z} \cong H^4(X;\mathbb{Z}[\pi]) \cong \mathbb{Z}$, by the Universal Coefficient spectral sequence and Poincaré duality, and so $c.d.\pi = 4$.

Parts (3) and (4) together imply that if $E^2\mathbb{Z} = 0$ then λ_X is nonsingular if and only if $E^3\mathbb{Z} = 0$ also. Does this remain the case without any conditions on $E^2\mathbb{Z}$? If Π is projective and λ_X is nonsingular then $\pi \cong \pi_1(Z)$ for some PD_4-complex Z with $\pi_2(Z) = 0$, by Theorem 5 below, and so $E^2\mathbb{Z} = E^3\mathbb{Z} = 0$.

We shall say that a based map $f : X \to Y$ between PD_4-complexes is a *degree-1 map* and write $f_*[X] = \pm[Y]$ if $f^*w_1(Y) = w_1(X) = w$ and the lift of f to a based map of universal covers induces an isomorphism $H_4(X;\mathbb{Z}^w) \cong H_4(Y;\mathbb{Z}^w)$. (Note that if we do not work with based maps the homomorphisms induced by different lifts may differ by sign – see [55] for an investigation of the subtleties involved.) The homomorphism $\pi_1(f)$ is then surjective, and Poincaré duality in X and Y determine *umkehr* homomorphisms $f_! : H_*(Y;\mathbb{Z}[\pi_1(Y)]) \to H_*(X; f^*\mathbb{Z}[\pi_1(Y)])$, which split the homomorphisms induced by f. The umkehr homomorphisms are well-defined up to sign [51, §10.3]. If $f : X \to Z$ is a 2-connected degree-1 map then cap product with $[X]$ induces an isomorphism from the *surgery cokernel* $K^2(f) = \overline{\mathrm{Cok}(H^2(f;\mathbb{Z}[\pi]))}$ to $K_2(f)$, and the induced pairing λ_f on $K^2(f) \times K^2(f)$ is nonsingular [60, Theorem 5.2].

We shall not usually specify a fundamental class $[X]$, and so we shall allow orientation-reversing homotopy equivalences of oriented PD_4-complexes, and isomorphisms of modules with pairings which are isometries after a change of sign. In particular, if Y is a second PD_4-complex we write $\lambda_X \cong \lambda_Y$ if there is an isomorphism $\theta : \pi_1(X) \cong \pi_1(Y)$ such that $w_1(X) = w_1(Y) \circ \theta$ and a $\mathbb{Z}[\pi]$-module isomorphism $\Theta : \pi_2(X) \cong \theta^*\pi_2(Y)$ inducing an isometry of cohomology intersection pairings (after changing the sign of $[Y]$, if necessary).

In [5] it is shown that a PD_4-complex X is determined by its algebraic 2-type (i.e., by $P_2(X)$) together with $w_1(X)$ and $f_{X*}[X]$. (The main step involves showing that if $h: P_2(X) \to P_2(Y)$ is a homotopy equivalence such that $h^*w_1(Y) = w_1(X)$ and $h_* f_{X*}[X] = f_{Y*}[Y]$ (up to sign) then $h = P_2(g)$ for some map $g: X \to Y$ such that $H_4(g;\mathbb{Z}^w)$ is an isomorphism.) Our goal is to show that under suitable conditions X is determined by the more accessible invariants encapsulated in the sextuple $[\pi, w, v_2(X), \Pi, k_1(X), \lambda_X]$. (This is the *quadratic 2-type* of X, as in [27], enhanced by the Wu classes; equivalently, by the Stiefel-Whitney classes.) If $\lambda_X \neq 0$ then λ_X determines w, since $\lambda_X(gu, gv) = w(g)g\lambda_X(u,v)g^{-1}$ for all u, v and g.

The *Wu classes* of a PD_n-complex P are the classes $v_i(P) \in H^i(P;\mathbb{F}_2)$ determined by Poincaré duality from the condition

$$u \smile v_i(P) = Sq^i u, \quad for\ all\ \ u \in H^{n-i}(P;\mathbb{F}_2).$$

If P is a manifold these are equivalent to the tangential Stiefel-Whitney classes, by the "Wu Formula" [54, Theorem 6.10.7]. (Spanier writes V_i for our v_i, and does not use the term Wu class.) In dimension 4 we can be quite explicit; if X is a PD_4-complex then $v_1 = w_1$ is the orientation character, and $v_2 = w_2 + w_1^2$. We choose to use v_2 rather than w_2 since it is the characteristic element for the intersection pairing on $H^2(X;\mathbb{F}_2)$.

It shall be useful to distinguish three "v_2-types" of PD_4-complexes:

I. $v_2(\widetilde{X}) \neq 0$ (i.e., $v_2(X)$ is not in the image of $H^2(\pi;\mathbb{F}_2)$ under c_X^*);
II. $v_2(X) = 0$;
III. $v_2(X) \neq 0$ but $v_2(\widetilde{X}) = 0$ (i.e., $v_2(X)$ is in $c_X^*(H^2(\pi;\mathbb{F}_2)) \setminus \{0\}$).

This trichotomy is due to Kreck, who formulated it in terms of Stiefel-Whitney classes of the stable normal bundle of a closed 4-manifold. The *refined* v_2-type (II and III) is given by the orbit of v_2 in $H^2(\pi;\mathbb{F}_2)$ under the action of automorphisms of π which fix the orientation character.

6 Minimal models

A *model* for a PD_4-complex X is a 2-connected degree-1 map $f: X \to Z$ to a PD_4-complex Z. (We shall also say that Z is a model for X.) The *surgery kernel* $K_2(f) = \mathrm{Ker}(\pi_2(f))$ is a finitely generated projective $\mathbb{Z}[\pi]$-module, and is an orthogonal direct summand of $\pi_2(X)$ with respect to the intersection pairing [60, Theorem 5.2]. If both complexes are finite then $K_2(f)$ is stably free. The PD_4-complex X is *order-minimal* if every such map is a homotopy equivalence, i.e., if X is minimal with respect to the order determined by such maps. It is *strongly minimal* if $\lambda_X = 0$, and is χ*-minimal* if $\chi(X) \leq \chi(Y)$, for Y any PD_4-complex with $(\pi_1(Y), w_1(Y)) \cong (\pi, w)$. We then let $q(\pi, w) = \chi(X)$ be this minimal value. (The definition of "strongly minimal" used here may be broader than the one used in [38], where we said that Z was strongly minimal if $\pi_2(Z)^\dagger = 0$. The two definitions are equivalent if $(E^2\mathbb{Z})^\dagger = 0$.)

Order minimality is the most natural property, and χ-minimality perhaps the one most easily established. It is clear that strongly minimal PD_4-complexes are order-minimal. We shall show that χ-minimality interpolates between these notions, when the L^2-Euler characteristic formula $\chi(X) = \Sigma(-1)^i\beta_i^{(2)}(X)$ applies. (Here $\beta_i^{(2)}(X)$ and $\beta_i^{(2)}(\pi)$ are the ith L^2-Betti numbers of the space X and group π. The book [45] is the definitive reference for L^2 homology; a brief outline is given in Sections 1.9 and 2.2 of [34].)

Theorem 2. *A PD_4-complex X with fundamental group π is strongly minimal if and only if $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$.*

Proof. The module $C^2(X;\mathbb{C}[\pi])$ may be identified with the group of cellular 2-cochains with compact support on $\widetilde{X}$, while the corresponding module $C^2_{(2)}(\widetilde{X})$ of L^2-cochains is the group of square-summable cellular 2-cochains on $\widetilde{X}$. The compactly supported cochains are dense in the square-summable cochains. For each $z \in \pi_2(X)$ the evaluation $ev_z : f \to f(z)$ is continuous as a linear map from $C^2_{(2)}(\widetilde{X})$ to $\mathbb{C}$. (See the proof of [34, Theorem 3.4]. If X is strongly minimal then $ev_z(f) = 0$ for all $f \in C^2(X;\mathbb{C}[\pi])$. Hence $ev_z = 0$ for all $z \in \pi_2(M)$. The L^2 analogue of the evaluation sequence (as in [19, §1.4]) then shows that c_X induces an isomorphism on the unreduced L^2-cohomology modules, and so $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$. The converse is part (3) of [34, Theorem 3.4].

The next two corollaries need a further hypothesis at present.

Corollary 3 *Suppose that either X is finite or π satisfies the Strong Bass Conjecture. Then if X is strongly minimal it is χ-minimal, and if it is χ-minimal it is order minimal.*

Proof. If X is finite or π satisfies the Strong Bass Conjecture we may use the L^2-Euler characteristic formula then $\chi(X) = \beta_2^{(2)}(X) - 2\beta_1^{(2)}(X)$ [20]. Since we may construct a $K(\pi,1)$ complex by adjoining cells of dimension > 2 to X, we have $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi)$, in general. Hence X strongly minimal implies that X is χ-minimal, by the Theorem.

Suppose that $f : X \to Y$ is a 2-connected degree-1 map and $\chi(X) = \chi(Y)$. Then $K_2(f)$ is a finitely generated projective $\mathbb{Z}[\pi]$-module and $\mathbb{Z}\otimes_{\mathbb{Z}[\pi]} K_2(f) = 0$. If X is finite then X is a stably free $\mathbb{Z}[\pi]$-module, so $K_2(f) = 0$, by a result of Kaplansky [52]. This also holds if π satisfies the Weak Bass Conjecture [18]. In either case, f is a homotopy equivalence, and so χ-minimality implies order minimality.

In particular, every sequence of 2-connected degree-1 maps

$$X_1 \to X_2 \to X_3 \to \dots$$

eventually becomes a sequence of homotopy equivalences. If $f : X \to Z$ is a 2-connected degree-1 map and Z is strongly minimal then $\lambda_f = \lambda_X$.

Corollary 4 *Suppose that either X is finite or π satisfies the Strong Bass Conjecture. If $\beta_1^{(2)}(X) = \chi(X) = 0$ then X is strongly minimal.*

Proof. In this case the L^2 Euler characteristic formula gives $\beta_2^{(2)}(X) = 0$. Hence $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$.

Strong minimality has the disadvantage of limited applicability. However, the case of greatest interest to us is when $c.d.\pi \leq 2$. The three notions of minimality are then equivalent, and order minimality is equivalent to strong minimality if and only if $c.d.\pi \leq 2$. (See Theorems 18 and 21 below, and [37] for π a free group.)

If $\pi \cong \mathbb{Z}^r$ and X is χ-minimal then X is order minimal. However, X can only be strongly minimal if $r = 1$, 2 or 4. The 4-torus $\mathbb{R}^4/\mathbb{Z}^4$ is the unique strongly minimal PD_4-complex with fundamental group $\mathbb{Z}^4$, since $E^s\mathbb{Z} = 0$ if $s \leq 3$ for this group. Hence $q(\mathbb{Z}^4) = 0$. Let K be the 2-complex corresponding to the standard presentation of $\mathbb{Z}^4$ with four generators and six relators, and let N be a regular neighbourhood of an embedding of K in $\mathbb{R}^5$. Then $M = \partial N$ is an orientable 4-manifold with $\pi_1(M) \cong \mathbb{Z}^4$ and $\chi(M) = 6$. If a 2-connected degree-1 map $f : M \to Y$ is not a homotopy equivalence then $\chi(Y) < \chi(M)$ and so $\beta_2(Y) < 12$. Since $c_Y^* H^2(\mathbb{Z}^4;\mathbb{Z})$ has rank 6 it follows easily from Poincaré duality in Y that $c_Y^* H^2(\mathbb{Z}^4;\mathbb{Z})$ cannot be self-annihilating with respect to cup product, and so c_Y has nonzero degree. However $c_{M*}[M] = 0$, since c_M factors through N, and so there can be no such map f. Thus M is order-minimal, but not χ-minimal, and not strongly minimal.

If Z is strongly minimal and $\pi \cong G_1 * G_2$ does Z decompose accordingly as a connected sum? If so, the hypothesis that π have one end would not be needed in our consideration later of groups of cohomologicial dimension 2. If M is a closed 4-manifold and $\pi_1(M) \cong G_1 * G_2$ then there is a simply-connected 4-manifold N such that $M\#N \cong P_1\#P_2$, where $\pi_1(P_i) \cong G_i$ for $i = 1,2$ [34, Theorem 14.10]. If $p_i : P_i \to Z_i$ are strongly minimal models then $p = p_1\#p_2 : M\#N \to Z_1\#Z_2$ is a strongly minimal model for $M\#N$. The image of $\pi_2(N)$ generates a projective direct summand of $\pi_2(M\#N)$ on which the intersection pairing is nonsingular, and so p factors through M, by the construction of Theorem 5 below. Thus M has a strongly minimal model which is a connected sum.

A strongly minimal 4-manifold M must be of type II or III, since $\alpha^* v_2(\widetilde{M})$ is the normal Stiefel-Whitney class $w_2(\nu_\alpha)$, for α an immersion of S^2 in $\widetilde{M}$ with normal bundle ν_α, and so $v_2(\widetilde{M})([\alpha])$ is the *mod*-2 self-intersection number of $[\alpha] \in \pi_2(M)$. Is there a purely homotopy-theoretic argument showing that all strongly minimal PD_4-complexes are of type II or III? (This is so if $c.d.\pi = 2$, by Theorem 20 below.)

Lemma 7. *Let $f : X \to Z$ be a 2-connected degree-1 map of PD_4-complexes with fundamental group π. If X is of type* II *or* III *then so is Z.*

Proof. Since f is 2-connected, $c_X = gc_Z f$, for some self homotopy equivalence g of $K(\pi,1)$. If $v_2(X) = c_X^* V$ for some $V \in H^2(\pi;\mathbb{F}_2)$ then

$$f^*(v_2(Z) \smile \alpha) = f^*(\alpha^2) = v_2(X) \smile f^*\alpha = f^*(c_Z^* g^* V \smile \alpha),$$

for all $\alpha \in H^2(Z;\mathbb{F}_2)$. Hence $v_2(Z) = c_Z^* g^* V$, since $H^4(f;\mathbb{F}_2)$ is an isomorphism.

The converse is false. For instance, the blowup of a ruled surface is of type I, but its minimal models are of type II or III. (See §14 below.)

If X has v_2-type I and $c.d.\pi = 2$ is there a model $f : X \to Z$ with $v_2(Z) = 0$?

Lemma 8. *Let Z be a PD_4-complex with fundamental group π, and let Z_ρ be the covering space associated to a subgroup ρ of finite index in π. Then Z is strongly minimal if and only if Z_ρ is strongly minimal.*

Proof. Let $\Pi = \pi_2(Z)$. Then $\pi_2(Z_\rho) \cong \Pi|_\rho$. Moreover, $H^2(\pi;\mathbb{Z}[\pi])|_\rho \cong H^2(\rho;\mathbb{Z}[\rho])$ and $Hom_{\mathbb{Z}[\pi]}(\Pi,\mathbb{Z}[\pi])|_\rho \cong Hom_{\mathbb{Z}[\rho]}(\Pi|_\rho,\mathbb{Z}[\rho])$, as right $\mathbb{Z}[\rho]$-modules, since $[\pi:\rho]$ is finite. The lemma follows from these observations.

7 Existence of strongly minimal models

In this section we shall obtain a criterion for the existence of a strongly minimal model, as a consequence of the following theorem, which may be thought of as a converse to the 4-dimensional case of Wall's Lemma 2.2 and Theorem 5.2.

Theorem 5. *Let X be a PD_4-complex with fundamental group π. If K is a finitely generated projective direct summand of $H^2(X;\mathbb{Z}[\pi])$ such that λ_X induces a nonsingular pairing on $K \times K$ then there is a PD_4-complex Z and a 2-connected degree-1 map $f : X \to Z$ with $K_2(f) = D(K)$.*

Proof. Suppose first that K is stably free and choose maps $m_i : S^2 \to X$ for $1 \le i \le s$ representing generators of $D(K)$, and such that the kernel of the corresponding epimorphism $m : \mathbb{Z}[\pi]^s \to D(K)$ is free of rank t. Attach s 3-cells to X along the m_i to obtain a cell complex Y with $\pi_1(Y) \cong \pi$, $\pi_2(Y) \cong \Pi/D(K)$ and $H_3(Y;\mathbb{Z}[\pi]) \cong H_3(X;\mathbb{Z}[\pi]) \oplus \mathbb{Z}[\pi]^t$. Since the Hurewicz map is onto in degree 3 for 1-connected spaces (such as $\widetilde{Y}$) we may then attach t 4-cells to Y along maps whose Hurewicz images form a basis for $H_3(Y,X;\mathbb{Z}[\pi])$ to obtain a cell complex Z with $\pi_1(Z) \cong \pi$ and $\pi_2(Z) \cong \Pi/D(K)$.

If K is not stably free then $K \oplus F \cong F$, where F is free of countable rank, and we first construct Y by attaching countably many 2- and 3-cells to X, and then attach countably many 4-cells to Y to obtain Z as before.

The inclusion $f : X \to Z$ is 2-connected and $\mathrm{Ker}(H_2(f;\mathbb{Z}[\pi])) = D(K)$. Comparison of the equivariant chain complexes for X and Z shows that $H_i(f;\mathbb{Z}[\pi])$ is an isomorphism for all $i \neq 2$, while $H^j(f;\mathbb{Z}[\pi])$ is an isomorphism for all $j \neq 2$ or 3, and $H^2(f;\mathbb{Z}[\pi])$ is a monomorphism. The connecting homomorphism in the long exact sequence for the cohomology of (Z,X) with coefficients $\mathbb{Z}[\pi]$ induces an isomorphism from the summand $K \le H^2(X;\mathbb{Z}[\pi])$ to $H^3(Z,X;\mathbb{Z}[\pi]) = Hom_{\mathbb{Z}[\pi]}(D(K),\mathbb{Z}[\pi])$. Therefore $H^3(Z;\mathbb{Z}[\pi]) = 0$. Let $[Z] = f_*[X] \in H_4(Z;\mathbb{Z}^w)$. Cap product with $[Z]$ gives isomorphisms $\overline{H^j(Z;\mathbb{Z}[\pi])} \cong H_{4-j}(Z;\mathbb{Z}[\pi])$ for $j \neq 2$, by the projection formula $f_*(f^*\alpha \frown [X]) = \alpha \frown [Z]$. This is also true when $j = 2$, for then $H^2(f;\mathbb{Z}[\pi])$ identifies $H^2(Z;\mathbb{Z}[\pi])$ with the orthogonal complement of K in

$H^2(X;\mathbb{Z}[\pi])$, and $f_*(- \frown [X])$ carries this isomorphically to $H_2(Z;\mathbb{Z}[\pi])$. Therefore Z is a PD_4-complex with fundamental class $[Z]$, f has degree 1 and $K_2(f) = D(K)$.

This construction derives from [38], via [39]. The main theorem of [32] includes a similar result, for X a closed orientable 4-manifold and K a free module.

Corollary 6 *The PD_4-complex X has a strongly minimal model if and only if H/E is a finitely generated projective $\mathbb{Z}[\pi]$-module and λ_X is nonsingular.*

Proof. If $f : X \to Z$ is a 2-connected degree-1 map then $K^2(f) = \mathrm{Cok}(H^2(f;\mathbb{Z}[\pi]))$ is a finitely generated projective direct summand of $\overline{H^2(X;\mathbb{Z}[\pi])}$ [60, Lemma 2.2]. If Z is strongly minimal the inclusion $E \to \overline{H^2(Z;\mathbb{Z}[\pi])}$ is an isomorphism, and so $H/E \cong \overline{K_2(f)}$. Hence the conditions are necessary. If they hold the construction of Theorem 5 gives a strongly minimal model for X.

The above conditions hold if $\Pi^\dagger$ is a finitely generated projective $\mathbb{Z}[\pi]$-module and $E^3\mathbb{Z} = 0$. In particular, they hold if $c.d.\pi \le 2$, by an elementary argument using Schanuel's Lemma and duality. (See Theorem 18 below). On the other hand, if $c.d.\pi = 3$ and $\mathbb{Z}$ has a finite projective resolution then no PD_4-complex with fundamental group π is strongly minimal. For if $\lambda_X = 0$ then $E^3\mathbb{Z} \cong (E^2\mathbb{Z})^\dagger$, by Lemma 6, and this condition cannot hold, by the next lemma.

Lemma 9. *Let π be a group such that the augmentation module $\mathbb{Z}$ has a finite projective resolution of length ≤ 3, and let $E = E^2\mathbb{Z}$. If $E^3\mathbb{Z} \cong E^\dagger$ then $c.d.\pi \le 2$.*

Proof. Let P_* be a projective resolution of $\mathbb{Z}$, of length 3. Then $\partial_3^\dagger : P_2^\dagger \to P_3^\dagger$ is a presentation for $E^3\mathbb{Z}$. Hence $(E^3\mathbb{Z})^\dagger = \mathrm{Ker}(\partial_3^{\dagger\dagger}) = \mathrm{Ker}(\partial_3) = 0$. But then $E^3\mathbb{Z} \cong E^\dagger \cong E^{\dagger\dagger\dagger} = 0$. Hence ∂_3 is a split injection, and so $c.d.\pi \le 2$.

Surgery on a factor of the 4-torus $\mathbb{R}^4/\mathbb{Z}^4$ gives a closed 4-manifold M with $\pi \cong \mathbb{Z}^3$ and $\chi(M) = 2$. This 4-manifold is χ-minimal [34, Lemma 3.11], and is order minimal, by Corollary 4, but cannot be strongly minimal, since $c.d.\pi = 3$.

The condition $E^3\mathbb{Z} \cong (E^2\mathbb{Z})^\dagger$ is far from characterizing the fundamental groups of strongly minimal PD_4-complexes. In §9–§14 we shall determine such groups within certain subclasses. In all cases considered, π has finitely many ends (i.e., π is virtually cyclic or $E^1\mathbb{Z} = 0$) and $E^3\mathbb{Z} = 0$.

Lemma 10. *Let $f : X \to Z$ be a 2-connected degree-1 map of PD_4-complexes with fundamental group π. Then $k_1(Z) = f_\#(k_1(X))$ and $k_1(X) = f_{!\#}k_1(Z)$, where $f_\#$ and $f_{!\#}$ are the change-of-coefficients homomorphisms induced by $\pi_2(f)$ and the umkehr homomorphism. If $E^3\mathbb{Z} = 0$ then these are mutually inverse isomorphisms.*

Proof. Since $K_2(f)$ is projective, $\pi_2(X) \cong \pi_2(Z) \oplus K_2(f)$, where the projection onto the first factor is given by $\pi_2(f)$ and is split by the umkehr map $f_!$.

Let $q : Q \to Z$ be the pullback of $P_3(f) : P_3(X) \to P_3(Z)$ over the inclusion of Z into $P_3(Z)$. Then q is a fibration with homotopy fibre $K(K_2(f), 2)$ and $f = qg$, where $g : X \to Q$ and $P_3(g)$ is a homotopy equivalence. Hence $\pi_2(g)$ is an isomorphism and $k_1(Q) = g_\# k_1(X)$. The fibration q is determined by Z and a k-invariant in $H^3(Z;K_2(f)) \cong H_1(Z;K_2(f))$, which is 0 since $K_2(f)$ is projective.

Hence $k_1(Q) = g_\# f_{!\#} k_1(Z)$. Therefore $k_1(X) = f_{!\#} k_1(Z)$, since $g_\#$ is an isomorphism, and so $f_\# k_1(X) = f_\# f_{!\#} k_1(Z) = k_1(Z)$.

The second assertion follows easily from the fact that $\pi_2(f)$ is an epimorphism with kernel $K_2(f)$ a finitely generated projective direct summand of $\Pi = \pi_2(X)$ and the hypothesis $E^3\mathbb{Z} = 0$, which implies that $H^3(\pi; K_2(f)) = 0$.

In particular, if Z is strongly minimal then $k_1(X)$ derives from $H^3(\pi; E^2\mathbb{Z})$. Are there such examples with $k_1(X) \neq 0$? The simplest examples for testing that we have found are the groups $\pi = A_3^2 *_C A_2^3$, where $A_n = \mathbb{Z}^n * \mathbb{Z}^n$ and C is either trivial or $\mathbb{Z}^4$. These groups have $c.d.\pi = 6$. Mayer-Vietoris arguments show that if $C = 1$ then $E^1\mathbb{Z} \cong E^2\mathbb{Z} \cong E^3\mathbb{Z} \cong \mathbb{Z}[\pi]$, while if $C = \mathbb{Z}^4$ then $E^1\mathbb{Z} = 0$ (i.e., π has one end) and $E^2\mathbb{Z} \cong E^3\mathbb{Z} \cong \mathbb{Z}[\pi]$. In each case it follows that $H^3(\pi; E^2\mathbb{Z}) \cong \mathbb{Z}[\pi]$. These groups are right angled Artin groups. Perhaps the "smallest" such group with similar cohomological properties is the one given by the 1-skeleton of a minimal triangulation of $S^2 \times S^1$, which has 10 generators and 40 relators but is less easily described explicitly. (This group has one end and $c.d. = 4$.)

8 Reduction

The main result of this section implies that when a PD_4-complex X has a strongly minimal model Z its homotopy type is determined by Z and λ_X. Recall the notation $B_M(-)$ from §2.

Lemma 11. *Let* $\beta_\xi = B_{\mathbb{Z}^n}(b_{(\mathbb{CP}^\infty)^n}(\xi))$, *for* $\xi \in H_4((\mathbb{CP}^\infty)^n; \mathbb{Z})$, *and let* G *be a group. Let* $u = \Sigma u_g g$ *and* $v = \Sigma v_h h \in H^2((\mathbb{CP}^\infty)^n; \mathbb{Z}[G]) \cong H^2((\mathbb{CP}^\infty)^n; \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Z}[G]$. *Then*

$$v(u \frown \xi) = \Sigma_{g,h \in G} \beta_\xi(u_g, v_h) g\bar{h}.$$

Proof. As each side of the equation is linear in ξ and $H_4((\mathbb{CP}^\infty)^n; \mathbb{Z})$ is generated by the images of homomorphisms induced by maps from $\mathbb{CP}^\infty$ or $(\mathbb{CP}^\infty)^2$, it suffices to assume $n = 1$ or 2. Since moreover each side of the equation is bilinear in u and v we may reduce to the case $G = 1$. As these functions have integral values and $2(x \otimes y) = (x+y) \otimes (x+y) - x \otimes x - y \otimes y$ in $H_4((\mathbb{CP}^\infty)^2; \mathbb{Z})$, for all $x, y \in \Pi \cong \mathbb{Z}^2$, we may reduce further to the case $n = 1$, which is easy.

Lemma 12. *Let* M *be a finitely generated projective* $\mathbb{Z}[\pi]$*-module and* $L = L_\pi(M, 2)$. *The secondary boundary homomorphism* b_L *determines an epimorphism* b' *from* $H_4(L; \mathbb{Z}^w)$ *to* $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$ *such that*

$$B_M(b'(x))(u, v) = v(u \frown x) \quad for all \;\; u, v \in M^\dagger \;\; and \;\; x \in H_4(L; \mathbb{Z}^w).$$

Proof. The homomorphism from $H_4(L; \mathbb{Z}^w)$ to $H_4(\pi; \mathbb{Z}^w)$ induced by c_L is an a epimorphism, since c_L has a section σ. Since $\widetilde{L} \simeq K(M, 2)$ the homomorphism $b_{\widetilde{L}}$ is an isomorphism and $H_3(\widetilde{L}; \mathbb{Z}) = 0$, while since M is projective $H_p(\pi; M) = 0$ for all

$p > 0$. Therefore it follows from the spectral sequence for the universal covering $\widetilde{L} \to L$ that the kernel of the epimorphism induced by c_L is $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} H_4(\widetilde{L};\mathbb{Z})$. Let $b'(x) = (1 \otimes b_{\widetilde{L}})(x - \sigma_* c_{L*}(x))$ for all $x \in H_4(L;\mathbb{Z}^w)$. Then b' is an epimorphism onto $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$.

Let $x \in H_4(L;\mathbb{Z}^w)$ and $u, v \in M^\dagger \cong H^2(L;\mathbb{Z}[\pi])$. Since M is the union of its finitely generated free abelian subgroups and homology commutes with direct limits there is an $n > 0$ and a map $k : (\mathbb{CP}^\infty)^n \to \widetilde{L}$ such that $b'(x)$ is the image of $k_*(\xi)$ for some $\xi \in H_4((\mathbb{CP}^\infty)^n;\mathbb{Z})$. Then $B_M(b'(x))(u,v) = ev_M(k_*\xi)(u,v)$.

Suppose that $k^*u = \Sigma u_g g$ and $k^*v = \Sigma v_h h$ in $H^2((\mathbb{CP}^\infty)^n;\mathbb{Z}[\pi])$. Then we have $ev_M(k_*\xi)(u,v) = \Sigma_{g,h\in G}\beta_\xi(u_g, v_h)g\bar{h}$, which is equal to $v(u \frown k_*\xi) = k^*v(k^*u \frown \xi)$, by Lemma 11. Now $x = k_*\xi + \sigma^*u \frown c_{L*}x$ and $u \frown \sigma_* c_{L*}x = \sigma_*(\sigma^*u \frown c_{L*}x) = 0$, since $H_2(\pi;\mathbb{Z}[\pi]) = 0$. Hence $B_M(b'(x))(u,v) = v(u \frown x)$, for all $u, v \in M^\dagger$ and $x \in H_4(L;\mathbb{Z}^w)$.

Theorem 7. *Let $g_X : X \to Z$ and $g_Y : Y \to Z$ be 2-connected degree-1 maps of PD_4-complexes with fundamental group π. If $w = w_1(Z)$ is trivial on elements of order 2 in π then there is a homotopy equivalence $h : X \to Y$ such that $g_Y h = g_X$ if and only if $\lambda_{g_X} \cong \lambda_{g_Y}$ (after changing the sign of $[Y]$, if necessary).*

Proof. The condition $\lambda_{g_X} \cong \lambda_{g_Y}$ is clearly necessary. Suppose that it holds.

Since g_X and g_Y induce isomorphisms on π_1, we may assume that $c_X = c_Z g_X$ and $c_Y = c_Z g_Y$. Since g_X and g_Y are 2-connected degree-1 maps, there are canonical splittings $\pi_2(X) = K_2(g_X) \oplus N$ and $\pi_2(Y) = K_2(g_Y) \oplus N$, where $N = \pi_2(Z)$, and $K_2(g_X)$ and $K_2(g_Y)$ are projective. The projections $\pi_2(g_X)$ and $\pi_2(g_Y)$ onto the second factors are split by the umkehr homomorphisms. We may identify $K_2(g_X)^\dagger$ and $K_2(g_Y)^\dagger$ with direct summands of $H^2(X;\mathbb{Z}[\pi])$ and $H^2(X;\mathbb{Z}[\pi])$, respectively [60, Lemma 2.2]. The homomorphism θ induces an isomorphism $K_2(Y) \cong M = K_2(X)$ such that $\lambda_{g_Y} = \lambda_{g_X}$ as pairings on $M^\dagger \times M^\dagger$. Hence $\pi_2(X) \cong \pi_2(Y) \cong \Pi = M \oplus N$. We may also assume that $M \neq 0$, for otherwise g_X and g_Y are homotopy equivalences.

Let $g : P = P_2(X) \to P_2(Z)$ be the 2-connected map induced by g_X. Then g is a fibration with fibre $K(M,2)$, and the inclusion of N as a direct summand of Π determines a section s for g. Since $\pi_2(X) \cong \pi_2(Y)$, and $k_1(X) = (g_{X!})_\#(k_1(Z))$ and $k_1(Y) = (g_{Y!})_\#(k_1(Z))$, by Lemma 10, we see that $P_2(Y) \simeq P$. We may choose the homotopy equivalence so that composition with g is homotopic to the map induced by g_Y. (This uses our knowledge of $E_\pi(P)$, as recorded in §3 above.)

The splitting $\Pi = M \oplus N$ also determines a projection $q : P \to L = L_\pi(M,2)$. We may construct L by adjoining 3-cells to X to kill the kernel of projection from Π onto M and then adjoining higher dimensional cells to kill the higher homotopy. Let $j : X \to L$ be the inclusion. Then $B_M(b'(j_*[X]))(u,v) = v(u \frown j_*[X])$ for all $u, v \in M^\dagger$, by Lemma 12. Using the projection formula and identifying $M^\dagger = H^2(L;\mathbb{Z}[\pi])$ with $K^2(X)$ we may equate this with $\lambda_{g_X}(u,v)$. Hence $f_{X*}[X]$ and $f_{Y*}[Y]$ have the same image $\lambda_{g_X} = \lambda_{g_Y}$ in $Her_w(M^\dagger)$.

Since $P_2(Z)$ is a retract of P comparison of the Cartan-Leray spectral sequences for the classifying maps c_P and $c_{P_2(Z)}$ shows that

$$\mathrm{Cok}(H_4(s;\mathbb{Z}^w)) \cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (\Gamma_W(\Pi)/\Gamma_W(N)).$$

Since π has no orientation reversing element of order 2 the homomorphism B_M is injective, by Theorem 1, and therefore since $\lambda_{g_X} = \lambda_{g_Y}$ the images of $f_{X*}[X]$ and $f_{Y*}[Y]$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (\Gamma_W(\Pi)/\Gamma_W(N))$ differ by an element of the subgroup $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (M \otimes N)$. Let $c \in M \otimes N$ represent this difference, and let $\gamma \in \Gamma_W(M)$ represent $b'(f_{X*}[X])$. Since $B_M(1 \otimes \gamma) = \lambda_{g_X}$ is nonsingular $\widetilde{B_M}(\gamma)$ is surjective, and so we may choose a homomorphism $\theta : M \to N$ such that $(\widetilde{B_M}(\gamma) \otimes 1)(t^{-1}(\theta)) = (d \otimes 1)(c)$. Hence $\Gamma_W(\alpha_\theta)(\gamma) - \gamma \equiv c \bmod \Gamma_W(N)$, by Lemma 3. Let $P(\theta)$ be the corresponding self homotopy equivalence of P. Then $gP(\theta) = g$ and $P(\theta)_* f_{Y*}[Y] = f_{X*}[X]$ *mod* $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(N)$. It follows that $P(\theta)_* f_{Y*}[Y] = f_{X*}[X]$ in $H_4(P;\mathbb{Z}^w)$, since $g_{X*}[X] = g_{Y*}[Y]$ in $H_4(Z;\mathbb{Z}^w)$ and so $(gf_X)_*[X] = (gf_Y)_*[Y]$ in $H_4(P_2(Z);\mathbb{Z}^w)$.

There is then a map $h : X \to Y$ with $f_Y h = f_X$, by the argument of [27, Lemma 1.3]. Since the orientation characters of X and Y are compatible, h lifts to a map $h^+ : X^+ \to Y^+$. Since f_X and f_Y are 3-connected $\pi_1(h^+)$, $\pi_2(h^+)$ and $H_2(h^+;\mathbb{Z})$ are isomorphisms. Since M is projective and nonzero, $\mathbb{Z} \otimes_{\mathrm{Ker}(w)} M$ is a nontrivial torsion free direct summand of $H_2(X^+;\mathbb{Z})$, and so h^+ has degree 1, by Poincaré duality. Hence h^+ is a homotopy equivalence, and therefore so is h.

The original version of this result [39, Theorem 11] assumed that $k_1(X) = k_1(Y) = 0$. This was relaxed to the condition that "$k_1(X) = (g_{X!})_\# k_1(Z)$ and $k_1(Y) = (g_{Y!})_\# k_1(Z)$" in an earlier version of the present paper [arXiv: 1303.5486v2]. The final step is due to Hegenbarth, Pamuk and Repovš, who noted that Poincaré duality in Z may be used to establish an equivalent condition [31]. (This observation has been used in the current version of Lemma 10 above.)

The argument for Theorem 7 breaks down when $\pi = \mathbb{Z}/2\mathbb{Z}$ and w is nontrivial, for then $B_M : \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M) \to Her_w(M^\dagger)$ is no longer injective, and the intersection pairing is no longer a complete invariant [28]. Thus the condition on 2-torsion is in general necessary.

Corollary 8 *If X has a strongly minimal model Z and π has no 2-torsion then the homotopy type of X is determined by Z and λ_X.* □

Corollary 9 *[32] If $g : X \to Z$ is a 2-connected degree-1 map of PD_4-complexes such that $w_1(Z)$ is trivial on elements of order 2 in $\pi_1(Z)$ then X is homotopy equivalent to $M\#Z$ with M 1-connected if and only if λ_g is extended from a nonsingular pairing over $\mathbb{Z}$.* □

The result of [32] assumes that X is orientable, π is infinite and either $E^2\mathbb{Z} = 0$ or π acts trivially on $\pi_2(Z)$. (In the latter case $Hom_{\mathbb{Z}[\pi]}(\pi_2(Z), \mathbb{Z}[\pi]) = 0$, and so Z is strongly minimal.)

9 Realization of pairings

In this short section we shall show that if Z is a strongly minimal PD_4-complex and $\mathrm{Ker}(w)$ has no element of order 2 every nonsingular w-hermitian pairing on a finitely generated projective $\mathbb{Z}[\pi]$-module is realized as λ_X for some PD_4-complex X with minimal model Z. This is an immediate consequence of the following stronger result.

Theorem 10. *Let Z be a PD_4-complex with fundamental group π and let $w = w_1(Z)$. Assume that $\mathrm{Ker}(w)$ has no element of order 2. Let N be a finitely generated projective $\mathbb{Z}[\pi]$-module and Λ be a nonsingular w-hermitian pairing on $N^\dagger$. Then there is a PD_4-complex X and a 2-connected degree-1 map $f : X \to Z$ such that $\lambda_f \cong \Lambda$.*

Proof. Suppose $N \oplus F_1 \cong F_2$, where F_1 and F_2 are free $\mathbb{Z}[\pi]$-modules with countable bases I and J, respectively. (These may be assumed finite if N is stably free.) We may assume $Z = Z_o \cup_\theta e^4$ is obtained by attaching a single 4-cell to a 3-complex Z_o [60, Lemma 2.9]. Construct a 3-complex X_o with $\pi_2(X_o) \cong \pi_2(Z_o) \oplus N$ by attaching J 3-cells to $Z_o \vee (\vee^I S^2)$, along sums of translates under π of the 2-spheres in $\vee^I S^2$, as in Theorem 5. Let $i : Z_o \to X_o$ be the natural inclusion. Collapsing $\vee^I S^2$ gives $X_o / \vee^I S^2 \simeq Z_o \vee (\vee^J S^3)$, and so there is a retraction $q : X_o \to Z_o$. Let $p : \Pi = \pi_2(X_o) \to N$ be the projection with kernel $\mathrm{Im}(\pi_2(i))$, and let $j : X_o \to L = L_\pi(N, 2)$ be the corresponding map. Then $\pi_2(ji) = 0$ and so ji factors through $K(\pi, 1)$. The map $B_N : \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(N) \to Her_w(N^\dagger)$ is an epimorphism, by Theorem 1. Therefore we may choose $\psi \in \pi_3(X_o)$ so that $B_N([j(\psi)]) = \Lambda$.

Let $\phi = \psi - iq\psi + i\theta$. Then $q\phi = \theta$ and $j(\phi) = j(\psi)$, so $B_N([j(\phi)]) = \Lambda$. Let $X = X_o \cup_\phi D^4$. The retraction q extends to a map $f : X \to Z$. Comparison of the exact sequences for these pairs shows that f induces isomorphisms on homology and cohomology in degrees $\neq 2$. In particular, $H_4(X; \mathbb{Z}^w) \cong H_4(Z; \mathbb{Z}^w)$. Let $[X] = f_*^{-1}[Z]$. Then $f_*(f^*(\alpha) \frown [X]) = \alpha \frown [Z]$ for all cohomology classes α on Z, by the projection formula. Therefore cap product with $[X]$ induces the Poincaré duality isomorphisms for Z in degrees other than 2. As it induces an isomorphism $\overline{H^2(X; \mathbb{Z}[\pi])} \cong H_2(X; \mathbb{Z}[\pi])$, by the assumption on Λ, X_ϕ is a PD_4-complex with $\lambda_X \cong \Lambda$.

10 Strongly minimal models with $\pi_2 = 0$

A PD_4-complex Z with $\pi_2(Z) = 0$ is clearly strongly minimal.

Lemma 13. *Let X be a PD_4-complex with fundamental group π. Then*

1. *$\Pi = 0$ if and only if X is strongly minimal and $E^2\mathbb{Z} = 0$, and then $E^3\mathbb{Z} = 0$;*
2. *if $\Pi = 0$ and π is infinite then the homotopy type of X is determined by π, w and $k_2(X) \in H^4(\pi; E^1\mathbb{Z})$.*

Proof. Part (1) follows from part (1) of Lemma 6. If $\Pi = 0$ then $P_2(X) \simeq K(\pi,1)$ and $\pi_3(Z) \cong E^1\mathbb{Z}$, by Poincaré duality. Hence (2) follows from Lemma 5.

Theorem 11. *Let π be a finitely presentable group with no 2-torsion and such that $E^2\mathbb{Z} = E^3\mathbb{Z} = 0$, and let $w : \pi \to \mathbb{Z}^\times$ be a homomorphism. Then two PD_4-complexes X and Y with fundamental group π, $w_1(X) = c_X^* w$, $w_1(Y) = c_Y^* w$ and $\pi_2(X)$ and $\pi_2(Y)$ projective $\mathbb{Z}[\pi]$-modules are homotopy equivalent if and only if*

1. *$c_{X*}[X] = \pm g^* c_{Y*}[Y]$ in $H_4(\pi;\mathbb{Z}^w)$, for some $g \in Aut(\pi)$ with $wg = w$; and*
2. *$\lambda_X \cong \lambda_Y$.*

Proof. The hypotheses imply that X and Y have strongly minimal models Z_X and Z_Y with $\pi_2(Z_X) = \pi_2(Z_Y) = 0$, and hence $P_2(Z_X) \simeq P_2(Z_Y) \simeq K(\pi,1)$. Moreover $H^3(\pi;\pi_2(X)) = H^3(\pi;\pi_2(Y)) = 0$, since $E^3\mathbb{Z} = 0$, and so the result follows by the argument of Theorem 7.

In particular, $Z_X \simeq Z_Y$. If π also has one end then the minimal model is aspherical. See Theorem 15 below.

Connected sums of complexes with $\pi_2 = 0$ again have $\pi_2 = 0$, and the fundamental groups of such connected sums usually have infinitely many ends. (The sole nontrivial exception is $\mathbb{RP}^4\#\mathbb{RP}^4$.) The arguments of [57] can be extended to this situation, to show that if π splits as a free product then Z has a corresponding connected sum decomposition [7]. (In particular, if π is torsion free then its free factors are one-ended or infinite cyclic, and so the summands are either aspherical or copies of $S^3 \times S^1$ or $S^3 \tilde{\times} S^1$, the non-orientable S^3-bundle space over S^1.)

In the next two sections we shall determine the groups π with finitely many ends which are fundamental groups of strongly minimal PD_4-complexes Z with $\pi_2(Z) = 0$. (Little is known about such complexes with π indecomposable and having infinitely many ends. It follows from the results of [15] that the centralizer of any element of finite order is finite or has two ends.)

11 Strongly minimal models with π virtually free

If π is virtually free (in particular, if it is finite or two-ended) then $E^s\mathbb{Z} = 0$ for all $s > 1$, and so a strongly minimal PD_4-complex Z with fundamental group π must have $\pi_2(Z) = 0$, by Lemma 13. Thus if π is finite $\widetilde{Z} \simeq S^4$, and so $Z \simeq S^4$ or $\mathbb{RP}^4$ [34, Lemma 12.1]. Every orientable PD_n-complex admits a degree-1 map to S^n. It is well known that the (oriented) homotopy type of a 1-connected PD_4-complex is determined by its intersection pairing and that every such pairing is realized by some 1-connected topological 4-manifold [24, page 161]. Thus the only finite group we need to consider is $\pi = \mathbb{Z}/2\mathbb{Z}$.

Theorem 12. *Let X be a PD_4-complex with $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$ and let $w = w_1(X)$. Then $\mathbb{RP}^4$ is a model for X if and only if $w^4 \neq 0$.*

Proof. The condition is clearly necessary. Conversely, we may assume that $X = X_o \cup e^4$ is obtained by attaching a single 4-cell to a 3-complex X_o [60, Lemma 2.9]. The map $c_X : X \to \mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$ factors through a map $f : X \to \mathbb{RP}^4$, and $w = f^*w_1(\mathbb{RP}^4)$, since $w \neq 0$. The degree of f is well-defined up to sign, and is odd since $w^4 \neq 0$. We may arrange that f is a degree-1 map, after modifying f on a disc, if necessary. (See [48].)

In particular, $\pi_2(X)$ is projective if and only if $w^4 \neq 0$. Can this be seen directly? The two $\mathbb{RP}^2$-bundles over S^2 provide contrasting examples. If $X = S^2 \times \mathbb{RP}^2$ then $w^3 = 0$ and $\Pi \cong \mathbb{Z} \oplus \mathbb{Z}^w$, which has no nontrivial projective $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-module summand. Thus $S^2 \times \mathbb{RP}^2$ is order minimal but not strongly minimal. On the other hand, if X is the nontrivial bundle space then $w^4 \neq 0$ and $\Pi \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.

Non-orientable topological 4-manifolds with fundamental group $\mathbb{Z}/2\mathbb{Z}$ are classified up to homeomorphism in [28], and it is shown there that the homotopy types are determined by the Euler characteristic, w^4, the v_2-type and an Arf invariant (for v_2-type III). The authors remark that their methods show that λ_X together with a quadratic enhancement $q : \Pi \to \mathbb{Z}/4\mathbb{Z}$ due to [42] is also a complete invariant for the homotopy type of such a manifold.

If $\pi = \pi_1(Z)$ has two ends and $\pi_2(Z) = 0$ then $\widetilde{Z} \simeq S^3$. Since π has two ends it is an extension of $\mathbb{Z}$ or the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ by a finite normal subgroup F. Since F acts freely on $\widetilde{Z}$ it has cohomological period dividing 4 and acts trivially on $\pi_3(Z) \cong H_3(Z;\mathbb{Z}[\pi])$, while the action $u : \pi \to \{\pm 1\} = Aut(\pi_3(Z))$ induces the usual action of π/F on $H^4(F;\mathbb{Z})$. The action u and the orientation character $w_1(Z)$ determine each other, and every such group π and action u is realized by some PD_4-complex Z with $\pi_2(Z) = 0$. The homotopy type of Z is determined by π, u and the first nontrivial k-invariant in $H^4(\pi;\mathbb{Z}^u)$. (See [34, Chapter 11].)

We shall use Farrell cohomology to show that any PD_4-complex X with $\pi_1(X) \cong \pi$ satisfying corresponding conditions has a strongly minimal model. We refer to the final chaper of [9] for more information on Farrell cohomology.

It is convenient to use the following notation. If R is a noetherian ring and M is a finitely generated R-module let $\Omega^1 M = \mathrm{Ker}(\phi)$, where $\phi : R^n \to M$ is any epimorphism, and define $\Omega^k M$ for $k > 1$ by iteration, so that $\Omega^{n+1}M = \Omega^1\Omega^n M$. We shall say that two finitely generated R-modules M_1 and M_2 are projectively equivalent ($M_1 \simeq M_2$) if they are isomorphic up to direct sums with a finitely generated projective module. Then these "syzygy modules" $\Omega^k M$ are finitely generated, and are well-defined up to projective equivalence, by Schanuel's Lemma.

Theorem 13. *Let X be a PD_4-complex such that $\pi = \pi_1(X)$ has two ends. Then X has a strongly minimal model if and only if π and the action u of π on $H_3(X;\mathbb{Z}[\pi]) \cong \mathbb{Z}$ are realized by some PD_4-complex Z with $\pi_2(Z) = 0$.*

Proof. If $\pi_2(Z) = 0$ then $\widetilde{Z} \simeq S^3$, by Poincaré duality and the Hurewicz and Whitehead Theorems, and the conditions on π are necessary, by Theorem 11.1 and Lemma 11.3 of [34].

Conversely, since π is virtually infinite cyclic the conditions imply that the Farrell cohomology of π has period dividing 4 [22]. We may assume that the chain complex

C_* for $\widetilde{X}$ is a complex of finitely generated $\mathbb{Z}[\pi]$-modules. Then the modules B_2, Z_2 Z_3 and Π are finitely generated, since $\mathbb{Z}[\pi]$ is noetherian. The chain complex C_* gives rise to four exact sequences:

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0,$$

$$0 \to Z_3 \to C_3 \to B_2 \to 0,$$

$$0 \to B_2 \to Z_2 \to \Pi \to 0$$

and

$$0 \to C_4 \to Z_3 \to \mathbb{Z}^u \to 0.$$

It is clear that $Z_2 \simeq \Omega^3\mathbb{Z}$ and $Z_3 \simeq \Omega^1 B_2$, while $\Omega^1 Z_3 \simeq \Omega^1(\mathbb{Z}^u)$. The standard construction of a resolution of the middle term of a short exact sequence from resolutions of its extremes, applied to the third sequence, gives a projective equivalence $\Omega^1 Z_2 \simeq \Omega^1 B_2 \oplus \Omega^1 \Pi$. The corresponding sequences for a strongly minimal complex with the same group π and action u give an equivalence $\Omega^1(\mathbb{Z}^u) \simeq \Omega^1(\Omega^4\mathbb{Z})$. (This is in turn equivalent to $\Omega^1\mathbb{Z}$, by periodicity.) Together these equivalences give

$$\Omega^5\mathbb{Z} \simeq \Omega^2 Z_2 \simeq \Omega^2 B_2 \oplus \Omega^2 \Pi \simeq \Omega^1 Z_3 \oplus \Omega^2 \Pi \simeq \Omega^5\mathbb{Z} \oplus \Omega^2 \Pi.$$

Hence $Ext^q_{\mathbb{Z}[\pi]}(\Omega^5\mathbb{Z}, N) \cong Ext^q_{\mathbb{Z}[\pi]}(\Omega^5\mathbb{Z}, N) \oplus Ext^q_{\mathbb{Z}[\pi]}(\Omega^2\Pi, N)$, for all $q > v.c.d.\pi = 1$, and any $\mathbb{Z}[\pi]$-module N. If N is finitely generated so is $Ext^q_{\mathbb{Z}[\pi]}(\Omega^1\mathbb{Z}, N)$, and so $Ext^{q+2}_{\mathbb{Z}[\pi]}(\Pi, N) = Ext^q_{\mathbb{Z}[\pi]}(\Omega^2\Pi, N) = 0$, for all $q > 1$. Since Π is finitely generated $Ext^r_{\mathbb{Z}[\pi]}(\Pi, -)$ commutes with direct limits and so is 0, for all $r > 3$. Therefore Π has finite projective dimension [9, Theorem X.5.3]. There is a Universal Coefficient spectral sequence

$$E_2^{pq} = Ext^q_{\mathbb{Z}[\pi]}(H_p(X; \mathbb{Z}[\pi]), \mathbb{Z}[\pi]) \Rightarrow H^{p+q}(X; \mathbb{Z}[\pi]).$$

Here $E_2^{pq} = 0$ unless $p = 0$, 2 or 3, and $E_2^{0q} = E_2^{3q} = 0$ if $q > 1$, since π is virtually infinite cyclic and $\Omega^1(\mathbb{Z}^u) \simeq \Omega^1\mathbb{Z}$. It follows easily from this spectral sequence and Poincaré duality that $Ext^s_{\mathbb{Z}[\pi]}(\Pi, \mathbb{Z}[\pi]) = 0$ for all $s \geq 1$. Since Π also has finite projective dimension it is projective. Hence X has a strongly minimal model, by Theorem 5.

Thus, for instance, an orientable PD_4-complex with fundamental group D_∞ does not have a strongly minimal model.

We shall summarize here the results of [37] on the case when $\pi \cong F(n)$, for some $n \geq 1$. All epimorphisms $w : F(n) \to \mathbb{Z}^\times$ are equivalent up to composition with an automorphism of $F(n)$. The ring $\mathbb{Z}[F(n)]$ is a coherent domain of global dimension 2, for which all projectives are free. There are just two homotopy types of χ-minimal PD_4-complexes Z with $\pi_1(Z) \cong F(n)$, namely $\#^n(S^3 \times S^1)$ (if $w = 0$) and $(S^3 \tilde{\times} S^1)\#(\#^{n-1}(S^3 \times S^1))$ (if $w \neq 0$). (These are strongly minimal, and so the notions of minimality coincide in this case.) If X is any PD_4-complex with $\pi_1(X) \cong F(n)$ then $\pi_2(X)$ is a finitely generated free $\mathbb{Z}[F(n)]$-module, and there is a degree-1 map from X to the minimal model with compatible w. Every w-hermitian pairing on a finitely generated free $\mathbb{Z}[F(n)]$-module is realizable by some such PD_4-complex, and two such complexes X and Y realizing $(F(n),w)$ are homotopy equivalent if and only if λ_X and λ_Y are isometric.

The key observation is that if X is a PD_4-complex with $\pi_1(X)\cong F(n)$ then its 3-skeleton is standard: if $\beta_2(X)=\beta$ then $X\simeq X_\psi = X_o\cup_\psi e^4$, where $X_o=\vee^n(S^1\vee S^3)\vee(\vee^\beta S^2)$ and $\psi\in\pi_3(X_o)$. (This is an easy homological argument, relying on Schanuel's Lemma and the theorems of Hurewicz and Whitehead.) The main results then follow on exploring how the group $E(X_o)$ acts on the attaching map ψ. This group is "large" and its action is easily analyzed. Most of these results (excepting for the determination of the minimal models) can also be proven by adapting the arguments of this paper.

Finitely generated virtually free groups provide a potentially broader class of examples. These groups are fundamental groups of finite graphs of finite groups. The arguments of [15] may be adapted to show that if Z is a strongly minimal PD_4-complex such that $\pi=\pi_1(Z)$ is virtually free (so $\pi_2(Z)=0$) and if π has no dihedral subgroup of order >2 then it is a free product of groups with two ends [7]. However, not much is known about criteria for 2-connected degree-1 maps to a specific minimal model.

12 Strongly minimal models with π one-ended

We begin this section with a general result on the case when π has one end.

Lemma 14. *Let G be a group. If T is a locally-finite normal subgroup of G then T acts trivially on $H^j(G;\mathbb{Z}[G])$, for all $j\geq 0$.*

Proof. If T is finite then $H^j(G;\mathbb{Z}[G])\cong H^j(G/T;\mathbb{Z}[G/T]))$, for all j, and the result is clear. Thus we may assume that T and G are infinite. Hence $H^0(G;\mathbb{Z}[G])=0$, and T acts trivially. We may write $T=\cup_{n\geq 1}T_n$ as a strictly increasing union of finite subgroups. Then there are short exact sequences [41]

$$0\to \varprojlim{}^1 H^{s-1}(T_n;\mathbb{Z}[\pi])\to H^s(T;\mathbb{Z}[\pi])\to \varprojlim H^s(T_n;\mathbb{Z}[\pi])\to 0.$$

Hence $H^s(T;\mathbb{Z}[\pi])=0$ if $s\neq 1$ and $H^1(T;\mathbb{Z}[\pi])=\varprojlim{}^1 H^0(T_n;\mathbb{Z}[\pi])$, and so the Lyndon-Hochschild-Serre spectral sequence collapses to give

$$H^j(G;\mathbb{Z}[G])\cong H^{j-1}(G/T;H^1(T;\mathbb{Z}[G])),\quad \text{for all } j\geq 1.$$

Let $g\in T$. We may assume that $g\in T_n$ for all n, and so g acts trivially on $H^0(T_n;\mathbb{Z}G)$, for all j and n. But then g acts trivially on $\varprojlim{}^1 H^0(T_n;\mathbb{Z}[\pi])$, by the functoriality of the construction. Hence every element of T acts trivially on $H^{j-1}(G/T;H^1(T;\mathbb{Z}[G]))$, for all $j\geq 1$.

Theorem 14. *Let X be an orientable, strongly minimal PD_4-complex. If $\pi = \pi_1(X)$ has one end then π has no non-trivial locally-finite normal subgroup.*

Proof. Suppose that π has a nontrivial locally-finite normal subgroup T. Since π has one end, $H_s(X;\mathbb{Z}[\pi]) = 0$ for $s \neq 0$ or 2. Since X is strongly minimal, $\Pi = H_2(X;\mathbb{Z}[\pi]) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$. Hence T acts trivially on Π, since it acts trivially on $H^2(\pi;\mathbb{Z}[\pi])$, by Lemma 14, and X is orientable.

Let $g \in T$ have prime order p, and let $C = \langle g \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Then C acts freely on $\widetilde{X}$, which has homology only in degrees 0 and 2. On considering the homology spectral sequence for the classifying map $c_{\widetilde{X}/C} : \widetilde{X}/C \to K(C,1)$, we see that $H_{i+3}(C;\mathbb{Z}) \cong H_i(C;\Pi)$, for all $i \geq 2$. (See [34, Lemma 2.10].) Since C has cohomological period 2 and acts trivially on Π, there is an exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \Pi \to \Pi \to 0.$$

On the other hand, since π is finitely presentable, $\Pi \cong H^2(\pi;\mathbb{Z}[\pi])$ is torsion-free [25, Proposition 13.7.1]. Hence T has no such element g and so π has no such finite normal subgroup.

As an immediate consequence, if X is strongly minimal, but not orientable, and π has one end, then either π has no nontrivial locally-finite normal subgroup or $\pi \cong \pi^+ \times \mathbb{Z}/2\mathbb{Z}^-$, and π^+ has no nontrivial locally-finite normal subgroup.

A finitely presentable group G is a *PD_4-group* if $K(G,1)$ is a PD_4-complex. Such a group has one end and $E^2\mathbb{Z} = 0$, and $K(G,1)$ is clearly strongly minimal. Conversely, if X is a strongly minimal complex, $\pi = \pi_1(X)$ has one end and $E^2\mathbb{Z} = 0$ then X is aspherical. Hence π is a PD_4-group and $K(\pi,1)$ is the unique strongly minimal model. The next theorem gives several equivalent conditions for a PD_4-complex with such a group to have a strongly minimal model.

Theorem 15. *Let X be a PD_4-complex with fundamental group π such that π has one end and $E^2\mathbb{Z} = 0$. Then the following are equivalent:*

1. *X has a strongly minimal model;*
2. *π is a PD_4-group and $\Pi = \pi_2(X)$ is projective:*
3. *π is a PD_4-group, $w_1(X) = c_X^* w_1(\pi)$ and c_X is a degree-1 map;*
4. *π is a PD_4-group, $w_1(X) = c_X^* w_1(\pi)$ and $k_1(X) = 0$.*

Proof. The equivalence $(1) \Leftrightarrow (2)$ follows from Corollary 6.

If Z is strongly minimal and $E^1\mathbb{Z} = E^2\mathbb{Z} = 0$ then $\pi_2(Z) = 0$ and $\pi_3(Z) = E^1\mathbb{Z} = 0$. Hence Z is aspherical, so π is a PD_4-group and $Z \simeq K = K(\pi,1)$. Any 2-connected map $f : X \to K$ is homotopic to c_X (up to composition with a self homotopy equivalence of K). Thus $w_1(X) = c_X^* w_1(\pi)$ and c_X is a degree-1 map. Conversely, if (3) holds then $K = K(\pi,1)$ is the unique strongly minimal PD_4-complex with fundamental group π, and c_X is a 2-connected degree-1 map. Thus $(3) \Leftrightarrow (1)$.

If (2) or (3) holds then $\Pi = \mathrm{Ker}(\pi_2(c_X))$ is projective, Since π is a PD_4-group, $H^3(\pi;M) = 0$ for any projective module M, and so $k_1(X) = 0$. Conversely, if (4) holds the map $c_P : P = P_2(X) \to K$ has a section s, since $k_1(X) = 0$. We may assume that $K = K_o \cup e^4$ and $X = X_o \cup e^4$, where K_o and X_o are 3-complexes. The restriction $s|_{K_o}$ factors through X_o, by cellular approximation, since $P = X_o \cup \{cells\ of\ dim \geq 4\}$. Thus K_o is a retract of X_o. The map c_X induces a commuting diagram of homomorphisms between the long exact sequences of the pairs (X, X_o) and (K, K_o), with coefficients $\mathbb{Z}[\pi]$. Hence the induced map from $H_4(X, X_0; \mathbb{Z}[\pi])$ to $H_4(K, K_0; \mathbb{Z}[\pi])$ is an isomorphism. The change of coefficients homomorphisms $\varepsilon_{w\#}$ induced by the w-twisted augmentation are epimorphisms. Since the natural maps from $H_4(X; \mathbb{Z}^w)$ to $H_4(X, X_o; \mathbb{Z}^w)$ and from $H_4(K; \mathbb{Z}^w)$ to $H_4 K, K_o; \mathbb{Z}^w)$ are isomorphisms, it follows that c_X has degree 1. Thus $(3) \Leftrightarrow (4)$.

If π has one end and Π is projective then $c.d.\pi = 4$ and $H^4(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z}$, by part (6) of Lemma 6. Must π be a PD_4-group? This is so if also $E^3\mathbb{Z} = 0$, for then X has a strongly minimal model, by Lemma 6 and Theorem 5, which must be aspherical. If X is strongly minimal and π is virtually an r-dimensional duality group then $r = 1, 2$ or 4, and in the latter case π is a PD_4-group.

The next result now follows from Corollary 20 and Theorem 15.

Corollary 16 *Let X and Y be PD_4-complexes with fundamental group π a PD_4-group, and such that $\pi_2(X)$ and $\pi_2(Y)$ are projective $\mathbb{Z}[\pi]$-modules, $w_1(X) = c_X^* w$ and $w_1(Y) = c_Y^* w$, where $w = w_1(\pi)$. Then X and Y are homotopy equivalent if and only if $\lambda_X \cong \lambda_Y$.* □

This corollary and the equivalence of (3) and (4) in the Theorem are from [12]. (It is assumed there that X and π are orientable.) Theorems 15 and 7 give an alternative proof of the main result of [12], namely that a PD_4-complex X with fundamental group π a PD_4-group and $w_1(X) = w_1(\pi)$ is homotopy equivalent to $M \# K(\pi, 1)$, for some 1-connected PD_4-complex M if and only if $k_1(X) = 0$ and λ_X is extended from a nonsingular pairing over $\mathbb{Z}$.

13 Semidirect products and mapping tori

In this section we shall determine which semidirect products $\nu \rtimes_\alpha \mathbb{Z}$ with ν finitely presentable are fundamental groups of strongly minimal PD_4-complexes.

Theorem 17. *Let ν be a finitely presentable group and let X be a PD_4-complex with fundamental group $\pi \cong \nu \rtimes_\alpha \mathbb{Z}$, for some automorphism α of ν. Then the following are equivalent:*

1. *X is the mapping torus of a self homotopy equivalence of a PD_3-complex N with fundamental group ν;*
2. *X is strongly minimal;*
3. *$\chi(X) = 0$.*

In general, X has a strongly minimal model if and only if $\Pi^\dagger$ is projective.

Proof. Let X_ν be the covering space of X corresponding to ν. Then X_ν is the homotopy fibre of a map from X to S^1 which corresponds to the projection of π onto $\mathbb{Z}$, and $H_q(X_\nu;k)=0$ for $q>3$ and all coefficients k. The Lyndon-Hochschild-Serre spectral sequence gives an isomorphism $H^2(\pi;\mathbb{Z}[\pi])|_\nu \cong H^1(\nu;\mathbb{Z}[\nu])$ of right $\mathbb{Z}[\nu]$-modules. Since ν is finitely presentable it is accessible, and hence $H^1(\nu;\mathbb{Z}[\nu])$ is finitely generated as a right $\mathbb{Z}[\nu]$-module. (See Theorems VI.6.3 and IV.7.5 of [16].)

Suppose first that X is the mapping torus of a self homotopy equivalence of a PD_3-complex N. Since $\pi_2(X)|_\nu = \pi_2(N) \cong \overline{H^1(\nu;\mathbb{Z}[\nu])}$ is finitely generated as a left $\mathbb{Z}[\nu]$-module, $Hom_{\mathbb{Z}[\pi]}(\pi_2(X),\mathbb{Z}[\pi])=0$, and so X is strongly minimal.

If X is strongly minimal then $\pi_2(X) \cong \overline{H^2(X;\mathbb{Z}[\pi])} = \overline{H^2(\pi;\mathbb{Z}[\pi])}$, and so $\pi_2(X_\nu)=\pi_2(X)|_\nu$ is finitely generated as a left $\mathbb{Z}[\nu]$-module. Since ν is finitely presentable, it follows that $\beta_q(X_\nu;\mathbb{F}_2)$ is finite for $q\le 2$. Poincaré duality in X gives an isomorphism $H_3(X_\nu;\mathbb{F}_2)\cong H^1(X;\mathbb{F}_2[\pi/\nu])=\mathbb{F}_2$. Hence $\beta_q(X_\nu;\mathbb{F}_2)$ is finite for all q, and so $\chi(X)=0$, by a Wang sequence argument applied to the fibration $X_\nu \to X \to S^1$.

If $\chi(X)=0$ then X is a mapping torus of a self homotopy equivalence of a PD_3-complex N with $\pi_1(N)=\nu$. (See [34, Chapter 4].)

The indecomposable factors G_i of $\nu = *G_i$ are either PD_3-groups or virtually free [15], and in either case $H^2(G_i;\mathbb{Z}[G_i])=0$. Therefore $H^2(\nu;\mathbb{Z}[\nu])=0$ and so $E^3\mathbb{Z}=0$. The final assertion now follows from the evaluation sequence, Lemma 6 and Theorem 5.

The condition that ν be the fundamental group of a PD_3-complex is quite restrictive. Mapping tori of self homotopy equivalences of PD_3-complexes are always strongly minimal, but other PD_4-complexes with such groups may be order-minimal but not χ-minimal, and so have no strongly minimal model. (See §5 above for an example with $\pi=\mathbb{Z}^4$ and $\chi(M)=6$.)

If ν is finite then π has two ends, and if ν has one end then π is a PD_4-group. If ν is torsion free and has two ends it is $\mathbb{Z}$, and so $\pi\cong\mathbb{Z}^2$ or $\mathbb{Z}\rtimes_{-1}\mathbb{Z}$. More generally, when ν is a finitely generated free group $F(n)$ (with $n>0$) then π has one end and $c.d.\pi=2$. This broader class of groups is the focus of the rest of this paper.

14 Groups of cohomological dimension 2

When $c.d.\pi=2$, we may drop the qualification "strongly", by the following theorem. (This is also so if π is a free group. The arguments below may be adapted to the latter case, which is well understood [37].)

Theorem 18. *Let X be a PD_4-complex with $\pi_1(X)\cong\pi$ such that $c.d.\pi=2$, and let $w=w_1(X)$. Then*

1. $C_*(X;\mathbb{Z}[\pi])$ is $\mathbb{Z}[\pi]$-chain homotopy equivalent to $D_*\oplus P[2]\oplus D^\dagger_{4-*}$, where D_* is a projective resolution of $\mathbb{Z}$, $P[2]$ is a finitely generated projective module P con-

centrated in degree 2 *and* $D^{\dagger}_{4-*}$ *is the conjugate dual of* D_**, shifted to terminate in degree* 2*;*

2. $\Pi=\pi_2(X)\cong P\oplus E^2\mathbb{Z}$*;*
3. $\chi(X)\geq 2\chi(\pi)$*, with equality if and only if* $P=0$*;*
4. $(E^2\mathbb{Z})^{\dagger}=0$*;*
5. $\pi_3(X)\cong\Gamma_W(\Pi)\oplus E^1\mathbb{Z}$.

Moreover, $P_2(X)\simeq L=L_\pi(\Pi,2)$*, and so the homotopy type of* X *is determined by* π*,* w*,* Π*, and the orbit of* $k_2(X)\in H^4(L;\pi_3(X))$ *under the actions of* $Aut_\pi(\pi_3(X))$ *and* $E_0(L)$*.*

Every nonsingular w*-hermitian pairing on a finitely generated projective* $\mathbb{Z}[\pi]$*-module is realized by some such* PD_4*-complex.*

Proof. Let $C_*=C_*(X;\mathbb{Z}[\pi])$, and let D_* be the chain complex with $D_0=C_0$, $D_1=C_1$, $D_2=\mathrm{Im}(\partial_2^C)$ and $D_q=0$ for $q>2$. Then

$$0\to D_2\to D_1\to D_0\to\mathbb{Z}\to 0$$

is a resolution of the augmentation module. Since $c.d.\pi\leq 2$ and D_0 and D_1 are free modules D_2 is projective, by Schanuel's Lemma. Therefore the epimorphism from C_2 to D_2 splits, and so C_* is a direct sum $C_*\cong D_*\oplus(C/D)_*$. Since X is a PD_4-complex C_* is chain homotopy equivalent to the conjugate dual $C^{\dagger}_{4-*}$. Assertions (1) and (2) follow easily.

On taking homology with simple coefficients $\mathbb{Q}$, we see that $\chi(X)=2\chi(\pi)+\dim_{\mathbb{Q}}\mathbb{Q}\otimes_\pi P$. Hence $\chi(X)\geq 2\chi(\pi)$. Since π satisfies the Weak Bass conjecture [18] and P is projective $P=0$ if and only if $\dim_{\mathbb{Q}}\mathbb{Q}\otimes_\pi P=0$.

Let $\delta:D_2\to D_1$ be the inclusion. Then $E^2\mathbb{Z}=\mathrm{Cok}(\delta^{\dagger})$ and so $(E^2\mathbb{Z})^{\dagger}=\mathrm{Ker}(\delta^{\dagger\dagger})$. But $\delta^{\dagger\dagger}=\delta$ is injective, and so $(E^2\mathbb{Z})^{\dagger}=0$.

The indecomposable free factors of π are either one-ended or infinite cyclic, and at least one factor has one end, since $c.d.\pi>1$. Thus $H_3(\widetilde{X};\mathbb{Z})\cong E^1\mathbb{Z}$ is a free $\mathbb{Z}[\pi]$-module, by Lemma 2. Hence $\pi_3(X)\cong\Gamma_W(\Pi)\oplus E^1\mathbb{Z}$.

Since $c.d.\pi=2$ the first k-invariant of X is trivial, and so $P_2(X)\simeq L=L_\pi(\Pi,2)$. Hence the next assertion follows from Lemma 5.

The realization result follows from Theorem 10.

It follows immediately from (2), (3) and Theorem 5 that "χ-minimal", "order-minimal" and "strongly minimal" are equivalent, when $c.d.\pi=2$. We shall henceforth use just "minimal" for such complexes.

It remains unknown whether every finitely presentable group π with $c.d.\pi=2$ has a finite 2-dimensional $K(\pi,1)$-complex. We shall write $g.d.\pi=2$ if this is so.

Corollary 19 *Let* X *and* Y *be* PD_4*-complexes with fundamental group* π *such that* $c.d.\pi=2$*, and* $w_1(X)=c_X^*w$ *and* $w_1(Y)=c_Y^*w$ *for some homomorphism* $w:\pi\to\mathbb{Z}^\times$*. Then* X *and* Y *are homotopy equivalent if and only if they have the same minimal model* Z *and* $\lambda_X\cong\lambda_Y$*.* □

The minimal model may not be uniquely determined! See §14 below.

Theorem 20. *Let Z be a minimal PD_4-complex with fundamental group π such that $c.d.\pi = 2$, and let $w = w_1(Z)$, $L = L_\pi(E^2\mathbb{Z}, 2)$ and $\pi_3 = \Gamma_W(E^2\mathbb{Z}) \oplus E^1\mathbb{Z}$. Then*

1. *the homotopy type of Z is determined by π, w and the orbit of $k_2(Z) \in H^4(L;\pi_3)$ under the actions of $Aut_\pi(\Gamma_W(E^2\mathbb{Z}) \oplus E^1\mathbb{Z})$ and $E_0(L)$;*
2. *if $\widehat{Z}$ is another such complex then $P_2(\widehat{Z}) \simeq P_2(Z)$ if and only if there is an isomorphism $f : \pi_1(\widehat{Z}) \cong \pi$ such that $w_1(\widehat{Z}) = f^*w$;*
3. *the v_2-type of Z is* II *or* III*, i.e., $v_2(Z) = c_Z^*V$ for some $V \in H^2(\pi;\mathbb{F}_2)$;*
4. *if Z is orientable then it has signature $\sigma(Z) = 0$;*
5. *for every $v \in H^2(\pi;\mathbb{F}_2)$ there is a minimal PD_4-complex Z with $\pi_1(Z) \cong \pi$, $w_1(Z) = c_Z^*w$ and $v_2(Z) = c_Z^*v$.*

Proof. The first assertion follows from Theorem 18, since $P_2(Z) \simeq L$.

If $f : \pi_1(\widehat{Z}) \cong \pi$ is an isomorphism such that $w_1(\widehat{Z}) = f^*w$ then $\pi_2(\widehat{Z}) \cong \Pi$ and so $P_2(\widehat{Z}) \simeq P_2(Z)$. Conversely, $Ext^2_{\mathbb{Z}[\pi]}(\Pi, \mathbb{Z}[\pi]) = \mathbb{Z}^w$, so π and Π determine w.

Let $H = c_Z^*H^2(\pi;\mathbb{F}_2)$. Then $\dim H^2(Z;\mathbb{F}_2) = 2\dim H$, since $\chi(Z) = 2\chi(\pi)$, and $H \cup H = 0$, since $c.d.\pi = 2$. In particular, $v_2(Z) \cup h = h \cup h = 0$ for all $h \in H$. Therefore $v_2(Z) \in H$, by the nonsingularity of Poincaré duality. If Z is orientable a similar argument with coefficients $\mathbb{Q}$ shows that $H^2(Z;\mathbb{Q})$ has a self-orthogonal summand of rank $\beta_2(\pi) = \frac{1}{2}\beta_2(Z)$, and so $\sigma(Z) = 0$.

We may use a finite presentation $\mathscr{P} = \langle X \mid R \rangle$ for π as a pattern for constructing a 5-dimensional handlebody $D^5 \cup_{x\in X} h_x^1 \cup_{r\in R} h_r^2 \simeq C(\mathscr{P})$, where the 1- and 2-handles are indexed by X and R, respectively, but we refine the construction by taking non-orientable 1-handles for generators x with $w(x) \neq 0$ and using $w_2 = v + w^2$ to twist the framings of the 2-handles corresponding to the relators. Let M be the boundary of the resulting 5-manifold. Then $\pi_1(M) \cong \pi$, $w_1(M) = c_M^*w$ and $v_2(M) = c_M^*v$. Since $E^3\mathbb{Z} = 0$ the pairing λ_M is nonsingular, by part (4) of Lemma 6. Hence M has a strongly minimal model Z, by Corollary 6. Since c_M factors through c_Z via a 2-connected degree-1 map, Z has the required properties.

The argument for realizing v is taken from [29], where it is shown that if $C(\mathscr{P})$ is aspherical then the manifold M is itself minimal.

How does $k_2(X)$ determine $v_2(X)$ (and conversely)? This seems to be a crucial question. We expect that the orbit of the k-invariant is detected by the refined v_2-type, but have only proven this in some cases. (See Theorems 25 and 27 below.)

Since the Postnikov third stage $f_{X,3}$ (defined in §3 above) is 4-connected, $H^4(f_{X,3};\mathbb{F}_2)$ is injective, and so it is an isomorphism if also $\beta_2(X;\mathbb{F}_2) > 0$, by the nondegeneracy of Poincaré duality. Thus the ring $H^*(X;\mathbb{F}_2)$ and hence $v_2(X)$ should be directly computable from $H^*(P_3(X);\mathbb{F}_2)$.

If X is of v_2-type II or III then any minimal model for X must have compatible v_2-type, by Lemma 7. What happens if $v_2(\widetilde{X}) \neq 0$? Does X have a minimal model Z with $v_2(Z) = 0$? (If π is a PD_2-group then X has minimal models of each type, by Theorem 24 below.)

We show next that the class of groups considered here is the largest for which every PD_4-complex with such a fundamental group has a strongly minimal model.

Theorem 21. *Let π be a finitely presentable group and $w:\pi\to\mathbb{Z}^\times$ be a homomorphism. Then the following are equivalent:*

1. *every PD_4-complex with fundamental group π and orientation character w has a strongly minimal model;*
2. *every order minimal PD_4-complex with fundamental group π and orientation character w is strongly minimal;*
3. *$c.d.\pi\le 2$.*

Proof. The equivalence $(1)\Leftrightarrow(2)$ is clear.

Suppose that (1) holds, and let K be a finite 2-complex with $\pi_1(K)=\pi$. Then K has a 4-dimensional thickening N which is a handlebody with only 0-, 1- and 2-handles, and with $w_1(N)=c_N^*w$. (Cf. the final paragraph of Theorem 30.) Let $M=D(N)$ be the closed 4-manifold obtained by doubling N, and let $j:N\to M$ be one of the canonical inclusions. Then $(\pi_1(M),w_1(M))\cong(\pi,w)$, and collapsing the double gives a retraction $r:M\to N$. We may assume that $c_M=c_Nr$.

Since N is a retract of $M=D(N)$, we have

$$H^2(M;\mathbb{Z}[\pi])\cong H^2(N;\mathbb{Z}[\pi])\oplus H^2(M,N;\mathbb{Z}[\pi]).$$

Let $E=E^2\mathbb{Z}$, and $H=\overline{H^2(M;\mathbb{Z}[\pi])}$. Since $c_M\sim c_Nr$, we have

$$H/E\cong(\overline{H^2(N;\mathbb{Z}[\pi])}/E)\oplus\overline{H^2(M,N;\mathbb{Z}[\pi])}.$$

Since M has a strongly minimal model H/E is projective, by Corollary 6. Hence so is the direct summand $\overline{H^2(M,N;\mathbb{Z}[\pi])}$. This summand is $\overline{H^2(M,N;\mathbb{Z}[\pi])}\cong H_2(N;\mathbb{Z}[\pi])$, by Poincaré-Lefshetz duality.

Now $H_2(N;\mathbb{Z}[\pi])\cong P=H_2(K;\mathbb{Z}[\pi])$, since $K\simeq N$. Hence the augmentation $\mathbb{Z}[\pi]$-module $\mathbb{Z}$ has a projective resolution of length 3, given by $C_*(K;\mathbb{Z}[\pi])$ in degrees ≤ 2 and by the module P in degree 3, with differential ∂_3 given by the natural inclusion of P as the submodule of 2-cycles. Thus $c.d.\pi\le 3$. We also have $E^3\mathbb{Z}\cong E^\dagger$, since there is a strongly minimal PD_4-complex realizing the pair (π,w). Therefore $c.d.\pi\le 2$, by Lemma 9.

The converse implication $(3)\Rightarrow(1)$ follows from Theorem 20.

The group π is a PD_2-group if and only if $E^2\mathbb{Z}$ is infinite cyclic [8]. The minimal PD_4-complexes are then the total spaces of S^2-bundles over aspherical closed surfaces [34, Theorem 5.10]. We shall review this case in §14 below.

Otherwise $E^2\mathbb{Z}$ is not finitely generated. If $\pi\cong\nu\rtimes\mathbb{Z}$, with ν finitely presentable, then $\nu\cong F(n)$ for some $n>0$ and π has one end. Let $S^2\tilde{\times}S^1$ be the mapping torus of the antipodal map of S^2.

Theorem 22. *Let $\pi=F(n)\rtimes_\alpha\mathbb{Z}$, where $n>0$, and let $w:\pi\to\mathbb{Z}^\times$ be a homomorphism. Then the minimal PD_4-complexes X with fundamental group π and $w_1(X)=c_X^*w$ are homotopy equivalent to mapping tori, and their homotopy types may be distinguished by their refined v_2-types.*

Proof. A PD_3-complex N with fundamental group $F(n)$ is homotopy equivalent to $\#^n(S^2\times S^1)$ (if it is orientable) or $\#^n(S^2\tilde{\times}S^1)$ (otherwise). There is a natural representation of $Aut(F(n))$ by isotopy classes of based homeomorphisms of N, and the group of based self homotopy equivalences $E_0(N)$ is a semidirect product $D\rtimes Aut(F(n))$, where D is generated by Dehn twists about nonseparating 2-spheres. If we identify D with $(\mathbb{Z}/2\mathbb{Z})^n = H^1(F(n);\mathbb{F}_2)$, we then see that $E_0(N) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes Aut(F(n))$, with the natural action of $Aut(F(n))$ [33].

Thus a minimal PD_4-complex X with $\pi_1(X)\cong\pi$ is homotopy equivalent to the mapping torus $M(f)$ of a based self-homeomorphism f of such an N, with $w_1(N) = w|_{F(n)}$, and f has image (d,α) in $E_0(N)$. Let $\delta(f)$ be the image of d in $H^2(\pi;\mathbb{F}_2) = H^1(F(n);\mathbb{F}_2)/(\alpha-1)H^1(F(n);\mathbb{F}_2)$. If g is another based self-homeomorphism of N with image (d',α) and $\delta(g)=\delta(f)$ then $d-d'=(\alpha-1)(e)$ for some $e\in D$. Hence (d,α) and (d',α) are conjugate, and so $M(g)\simeq M(f)$.

All minimal PD_4-complexes X with $\pi_1(X)=\pi$ and $w_1(X)=w$ have the same Postnikov 2-stage $L=P_2(X)$, all have v_2-type II or III, and there is such a PD_4-complex X with $v_2(X)=V$, for every $V\in H^2(\pi;\mathbb{F}_2)$, by Theorem 18 and its corollary. Hence the refined v_2-type is a complete invariant.

If $\beta_1(\pi)>1$ then N may not be determined by $M(f)$. For instance if $N=S^2\tilde{\times}S^1$ then $M(id_N)=N\times S^1$ is also the mapping torus of an orientation reversing self homeomorphism of $S^2\times S^1$. It is a remarkable fact that if $\pi=F(n)\rtimes_\alpha\mathbb{Z}$, $n>1$ and $\beta_1(\pi)\geq 2$ then π is such a semidirect product for infinitely many distinct values of n [11]. However this does not affect our present considerations.

The refined v_2-type is also a complete invariant of the homotopy type of a minimal PD_4-complex when π is a PD_2-group. This case is treated in §15 below. The argument given there is generalized in Theorem 27 to other 2-dimensional duality groups, subject to a technical algebraic condition. This condition holds if $w=1$ and π is an ascending HNN extension $\mathbb{Z}*_m$, by Theorem 30, while if m is even there is an unique minimal model, by Corollary 28.

15 Realizing k-invariants

For the rest of this paper we shall assume that π is a finitely presentable, 2-dimensional duality group (i.e., π has one end and $c.d.\pi=2$). The homotopy type of a minimal PD_4-complex X with $\pi_1(X)=\pi$ is determined by π, w and the orbit of $k_2(X)$ under the actions of $E_0(L)$ and $Aut(\Gamma_W(\Pi))$, by Corollary 26. We would like to find more explicit and accessible invariants that characterize such orbits. We would also like to know which k-invariants give rise to PD_4-complexes. Note first that $H_3(\widetilde{X};\mathbb{Z})=H_4(\widetilde{X};\mathbb{Z})=0$, since π has one end.

Theorem 23. *Let π be a finitely presentable, 2-dimensional duality group, and let $w:\pi\to\mathbb{Z}^\times$ be a homomorphism. Let $\Pi=E^2\mathbb{Z}$ and let $k\in H^4(L;\Gamma_W(\Pi))$. Then*

1. *there is a* 4*-complex Y with $\pi_1(Y)\cong\pi$, $\pi_2(Y)\cong\Pi$, $\pi_3(Y)\cong\Gamma_W(\Pi)$, $k_2(Y)=k$ and $H_3(\widetilde{Y};\mathbb{Z})=H_4(\widetilde{Y};\mathbb{Z})=0$ if and only if the homomorphism determined by p_L^*k from $H_4(K(\Pi,2);\mathbb{Z})$ to $\Gamma_W(\Pi)$ is an isomorphism ;*
2. *any such complex Y is finitely dominated, and we may assume that Y is a finite complex if π is of type FF;*
3. *$\overline{H^2(Y;\mathbb{Z}[\pi])}\cong\Pi$;*
4. *$H_4(Y;\mathbb{Z}^w)\cong\mathbb{Z}$ and cap product with a generator induces isomorphisms $\overline{H^p(Y;\mathbb{Z}[\pi])}\cong H_{4-p}(Y;\mathbb{Z}[\pi])$, for $p\neq 2$.*

Proof. If Y is such a 4-complex then p_L^*k is an isomorphism, by the exactness of the Whitehead sequence.

Suppose, conversely, that p_L^*k is an isomorphism. Let $P(k)$ denote the Postnikov 3-stage determined by $k\in H^4(L;\Gamma_W(\Pi))$, and let $P=P(k)^{[4]}$. Let $C_*=C_*(\widetilde{P})$ be the equivariant cellular chain complex for $\widetilde{P}$, and let $B_q\le Z_q\le C_q$ be the submodules of q-boundaries and q-cycles, respectively. Clearly $H_1(C_*)=0$ and $H_2(C_*)\cong\Pi$, while $H_3(C_*)=0$, since p_L^*k is an isomorphism. Hence there are exact sequences

$$0\to B_1\to C_1\to C_0\to\mathbb{Z}\to 0,$$

$$0\to B_3\to C_3\to Z_2\to\Pi\to 0$$

and

$$0\to H_4(C_*)=Z_4\to C_4\to B_3\to 0.$$

Schanuel's Lemma implies that B_1 is projective, since $c.d.\pi=2$. Hence $C_2\cong B_1\oplus Z_2$ and so Z_2 is also projective. It then follows that B_3 is also projective, and so $C_4\cong B_3\oplus Z_4$. Thus $H_4(C_*)=Z_4$ is a projective direct summand of C_4.

After replacing P by $P\vee W$, where W is a wedge of copies of S^4, if necessary, we may assume that $Z_4=H_4(P;\mathbb{Z}[\pi])$ is free. Since $\Gamma_W(\Pi)\cong\pi_3(P)$ the Hurewicz homomorphism from $\pi_4(P)$ to $H_4(P;\mathbb{Z}[\pi])$ is onto, by the exactness of the Whitehead sequence. We may then attach 5-cells along maps representing a basis for Z_4 to obtain a countable 5-complex Q with 3-skeleton $Q^{[3]}=P(k)^{[3]}$ and with $H_q(\widetilde{Q};\mathbb{Z})=0$ for $q\ge 3$. The inclusion of P into $P(k)$ extends to a 4-connected map from Q to $P(k)$.

Let D_* be the finite projective resolution of $\mathbb{Z}$ determined by a finite presentation for π. Dualizing gives a finite projective resolution $E_*=D_{2-*}^{\dagger}$ for $\Pi=E^2\mathbb{Z}$. Then $C_*(\widetilde{Q})$ is chain homotopy equivalent to $D_*\oplus E_*[2]$, which is a finite projective chain complex. It follows from the finiteness conditions of Wall that Q is homotopy equivalent to a finitely dominated complex Y of dimension ≤ 4 [59]. (The splitting reflects the fact that c_Y is a retraction, since $k_1(Y)=0$.) The homotopy type of Y is uniquely determined by the data, as in Lemma 5.

If π is of type FF then B_1 is stably free, by Schanuel's Lemma. Hence Z_2 is also stably free. Since dualizing a finite free resolution of $\mathbb{Z}$ gives a finite free resolution of $\Pi=E^2\mathbb{Z}$ we see in turn that B_3 must be stably free, and so $C_*(\widetilde{Y})$ is chain homotopy equivalent to a finite free complex. Hence Y is homotopy equivalent to a finite 4-complex [59].

Condition (3) follows immediately from the 4-term evaluation sequence, since $\Pi^\dagger = E^2\mathbb{Z}^\dagger = 0$, by part (4) of Theorem 18.

We see easily that $\overline{H^4(Y;\mathbb{Z}[\pi])} = E^2\Pi \cong \mathbb{Z}$ and $H^4(Y;\mathbb{Z}^w) \cong Ext^2(\Pi;\mathbb{Z}^w) \cong \mathbb{Z}$. The homomorphism $\varepsilon_{w\#} : H^4(Y;\mathbb{Z}[\pi]) \to H^4(Y;\mathbb{Z}^w)$ induced by ε_w is surjective, since Y is 4-dimensional, and therefore is an isomorphism. We also have $H_4(Y;\mathbb{Z}^w) \cong Tor_2(\mathbb{Z}^w;\Pi) \cong \mathbb{Z}^w \otimes_\pi \mathbb{Z}[\pi] \cong \mathbb{Z}$. Let $[Y]$ be a generator of $H_4(Y;\mathbb{Z}^w)$. Then evaluation on $[Y]$ induces an isomorphism from $\overline{H^4(Y;\mathbb{Z}[\pi])}$ to $H_0(Y;\mathbb{Z}[\pi])$. Hence $- \frown [Y]$ induces isomorphisms from $\overline{H^p(Y;\mathbb{Z}[\pi])}$ to $H_{4-p}(Y;\mathbb{Z}[\pi])$ for all $p \neq 2$, since $H^p(Y;\mathbb{Z}[\pi]) = H_{4-p}(Y;\mathbb{Z}[\pi]) = 0$ if $p \neq 2$ or 4.

Since $Hom_{\mathbb{Z}[\pi]}(\overline{H^2(Y;\mathbb{Z}[\pi])}, H_2(Y;\mathbb{Z}[\pi])) \cong Hom_{\mathbb{Z}[\pi]}(E^2\mathbb{Z}, E^2\mathbb{Z})$ and $End(E^2\mathbb{Z}) = \mathbb{Z}$, by Lemma 1, cap product with $[Y]$ in degree 2 is determined by an integer. The 4-complex Y is a PD_4-complex if and only if this integer is ± 1. The obvious question is: what is this integer? Is it always ± 1? The complex C_* is chain homotopy equivalent to its dual, but is the chain homotopy equivalence given by slant product with $[Y]$?

If π is either a semidirect product $F(n) \rtimes \mathbb{Z}$ or the fundamental group of a Haken 3-manifold M then $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$, i.e., projective $\mathbb{Z}[\pi]$-modules are stably free [58]. (This is not yet known for all torsion free one relator groups.) In such cases finitely dominated complexes are homotopy finite.

16 PD_2-groups

The case of most natural interest is when π is a PD_2-group, i.e., is the fundamental group of an aspherical closed surface F. If Z is the minimal model for such a PD_4-complex X then $\Pi = \pi_2(Z)$ and $\Gamma_W(\Pi)$ are infinite cyclic, and Z is homotopy equivalent to the total space of a S^2-bundle over a closed aspherical surface. (The action $u : \pi \to Aut(\Pi)$ is given by $u(g) = w_1(\pi)(g)w(g)$ for all $g \in \pi$ [34, Lemma 10.3], while the induced action on $\Gamma_W(\Pi)$ is trivial.) There are two minimal models for each pair (π, w), distinguished by their v_2-type. This follows easily from the fact that the inclusion of $O(3)$ into the monoid of self-homotopy equivalences $E(S^2)$ induces a bijection on components and an isomorphism on fundamental groups [34, Lemma 5.9]. It is instructive to consider this case from the point of view of k-invariants also, as we shall extend the argument of this section to other groups in Theorem 27 below. In this case we may take F as an exemplar of $K = K(\pi,1)$.

Suppose first that π acts trivially on Π. Then $L \simeq K \times \mathbb{CP}^\infty$. Fix generators t, x, η and z for $H^2(\pi;\mathbb{Z})$, Π, $\Gamma_W(\Pi)$ and $H^2(\mathbb{CP}^\infty;\mathbb{Z}) = Hom(\Pi,\mathbb{Z})$, respectively, such that $z(x) = 1$ and $2\eta = [x,x]$. (These groups are all infinite cyclic, but we should be careful to distinguish the generators, as the Whitehead product pairing of Π with itself into $\Gamma_W(\Pi)$ is not the pairing given by multiplication.) Let t, z denote also the generators of $H^2(L;\mathbb{Z})$ induced by the projections to K and $\mathbb{CP}^\infty$, respectively. Then $H^2(\pi;\Pi)$ is generated by $t \otimes x$, while $H^4(L;\Gamma_W(\Pi))$ is generated by $tz \otimes \eta$ and $z^2 \otimes \eta$. (Note that t has order 2 if $w_1(\pi) \neq 0$.)

Lemma 15. *The k-invariant $k_2(S^2)$ generates $H^4(\mathbb{CP}^\infty;\mathbb{Z})$.*

Proof. Let $h:\mathbb{CP}^\infty\to K(\mathbb{Z},4)$ be the fibration with homotopy fibre $P_3(S^2)$ corresponding to $k_2(S^2)$. Since $P_3(S^2)$ may be obtained by adjoining cells of dimension ≥ 5 to S^2 we see that $H^4(P_3(S^2);\mathbb{Z})=0$. It follows from the spectral sequence of the fibration that h^* maps $H^4(K(\mathbb{Z},4);\mathbb{Z})$ onto $H^4(\mathbb{CP}^\infty;\mathbb{Z})$, and so $k_2(S^2)=h^*\iota_{\mathbb{Z},4}$ generates $H^4(\mathbb{CP}^\infty;\mathbb{Z})$.

Since $\widetilde{Z}\simeq S^2$, the image of $k_2(Z)$ in $H^4(\widetilde{L};\mathbb{Z})\cong\mathbb{Z}$ generates this group. Hence $k_2(Z)=\pm(z^2\otimes\eta+mtz\otimes\eta)$ for some $m\in\mathbb{Z}$. The action of $[K,L]_K=[K,\mathbb{CP}^\infty]\cong H^2(\pi;\mathbb{Z})$ on $H^2(L;\mathbb{Z})$ is determined by $t\mapsto t$ and $z\mapsto z+t$, and so its action on $H^4(L;\Gamma_W(\Pi))$ is given by $tz\otimes\eta\mapsto tz\otimes\eta$ and $z^2\otimes\eta\mapsto z^2\otimes\eta+2tz\otimes\eta$. There are thus two possible $E_0(L)$-orbits of k-invariants, and each is in fact realized by the total space of an S^2-bundle over the surface K.

If the action $u:\pi\to Aut(\Pi)$ is nontrivial these calculations go through essentially unchanged with coefficients $\mathbb{F}_2$ instead of $\mathbb{Z}$. There are again two possible $E_\pi(L)$-orbits of k-invariants, and each is realized by an S^2-bundle space.

In all cases the orbits of k-invariants correspond to the elements of $H^2(\pi;\mathbb{F}_2)=\mathbb{Z}/2\mathbb{Z}$. In fact the k-invariant may be detected by the Wu class. Let $[c]_2$ denote the image of a cohomology class under reduction *mod* (2). Since $k_2(Z)$ has image 0 in $H^4(Z;\Pi)$ it follows that $[z]_2^2\equiv m[tz]_2$ in $H^4(Z;\mathbb{F}_2)$. This holds also if the PD_2-group π is non-orientable (i.e., the surface F is non-orientable) or the action u is nontrivial, and so $v_2(Z)=m[z]_2$ and the orbit of $k_2(Z)$ determine each other.

If X is not minimal and $v_2(\widetilde{X})\neq 0$ then the minimal model Z is not uniquely determined by X. Nevertheless we have the following results.

Theorem 24. *Let E be the total space of an S^2-bundle over an aspherical closed surface F, and let X be a PD_4-complex with fundamental group $\pi\cong\pi_1(F)$. Let τ be the image of the generator of $H^2(\pi;\mathbb{F}_2)$ in $H^2(X;\mathbb{F}_2)$. Then there is a 2-connected degree-1 map $h:X\to E$ such that $c_E=c_Xh$ if and only if*

1. $(c_X^*)^{-1}w_1(X)=(c_E^*)^{-1}w_1(E)$*; and*
2. $\xi\smile\tau\neq 0$ *for some* $\xi\in H^2(X;\mathbb{F}_2)$ *such that* $\xi^2=0$ *if* $v_2(E)=0$ *and* $\xi^2\neq 0$ *if* $v_2(E)\neq 0$.

Proof. See Theorem 10.17 of the current version of [34].

This is consistent with Lemma 7, for if $v_2(X)=0$ then $\xi^2=0$ and $v_2(E)=0$, while if $v_2(X)=\tau$ then $\xi^2\neq 0$, and thus $v_2(E)\neq 0$ also.

If $w_1(X)=c_X^*w$, where $w=w_1(\pi)$, and $v_2(X)=0$ then E must be $F\times S^2$, and we may construct a degree-1 map as follows. Let Ω generate $H^2(\pi;\mathbb{Z}^w)$. We may choose $y\in H^2(X;\mathbb{Z})$ so that $(y\smile c_X^*\Omega)\cap[X]=1$, by Poincaré duality for X. Then $[y]_2^2=0$, since $v_2(X)=0$. Therefore if F is non-orientable $y^2=0$ in $H^4(X;\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$; if F is orientable then $y^2=2k(y\smile c_X^*\Omega)$ for some k, and we may replace y by $y'=y-kc_X^*\Omega$ to obtain a class with square 0. Such a class may be realized by a map $d:X\to S^2$ [54, Theorem 8.4.11], and we may set $h=(c_X,d):X\to F\times S^2$.

If $v_2(X) \neq 0$ or τ then there is a $\xi \in H^2(X;\mathbb{F}_2)$ such that $\xi \smile \tau \neq 0$ but $\xi^2 = 0$. There is also a class ζ such that $\zeta \smile (\tau - v_2(X)) = 0$ but $\zeta \smile \tau \neq 0$. Hence $\zeta^2 = \zeta \smile \tau \neq 0$. Thus X has minimal models of each v_2-type.

In particular, if C is a smooth projective complex curve of genus ≥ 1 and $X = (C \times \mathbb{CP}^1)\#\overline{\mathbb{CP}^2}$ is a blowup of the ruled surface $C \times \mathbb{CP}^1 = C \times S^2$ then each of the two orientable S^2-bundles over C is a minimal model for X. In this case they are also minimal models in the sense of complex surface theory. (See [1, Chapter VI].) Many of the other minimal complex surfaces in the Enriques-Kodaira classification are aspherical, and hence strongly minimal in our sense. However 1-connected complex surfaces are never minimal in our sense, since S^4 is the unique minimal 1-connected PD_4-complex and S^4 has no complex structure, by a classical result of Wu [1, Proposition IV.7.3].

Theorem 25. *The homotopy type of a PD_4-complex X with fundamental group π a PD_2-group is determined by π, $w_1(X)$, λ_X and the v_2-type.*

Proof. Let $v = w_1(\pi)$, $u = w_1(X) + c_X^* v$, and let Ω generate $H^2(\pi;\mathbb{Z}^v)$. Then $[\Omega]_2$ generates $H^2(\pi;\mathbb{F}_2)$, and $\tau = c_X^*[\Omega]_2 \neq 0$. If $v_2(X) = m\tau$ and $p : X \to Z$ is a 2-connected degree-1 map then $v_2(Z) = mc_Z^*[\Omega]_2$, and so there is an unique minimal model for X. Otherwise $\tau \neq v_2(X)$, and so there are elements $y, z \in H^2(X;\mathbb{F}_2)$ such that $y \smile \tau \neq y^2$ and $z \smile \tau \neq 0$. If $y \smile \tau = 0$ and $z^2 \neq 0$ then $(y+z) \smile \tau \neq 0$ and $(y+z)^2 = 0$. Taking $\xi = y, z$ or $y+z$ appropriately, we have $\xi \smile \tau \neq 0$ and $\xi^2 = 0$. Hence X has a minimal model Z with $v_2(Z) = 0$, by Theorem 24. In all cases the theorem now follows from Theorem 7.

If Z is strongly minimal and $E^2\mathbb{Z}$ is finitely generated but not 0 then $E^2\mathbb{Z}$ is infinite cyclic [8] and the kernel κ of the natural action of π on $\pi_2(Z) \cong \mathbb{Z}$ is a PD_2-group [34, Theorem 10.1]. Thus π is either a PD_2-group or a semidirect product $\kappa \rtimes (\mathbb{Z}/2\mathbb{Z})$. (In particular, π has one end).

17 Cup products

In Theorem 27 below we shall use a "cup-product" argument to relate cohomology in degrees 2 and 4. Let G be a group and let $\Gamma = \mathbb{Z}[G]$. Let C_* and D_* be chain complexes of left Γ-modules and $\mathscr{A}$ and $\mathscr{B}$ left Γ-modules. Using the diagonal homomorphism from G to $G \times G$ we may define *internal products*

$$H^p(Hom_\Gamma(C_*,\mathscr{A})) \otimes H^q(Hom_\Gamma(D_*,\mathscr{B})) \to H^{p+q}(Hom_\Gamma(C_* \otimes D_*, \mathscr{A} \otimes \mathscr{B}))$$

where the tensor products of Γ-modules taken over $\mathbb{Z}$ have the diagonal G-action. (See [14, Chapter XI.§4].) If C_* and D_* are resolutions of $\mathscr{C}$ and $\mathscr{D}$, respectively, we get pairings

$$Ext^p_\Gamma(\mathscr{C},\mathscr{A}) \otimes Ext^q_\Gamma(\mathscr{D},\mathscr{B}) \to Ext^{p+q}_\Gamma(\mathscr{C} \otimes \mathscr{D}, \mathscr{A} \otimes \mathscr{B}).$$

When $\mathscr{A}=\mathscr{B}=\mathscr{D}$, $\mathscr{C}=\mathbb{Z}$ and $q=0$ we get pairings

$$H^p(G;\mathscr{A})\otimes End_G(\mathscr{A})\to Ext^p_{\mathbb{Z}[G]}(\mathscr{A},\mathscr{A}\otimes\mathscr{A}).$$

If instead $C_*=D_*=C_*(\widetilde{S})$ for some space S with $\pi_1(S)\cong G$ composing with an equivariant diagonal approximation gives pairings

$$H^p(S;\mathscr{A})\otimes H^q(S;\mathscr{B})\to H^{p+q}(S;\mathscr{A}\otimes\mathscr{B}).$$

These pairings are compatible with the Universal Coefficient spectral sequences $Ext^q_\Gamma(H_p(C_*),\mathscr{A})\Rightarrow H^{p+q}(C^*;\mathscr{A})=H^{p+q}(Hom_\Gamma(C_*,\mathscr{A}))$, etc. We shall call these pairings “cup products”, and use the symbol $\smile$ to express their values.

We wish to show that if π is a finitely presentable, 2-dimensional duality group then cup product with id_Π gives an isomorphism

$$c^2_{\pi,w}:H^2(\pi;\Pi)\to Ext^2_{\mathbb{Z}[\pi]}(\Pi,\Pi\otimes\Pi).$$

The next lemma shows that these groups are isomorphic; we state it in greater generality than we need, in order to clarify the hypotheses on the group.

Lemma 16. *Let G be a group for which the augmentation (left) module $\mathbb{Z}$ has a finite projective resolution P_* of length n, and such that $H^j(G;\Gamma)=0$ for $j<n$. Let $\mathscr{D}=H^n(G;\Gamma)$, $w:G\to\mathbb{Z}^\times$ be a homomorphism and $\mathscr{A}$ be a left Γ-module. Then there are natural isomorphisms*

1. *$\alpha_\mathscr{A}:\mathscr{D}\otimes_\Gamma\mathscr{A}\to H^n(G;\mathscr{A})$; and*
2. *$e_\mathscr{A}:Ext^n_\Gamma(\overline{\mathscr{D}},\mathscr{A})\to\mathbb{Z}^w\otimes_\Gamma\mathscr{A}=\mathscr{A}/I_w\mathscr{A}$.*

Hence $\theta_\mathscr{A}=\alpha_\mathscr{A}e_{\overline{\mathscr{D}}\otimes\mathscr{A}}:Ext^n_\Gamma(\overline{\mathscr{D}},\overline{\mathscr{D}}\otimes\mathscr{A})\to H^n(G;\mathscr{A})$ is an isomorphism.

Proof. If P is a finitely generated projective left Γ-module then $Q=Hom_\Gamma(P,\Gamma)$ is a finitely generated *right* module. There is a natural isomorphism $P\cong Hom_\Gamma(\overline{Q},\Gamma)$, given by $p\mapsto(:f\mapsto\overline{f(p)})$, for all $p\in P$ and $f\in\overline{Q}$. There are also bifunctorial natural isomorphisms of abelian groups $A_{P\mathscr{A}}:Hom_\Gamma(P,\Gamma)\otimes_\Gamma\mathscr{A}\to Hom_\Gamma(P,\mathscr{A})$ given by $A_{P\mathscr{A}}(q\otimes_\Gamma a)(p)=q(p)a$ for all $a\in\mathscr{A}$, $p\in P$ and $q\in Hom_\Gamma(P,\Gamma)$.

We may assume that $P_0=\Gamma$. Let $Q_j=Hom_\Gamma(P_{n-j},\Gamma)$ and $\partial^Q_i=Hom_\Gamma(\partial^P_{n-j},\Gamma)$. This gives a resolution Q_* for $\mathscr{D}$ with $Q_n=\Gamma$. The isomorphisms $A_{P_*\mathscr{A}}$ and $A_{\overline{Q_*}\mathscr{A}}$ induce isomorphisms of chain complexes $Q_*\otimes_\Gamma\mathscr{A}\to Hom_\Gamma(P_{n-*},\mathscr{A})$, and $\overline{P_*}\otimes_\Gamma\mathscr{A}\to Hom_\Gamma(\overline{Q_{n-*}},\mathscr{A})$, respectively, from which the first two isomorphisms follow. The final assertion follows since $\mathbb{Z}^w\otimes_\Gamma(\overline{\mathscr{D}}\otimes\mathscr{A})\cong\mathscr{D}\otimes_\Gamma\mathscr{A}$.

If G is finitely presentable, has one end and $n=2$ then G is a 2-dimensional duality group. It is not known whether all the groups considered in the lemma are duality groups.

Lemma 17. *If G satisfies the hypotheses of Lemma 16 and H is a subgroup of finite index in G then cup product with $id_{\overline{\mathscr{D}}}$ is an isomorphism for (G,w) if and only if it is so for $(H,w|_H)$.*

Proof. If $\mathscr{A}$ is a left $\mathbb{Z}[G]$-module then $H^n(G;\mathscr{A}) \cong H^n(H;\mathscr{A}|_H)$, by Shapiro's Lemma. Thus if G satisfies the hypotheses of Lemma 16 the corresponding module for H is $\overline{\mathscr{D}}|_H$. Further applications of Shapiro's Lemma then give the result.

In particular, it shall suffice to consider the orientable cases.

Let $\eta : Q_0 \to \mathscr{D}$ be the canonical epimorphism, and let $[\xi] \in H^n(G;\overline{\mathscr{D}})$ be the image of $\xi \in Hom_\Gamma(P_n,\overline{\mathscr{D}})$. Then $\xi \otimes \eta : P_n \otimes \overline{Q_0} \to \overline{\mathscr{D}} \otimes \overline{\mathscr{D}}$ represents $[\xi] \smile id_{\mathscr{D}}$ in $Ext^n_\Gamma(\overline{\mathscr{D}},\overline{\mathscr{D}} \otimes \overline{\mathscr{D}})$. If $\xi = A_{P_n\overline{\mathscr{D}}}(q \otimes_\Gamma \delta)$ then $\alpha_{\overline{\mathscr{D}}}(\eta(q) \otimes_\Gamma \delta) = [\xi]$. There is a chain homotopy equivalence $j_* : \overline{Q}_* \to P_* \otimes \overline{Q}_*$, since P_* is a resolution of $\mathbb{Z}$. Given such a chain homotopy equivalence, $e_{\overline{\mathscr{D}}\otimes\overline{\mathscr{D}}}([\xi] \smile id_{\overline{\mathscr{D}}})$ is the image of $(\xi \otimes \eta)(j_n(1^*))$, where 1^* is the canonical generator of $\overline{Q}_n$, defined by $1^*(1) = 1$.

Theorem 26. *Let G be a finitely presentable, 2-dimensional duality group, and let $w : G \to \mathbb{Z}^\times$ be a homomorphism. Then $c^2_{G,w}$ is an isomorphism.*

Proof. Note first that G satisfies the hypothesis of Lemma 16, with $n = 2$. Let $\mathscr{P} = \langle X \mid R\rangle^\varphi$ be a finite presentation for G. (We shall suppress the defining epimorphism $\varphi : F(X) \to G$ where possible.) After introducing new generators x' and relators $x'x$, if necessary, we may assume that each relator is a product of distinct generators, with all the exponents positive. The new presentation $\mathscr{P}'$ has the same deficiency as $\mathscr{P}$. We may also assume that $w = 1$, after replacing G by $H = \mathrm{Ker}(w)$ if necessary, by Lemma 17.

The Fox-Lyndon resolution associated to $\mathscr{P}$ gives an exact sequence

$$0 \to P_3 = \pi_2(C(\mathscr{P})) \to P_2 \to P_1 \to P_0 = \Gamma \to \mathbb{Z} \to 0$$

in which P_1 and P_2 are free left Γ-modules with bases $\langle p^1_x; x \in X\rangle$ and $\langle p^2_r; r \in R\rangle$, respectively. The differentials are given by $\partial p^1_x = x - 1$ and $\partial p^2_r = \Sigma_{x\in X} r_x p^1_x$, where $r_x = \frac{\partial r}{\partial x}$, for $r \in R$ and $x \in X$. Moreover, P_3 is projective and ∂_3 is a split monomorphism, since $c.d.G = 2$.

Suppose first that the 2-complex $C(\mathscr{P})$ associated to the presentation is aspherical. (This assumption is not affected by our normalization of the presentations, for if $C(\mathscr{P})$ is aspherical then G is efficient, and $\chi(C(\mathscr{P}')) = def(\mathscr{P}') = \chi(C(\mathscr{P}))$. Hence $C(\mathscr{P}')$ is also aspherical [34, Theorem 2.8]. Then $P_3 = 0$ and the above sequence is a free resolution of $\mathbb{Z}$. Let $Q_j = Hom_\Gamma(P_{2-j},\Gamma)$ and $\partial^Q_i = Hom_\Gamma(\partial^P_{2-j},\Gamma)$. Then $\overline{Q}_0 = P^\dagger_2$ and $\overline{Q}_1 = P^\dagger_1$ have dual bases $\{q^0_x\}$ and $\{q^1_r\}$, respectively. (Thus $q^1_x(p^1_y) = 1$ if $x = y$ and 0 otherwise, and $q^0_r(p^2_s) = 1$ if $r = s$ and 0 otherwise.) Then $\partial 1^* = \Sigma_{x\in X}(x^{-1} - 1)q^1_x$ and $\partial q^1_x = \Sigma_{r\in R}\overline{r_x} q^0_r$. After our normalization of the presentation, each r_x is either 0 or in $F(X)$, for all $r \in R$ and $x \in X$, and so $r_x - 1 = \partial(\Sigma_{y\in X} \frac{\partial r_x}{\partial y} p^1_y)$.

Define homomorphisms $j_i : \overline{Q}_i \to (P_* \otimes \overline{Q}_*)_i$, for $i = 0,1,2$, by setting

$$j_0(q^0_r) = 1 \otimes q^0_r \quad \text{for} \quad r \in R,$$

$$j_1(q^1_x) = 1 \otimes q^1_x - \Sigma_{r,y}\overline{r_x}(\frac{\partial r_x}{\partial y} p^1_y \otimes q^0_r) \quad \text{for} \quad x \in X, \quad \text{and}$$

$$j_2(1^*) = 1\otimes 1^* - \Sigma_{x\in X}x^{-1}(p^1_x\otimes q^1_x) - \Sigma_{r\in R}(p^2_r\otimes q^0_r).$$

Then

$$\partial j_1(q^1_x) - j_0(\partial q^1_x) = \Sigma_{r\in R}(1\otimes \overline{r_x}q^0_r) - \Sigma_{r,y}\overline{r_x}(\frac{\partial r_x}{\partial y}(y-1)\otimes q^0_r) - \Sigma_{r\in R}\overline{r_x}(1\otimes q^0_r)$$

$$= \Sigma_{r\in R}[(1\otimes \overline{r_x}q^0_r) - \overline{r_x}((r_x-1)\otimes q^0_r) - \overline{r_x}(1\otimes q^0_r)] = 0,$$

and so $\partial j_1 = j_0\partial$. Similarly,

$$\partial j_2(1^*) - j_1(\partial 1^*) = \Sigma_x[1\otimes (x^{-1}-1)q^1_x - x^{-1}((x-1)\otimes q^1_x)]+$$

$$\Sigma_x\Sigma_r[(x^{-1}(p^1_x\otimes \overline{r_x}q^0_r) - r_x p^1_x\otimes q^0_r)] - \Sigma_x(x^{-1}-1)[1\otimes q^1_x - \Sigma_{r,y}\overline{r_x}(\frac{\partial r_x}{\partial y}p^1_y\otimes q^0_r)]$$

$$= \Sigma_{r,x}[x^{-1}(p^1_x\otimes \overline{r_x}q^0_r) - r_x p^1_x\otimes q^0_r + \Sigma_y(x^{-1}-1)\overline{r_x}(\frac{\partial r_x}{\partial y}p^1_y\otimes q^0_r)].$$

It shall clearly suffice to show that the summand corresponding to each relator r is 0. After our normalization of the presentation, we may assume that $r = x_1\dots x_m$ for some distinct $x_1,\dots,x_m\in X$. Let $r_i = r_{x_i}$, for $1\le i\le m$. Then $r_i = x_1\dots x_{i-1}$, for $1\le i\le m$, so $r_i x_i = r_{i+1}$ if $i<m$ and $r_m x_m = r = 1$ in G. Moreover, $\frac{\partial r_i}{\partial y} = r_j$ if $y = x_j$, for some $1\le j<i$, and is 0 otherwise. Let $S_{i,j} = r_i^{-1}(r_j p^1_{x_j}\otimes q^0_r)$, for $1\le j\le i\le m$. Then $x_m^{-1}S_{m,j} = S_{1,j}$, for all $j\le m$, and so the summand corresponding to the relator r in $\partial j_2(1^*) - j_1(\partial 1^*)$ is

$$\Sigma_{i\le m}(x_i^{-1}S_{i,i} - S_{1,i} + \Sigma_{j<i}(x_i^{-1}S_{i,j} - S_{i,j}))$$

$$= \Sigma_{i<m}(S_{i+1,i} - S_{1,i}) + \Sigma_{i\le m}\Sigma_{j<i}(S_{i+1,j} - S_{i,j}).$$

This sum collapses to 0, and so $\partial j_2 = j_1\partial$. Thus j_* is a chain homomorphism. Since $\overline{Q}_*$ and $P_*\otimes\overline{Q}_*$ are resolutions of $\mathbb{Z}$ and j_* induces the identity on $\mathbb{Z}$, it is a chain homotopy equivalence.

We then have

$$(A_{P_2\overline{\mathscr{D}}}(q^0_s\otimes_\Gamma \delta)\otimes\eta)(j_*(1^*)) = -\Sigma_{r\in R}(q^0_s(p^2_r)\delta\otimes_\Gamma \eta(q^0_r)),$$

which has image $-\delta\otimes_\Gamma\eta(q^0_s)$ in $\mathscr{D}\otimes_\Gamma\overline{\mathscr{D}}$. Let τ be the ($\mathbb{Z}$-linear) involution of $H^2(G;\overline{\mathscr{D}})$ given by $\tau(\alpha_{\overline{\mathscr{D}}}(\rho\otimes_\Gamma\alpha)) = \alpha_{\overline{\mathscr{D}}}(\alpha\otimes_\Gamma\rho)$. Then

$$[\xi]\smile id_{\overline{\mathscr{D}}} = -\theta_{\overline{\mathscr{D}}}(\tau([\xi]))\quad for\quad \xi\in H^2(G;\overline{\mathscr{D}}),$$

and so $c^2_{G,w}$ is an isomorphism.

If $C(\mathscr{P})$ is not aspherical we modify the definition of the dual complex Q_* by setting $Q_1 = Hom_\Gamma(P_1,\Gamma)\oplus Hom_\Gamma(P_3,\Gamma)$ and extending the differential by $s^\dagger$, where $s\partial_3 = id_{P_3}$. Let $f: P_3^\dagger\to\Gamma^s$ be a split monomorphism, with left inverse $g:\Gamma^s\to P_3^\dagger$.

Fix a basis $\{e_1,\dots,e_s\}$ for Γ^s, and define a homomorphism $h:\Gamma\to\Gamma\otimes\Gamma^s$ by $h(e_i)=1\otimes e_i$. Then we may extend j_1 by setting $j_1=(1\otimes g)hf$ on $P_3^\dagger$.

In [40] we gave closed formulae for $j_2(1^*)$ for some simple (un-normalized) presentations of groups of particular interest. We should have also given the appropriate form of j_1 explicitly, for there we used the relators to simplify the derivatives r_x, which in general are sums of monomials $\Sigma_k \pm r_{xk}$, and such simplifications affect the second derivatives $\frac{\partial r_{xk}}{\partial y}$. It is safer to calculate such derivatives in $\mathbb{Z}[F(X)]$ *before* using the relators to simplify their images in Γ.

Similar formulae show that $c^1_{F,w}$ is an isomorphism for F free of finite rank $r\geq 1$.

18 Orbits of the k-invariant

In this section we shall attempt to extend the argument sketched in §15 above for the case of PD_2-groups to other finitely presentable, 2-dimensional duality groups. The hypothesis on 2-torsion in Theorem 27 below seems necessary for our argument, but does not hold in some cases where the result is known by other means.

Lemma 18. *Let π be a finitely presentable group such that $c.d.\pi=2$, and let $w:\pi\to\mathbb{Z}^\times$ be a homomorphism. Let $\Pi=E^2\mathbb{Z}$. Then there is an exact sequence*

$$\Pi\odot_\pi\Pi\to\mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]}\Gamma_W(\Pi)\to H^2(\pi;\mathbb{F}_2)\to 0.$$

If $\Pi\odot_\pi\Pi$ is 2-torsion free this sequence is short exact. If, moreover, for every $x\in\Pi$ either $x\in(2,I_w)\Pi$ or $x\odot x\notin(2,I_w)(\Pi\odot\Pi)$ then $\mathbb{Z}^w\otimes_{\mathbb{Z}[\Pi]}\Gamma_W(\pi)$ is 2-torsion free.

Proof. Since Π is torsion free as an abelian group. it is a direct limit of free abelian groups, and so the natural map from $\Pi\odot\Pi$ to $\Gamma_W(\Pi)$ is injective. Applying $\mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]}-$ to the exact sequence

$$0\to\Pi\odot\Pi\xrightarrow{s}\Gamma_W(\Pi)\xrightarrow{q_\Pi}\Pi/2\Pi\to 0.$$

gives the above sequence, since $\mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]}\Pi/2\Pi\cong\Pi/(2,I_w)\Pi\cong H^2(\pi;\mathbb{F}_2)$. The kernel on the left in this sequence is the image of $Tor_1^{\mathbb{Z}[\pi]}(\mathbb{Z}^w,\Pi/2\Pi)$, which is a 2-torsion group.

If $\Pi\odot_\pi\Pi$ is 2-torsion free this sequence is short exact, and nontrivial 2-torsion in $\mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]}\Gamma_W(\Pi)$ has nontrivial image in $\Pi/(2,I_w)\Pi$. If there is such torsion there are $x,y_i,z_i\in\Pi$ such that $x\notin(2,I_w)\Pi$ but $2[\gamma_\Pi(x)+s(\Sigma y_i\odot z_i)]=0$ in $\Pi\odot_\pi\Pi$. Since $2\gamma_\Pi(x)=s(x\odot x)$ in $\Gamma_W(\Pi)$, we then have $s(x\odot x)\equiv 2(-s(\Sigma y_i\odot z_i))$ *mod* $I_w(\Pi\odot\Pi)$, and so $x\odot x\in(2,I_w)(\Pi\odot\Pi)$.

The final condition in the lemma depends only on the image of x in $\Pi/(2,I_w)\Pi$.

Let X be a PD_4-complex with $\pi_1(X)=\pi$ and $\pi_2(X)=\Pi$, and let $L=L_\pi(\Pi,2)$. Then $\widetilde{L}\simeq K(\Pi,2)$, and so it follows from the Whitehead sequence that $H_3(\widetilde{L};\mathbb{Z})=0$

and $H_4(\widetilde{L};\mathbb{Z}) \cong \Gamma_W(\Pi)$. Let $\mathscr{A}$ be a left $\mathbb{Z}[\pi]$-module. Since π is a 2-dimensional duality group with dualizing module $\overline{\Pi}$, Lemma 16 gives canonical isomorphisms

$$H^2(\pi;\mathscr{A}) = Ext^2_{\mathbb{Z}[\pi]}(\mathbb{Z},\mathscr{A}) \cong \overline{\Pi} \otimes_{\mathbb{Z}[\pi]} \mathscr{A}$$

and

$$Ext^2_{\mathbb{Z}[\pi]}(\Pi,\mathscr{A}) = \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \mathscr{A}.$$

Moreover, $H^2(\widetilde{L};\mathscr{A}) = Hom_{\mathbb{Z}}(\Pi,\mathscr{A})$ and $H^4(\widetilde{L};\mathscr{A}) = Hom_{\mathbb{Z}}(\Gamma_W(\Pi),\mathscr{A})$. Hence the spectral sequence for the universal covering $p_L : \widetilde{L} \to L$ gives exact sequences

$$0 \to H^2(\pi;\mathscr{A}) \to H^2(L;\mathscr{A}) \to Hom_{\mathbb{Z}[\pi]}(\Pi,\mathscr{A}) \to 0$$

(split by the homomorphism $H^2(\sigma;\mathscr{A})$ induced by a section σ for c_L), and

$$0 \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \mathscr{A} \to H^4(L;\mathscr{A}) \xrightarrow{p_L^*} Hom_{\mathbb{Z}[\pi]}(\Gamma_W(\Pi),\mathscr{A}) \to 0.$$

The right hand homomorphisms are induced by p_L, in each case. Since $H_q(\widetilde{X};\mathbb{Z}) = 0$ for $q > 2$, the spectral sequence for $p_X : \widetilde{X} \to X$ gives an isomorphism $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \mathscr{A} = Ext^2_{\mathbb{Z}[\pi]}(\Pi,\mathscr{A})) \cong H^4(X;\mathscr{A})$, and so $f_{X,2}$ induces a (non-canonical?) splitting of the second of these sequences.

In the next theorem and subsequent comments p_L^* is used variously for the homomorphisms determined by $H^4(p_L;\Gamma_W(\Pi))$, $H^2(p_L;\Pi)$ and $H^4(p_L;\Pi/2\Pi)$.

Theorem 27. *Let π be a finitely presentable, 2-dimensional duality group, and let $w : \pi \to \mathbb{Z}^\times$ be a homomorphism. Let $\Pi = E^2\mathbb{Z}$. Assume that the image of $\Pi \odot_\pi \Pi$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ is 2-torsion free. Then the homotopy type of a minimal PD_4-complex Z with $(\pi_1(Z), w_1(Z)) \cong (\pi, w)$ is determined by its refined v_2-type.*

Proof. Let Z be a minimal PD_4-complex with $\pi_1(Z) \cong \pi$ and $w_1(Z) = c_Z^* w$. Then $\pi_2(Z) \cong \Pi$ and $\pi_3(Z) \cong \Gamma_W(\Pi)$, since π has one end, and the homotopy type of Z is determined by $k = k_2(Z) \in H^4(L;\Gamma_W(\Pi))$, where $\Pi = E^2\mathbb{Z}$ and $L = P_2(Z) = L_\pi(\Pi)$. This class is only well defined up to the actions of $Aut(\Gamma_W(\Pi))$ and $E_0(L)$. Since $p_L^* k = k_2(\widetilde{Z})$ is an automorphism (considered as an endomorphism of $\Gamma_W(\Pi)$), by part (1) of Theorem 23, we may assume that $p_L^* k = id_{\Gamma_W(\Pi)}$, after applying an automorphism of $\Gamma_W(\Pi)$. Now $E_0(L) \cong E_\pi(L) \rtimes Aut(\pi)$ and $E_\pi(L) \cong H^2(\pi;\Pi) \rtimes Aut(\Pi)$. (See §3 above). We shall consider the action of $Aut(\pi)$ in the final paragraph of the proof. Since $Aut(\Pi) = \{\pm 1\}$ acts trivially on $\Gamma_W(\Pi)$, the main task is to consider the action of $H^2(\pi;\Pi)$ on k. We shall show that this action is closely related to the cup product homomorphism $c^2_{\pi,w}$. Note also that since Z is minimal, $v_2(Z) = c_Z^* v$ for some $v \in H^2(\pi;\mathbb{F}_2)$, by Theorem 20, and $E_\pi(L)$ fixes classes induced from $K = K(\pi,1)$, such as $c_L^* v$.

Let $\phi \in H^2(\pi;\Pi)$ and let $s_\phi \in [K,L]_K$ and $h_\phi \in [L,L]_K$ be as defined in Lemma 4. Let $M = L_\pi(\Pi,3)$. Then $[M,M]_K = H^3(M;\Pi) \cong End(\Pi)$, since $c.d.\pi = 2$. Let $\overline{\Omega} : [M,M]_K \to [L,L]_K$ be the loop map. Let $g \in [M,M]_K$ have image $[g] = \pi_3(g) \in End(\Pi)$ and let $f = \overline{\Omega} g$. Then $\omega([g]) = f^* \iota_{\Pi,2}$ defines a homomorphism

$\omega : End(\Pi) \to H^2(L;\Pi)$ such that $p_L^*\omega([g]) = [g]$ for all $[g] \in End(\Pi)$. Moreover $f\mu = \mu(f,f)$, since $f = \overline{\Omega}g$, and so $fh_\phi = \mu(fs_\phi c_L, f)$. Hence

$$h_\phi^*\xi = \xi + c_L^* s_\phi^* \xi$$

for $\xi = \omega([g]) = f^*\iota_{\Pi,2}$. Naturality of the isomorphisms $H^2(X;\mathscr{A}) \cong [X, L_\pi(\mathscr{A},2)]_K$ for X a space over K and $\mathscr{A}$ a left $\mathbb{Z}[\pi]$-module implies that

$$s_\phi^*\omega([g]) = [g]_\# s_\phi^* \iota_{\Pi,2} = [g]_\#\phi$$

for all $\phi \in H^2(\pi;\Pi)$ and $g \in [M,M]_K$. (See [2, Chapter 5.§4].)

Using our present hypotheses, the exact sequences above give sequences

$$0 \to H^2(\pi;\Pi) \xrightarrow{c_L^*} H^2(L;\Pi) \xrightarrow{p_L^*} End(\Pi) \to 0$$

(split by ω and the homomorphism $H^2(\sigma;\Pi)$ induced by a section σ for c_L), and

$$0 \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi) \to H^4(L;\Gamma_W(\Pi)) \xrightarrow{p_L^*} End(\Gamma_W(\Pi)) \to 0.$$

We shall identify the modules on the left with their images, to simplify the notation.

If $u \in H^2(\pi;\Pi)$ then $h_\phi^*(u) = u$, since $c_L h_\phi = c_L$. The induced automorphism of the quotient $End(\Pi) = H^0(\pi;(H^2(\widetilde{L};\Pi))$ is also the identity, since the lifts of h_ϕ are (non-equivariantly) homotopic to the identity in $\widetilde{L}$. Hence there is a homomorphism

$$\delta_\phi : End(\Pi) \to H^2(\pi;\Pi)$$

such that $h_\phi^*(\xi) = \xi + c_L^*\delta_\phi(p_L^*\xi)$ for all $\xi \in H^2(L;\Pi)$. Since $p_L^* c_L^* = 0$ and $h_{\phi+\psi} = h_\phi h_\psi$ it follows that δ_ϕ is additive as a function of ϕ. Since π is a 2-dimensional duality group, $H^2(\pi;\Pi) \cong \overline{\Pi} \otimes_{\mathbb{Z}[\pi]} \Pi$, and so $\phi = \rho \otimes_\pi \alpha$ for some $\rho \in \overline{\Pi}$ and $\alpha \in \Pi$. If $g \in [M,M]_K$ then

$$\delta_\phi([g]) = \delta_\phi(p_L^*\omega([g])) = s_\phi^*\omega([g]) = \rho \otimes_\pi [g](\alpha). \tag{1}$$

In particular, $\delta_\phi(id_\Pi) = \phi$.

Similarly, the automorphism of $H^4(L;\Gamma_W(\Pi))$ induced by h_ϕ fixes the subgroup $G = \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$, and induces the identity on the quotient $End(\Gamma_W(\Pi)) = H^0(\pi;H^4(\widetilde{L};\Gamma_W(\Pi)))$. Then there is a homomorphism

$$f_\phi : H^4(L;\Gamma_W(\Pi)) \to G$$

such that $h_\phi^*(u) = u + f_\phi(u)$ for all $u \in H^4(L;\Gamma_W(\Pi))$, and such that $f_\phi|_G = 0$. Moreover, f_ϕ is additive as a function of ϕ, so we may define $\widehat{f} : H^2(\pi;\Pi) \to G$ by

$$\widehat{f}(\phi) = f_\phi(k), \quad \text{for all } \phi \in H^2(\pi;\Pi).$$

When $S = L$, $\mathscr{A} = \mathscr{B} = \Pi$, and $p = q = 2$ the construction of §15 gives a cup product pairing of $H^2(L;\Pi)$ with itself with values in $H^4(L;\Pi\otimes\Pi)$. Since $c.d.\pi = 2$ this pairing is trivial on the image of $H^2(\pi;\Pi)\otimes H^2(\pi;\Pi)$. The maps c_L and σ induce a splitting $H^2(L;\Pi)\cong H^2(\pi;\Pi)\oplus End(\Pi)$, and this pairing restricts to the cup product pairing of $H^2(\pi;\Pi)$ with $End(\Pi)$ with values in $Ext^2_{\mathbb{Z}[\pi]}(\Pi,\Pi\otimes\Pi)$. We may also compose with the natural homomorphisms from $\Pi\otimes\Pi$ to $\Pi\odot\Pi$ and $\Gamma_W(\Pi)$ to get pairings with values in $H^4(L;\Pi\odot\Pi)$ and $H^4(L;\Gamma_W(\Pi))$.

Since $h^*_\phi(\xi\smile\xi') = h^*_\phi\xi\smile h^*_\phi\xi'$ we have also

$$f_\phi(\xi\smile\xi') = \delta_\phi(p^*_L\xi')\smile\xi + \delta_\phi(p^*_L\xi)\smile\xi', \tag{2}$$

for all $\xi,\xi'\in H^2(L;\Pi)$. On passing to $\widetilde{L}\simeq K(\Pi,2)$ we find that

$$p^*_L(\xi\smile\xi')(\gamma_\Pi(x)) = p^*_L\xi(x)\odot p^*_L\xi'(x), \tag{3}$$

for all $\xi,\xi'\in H^2(L;\Pi)$ and $x\in\Pi$. (To see this, note that the inclusion of x determines a map from $\mathbb{CP}^\infty$ to $K(\Pi,2)$, since $[\mathbb{CP}^\infty,K(\Pi,2)] = Hom(\mathbb{Z},\Pi)$. Hence we may use naturality of cup products to reduce to the case when $K(\Pi,2) = \mathbb{CP}^\infty$ and x is a generator of $\Pi = \mathbb{Z}$.)

Let P be the image of $\Pi\odot_\pi\Pi$ in G. Since $c^2_{\pi,w}$ is an isomorphism, by Theorem 26, the induced map $\widehat{c}: H^2(\pi;\Pi)\to P$ is an epimorphism. Let $e = \widehat{f}-\widehat{c}$.

If $\Xi = \lambda\smile\lambda$ with $p^*_L\lambda = id_\Pi$ then $p^*_L(\Xi)(\gamma_\Pi(x)) = x\odot x = 2\gamma_\Pi(x)$, for all $x\in\Pi$, by Equation (5), while $f_\phi(\Xi) = 2(\phi\smile\lambda) = 2\phi\smile id_\Pi$, by Equation (4) and by the triviality of the cup product on the image of $H^2(\pi;\Pi)\otimes H^2(\pi;\Pi)$. Hence

$$p^*_L(\Xi) = 2id_{\Gamma_W(\Pi)} \quad\text{and}\quad f_\phi(\Xi) = 2\widehat{c}(\phi).$$

Since $p^*_L k = id_{\Gamma_W(\Pi)}$, we have $p^*_L(2k-\Xi) = 0$, and so $2k-\Xi\in G$, by the exactness of sequence (2) above. Then

$$2e(\phi) = f_\phi(2k-\Xi) = 0,$$

since $f_\phi|_G = 0$. Hence e has image in the 2-torsion subgroup ${}_2G$.

We invoke the hypothesis on 2-torsion at this point. Since ${}_2G\cap P = 0$, it follows easily that $|\mathrm{Cok}(\widehat{f})|\le|G/P| = |H^2(\pi;\mathbb{F}_2)|$. As ϕ varies in $H^2(\pi;\Pi)$ the values of $h_\phi(k)$ sweep out a coset of $\mathrm{Im}(\widehat{f})$ in $k+G = (p^*_L)^{-1}(id_{\Gamma_W(\Pi)})$, and there are at most 2^β cosets, where $\beta = \beta_2(\pi;\mathbb{F}_2)$.

For each $v\in H^2(\pi;\mathbb{F}_2)$ there is a minimal PD_4-complex Z such that $v_2(Z) = c^*_Z v$, by Theorem 18. The group $Aut(\pi)$ acts on K and L through based self-homotopy equivalences, and hence acts on the classifying maps c_Z and $f_{Z,2}$ by composition. These actions induce actions on $H^2(\pi;\mathbb{F}_2)$ and Π, and hence on $H^4(L;\Gamma_W(\Pi))$. The association $k\mapsto v_2(Z)$ defines a $Aut(\pi)$-equivariant surjection from $(p^*_L)^{-1}(id_{\Gamma_W(\Pi)}) = k+G$ to $H^2(\pi;\mathbb{F}_2)$, which is constant on cosets of $\mathrm{Im}(\widehat{f})$, since $E_\pi(L)$ acts trivially on $H^2(\pi;\mathbb{F}_2)$. It follows that the refined v_2-type is a complete invariant for the homotopy types of such complexes.

If $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ is 2-torsion free then $\widehat{f} = \widehat{c}$ (since $e = 0$), and the argument can be simplified slightly.

The hypothesis on 2-torsion holds if π is a PD_2-group, for then $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi) \cong \mathbb{Z}$ if $w = 1$ and has order 2 otherwise. (Note that in this case $\Pi \cong \mathbb{Z}^u$, where $u = w + w_1(\pi)$. We do not assume here that $w = w_1(\pi)$!) It holds also if $\pi = Z*_m$ with $|m| > 1$, by Theorem 30 below. On the other hand, if $\pi = F(r) \times \mathbb{Z}$ and $w(t) = -1$, where $t \in \pi$ generates the central $\mathbb{Z}$ factor, then $\Pi \odot_\pi \Pi$ and $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ have exponent 2, since t acts through ± 1 on Π. If $r > 1$ these groups are not finitely generated, and so the hypothesis of Theorem 27 does not hold.

Corollary 28 *If $H^2(\pi;\mathbb{F}_2) = 0$ and $\Pi \odot_\pi \Pi$ is 2-torsion free there is an unique minimal PD_4-complex realizing (π, w).* □

Hence two PD_4-complexes X and Y with fundamental group π are homotopy equivalent if and only if $\lambda_X \cong \lambda_Y$ (i.e., there is an isomorphism $\theta : \pi_1(X) \cong \pi_1(Y)$ such that $w_1(X) = w_1(Y) \circ \theta$ and an isometry of the pairings, up to sign.)

The hypothesis $H^2(\pi;\mathbb{F}_2) = 0$ holds if π is the group of a link of 2-spheres in an homology 4-sphere, in particular, if it is a 2-knot group or is the fundamental group of an homology 4-sphere.

Corollary 29 *If $H^2(\pi;\mathbb{F}_2) = \mathbb{F}_2$ and the image of $\Pi \odot_\pi \Pi$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ is 2-torsion free there are two minimal PD_4-complexes realizing (π, w), distinguished by whether $v_2(X) = 0$ or not.* □

The work of [29] suggests that the refined v_2-type should be a complete homotopy invariant, without the technical hypothesis on 2-torsion or the restriction that π have one end. If, moreover, $g.d.\pi = 2$ then every such minimal PD_4-complex should be homotopy equivalent to a closed 4-manifold, by Theorem 18. This is so if π is a semidirect product $F(r) \rtimes \mathbb{Z}$ or a PD_2-group, by Theorems 17 and 25. Can the connection between k_2 and v_2 be made more explicit? The canonical epimorphism $q_\Pi : \Gamma_W(\Pi) \to \Pi/2\Pi$ determines a change of coefficients homomorphism $q_{\Pi\#}$ from sequence (2) above to the parallel sequence

$$0 \to H^2(\pi;\mathbb{F}_2) \to H^4(L;\Pi/2\Pi) \xrightarrow{p_L^*} Hom_{\mathbb{Z}[\pi]}(\Gamma_W(\Pi), \Pi/2\Pi) \to 0.$$

Thus $q_{\Pi\#}(k_2(Z))$ lies in the $H^2(\pi;\mathbb{F}_2)$-coset $(p_L^*)^{-1}(q_\Pi)$.

Does Theorem 24 have an analogue for other 2-dimensional duality groups? Let X and Z be PD_4-complexes with such a fundamental group π, with Z minimal, and such that $(c_X^*)^{-1} w_1(X) = (c_Z^*)^{-1} w_1(Z)$. Then $[X,Z]_K$ maps onto $[X, P_3(Z)]_K$, by cellular approximation, and hence onto $\{f \in [X,L]_K \mid f^* k_2(Z) = 0\}$. Can the condition $f^* k_2(Z) = 0$ be made more explicit? The map f corresponds to a class in $H^2(X;\Pi)$ and $H^4(X;\Gamma_W(\Pi)) \cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi))$, by Poincaré duality for X. Theorem 24 suggests that we should consider the image of $f^* k_2(Z)$ in $H^2(\pi;\mathbb{F}_2)$, under the epimorphism of Lemma 18. Apart from this, we must determine when such a map f has a degree-1 representative $g : X \to Z$.

19 Verifying the torsion condition for $Z*_m$

If π is a 2-dimensional duality group but not a PD_2-group then $\Pi = E^2\mathbb{Z}$ is finitely generated as a left $\mathbb{Z}[\pi]$-module, but is not finitely generated as an abelian group. The associated groups $\Pi \odot_\pi \Pi$ and $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi))$ are infinitely generated abelian groups with no natural module structure. In this section we shall investigate the 2-torsion condition.

We consider first groups which have a one-relator presentation $\mathscr{P} = \langle X \mid r\rangle$. It is well-known that if the relator r is not conjugate to a proper power then the associated 2-complex $C(\mathscr{P})$ is aspherical, and so $g.d.\pi \leq 2$. (See §§9-11 of Chapter III of [46], or [17].)

Lemma 19. *Let π be a group with a finite one-relator presentation $\langle X \mid r\rangle$ and $c.d.\pi = 2$, and let $w = 1$. Let $\Pi = E^2\mathbb{Z}$. Then*

$$\Pi \odot_\pi \Pi \cong \mathbb{Z}[\pi]/(U + \overline{\Delta}),$$

where Δ is the right ideal generated by the free derivatives $\frac{\partial r}{\partial x}$, for all $x \in X$, and U is the subgroup of $\mathbb{Z}[\pi]$ generated by $g - g^{-1}$, for all $g \in \pi$.

Proof. On dualizing the Fox-Lyndon resolution of $\mathbb{Z}$ associated to the presentation $\langle X \mid r\rangle$, we see that $H^2(\pi;\mathbb{Z}[\pi]) \cong \mathbb{Z}[\pi]/\Delta$, and so $\Pi \cong \mathbb{Z}[\pi]/\overline{\Delta}$.

Define a function $T : \mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi]$ by $T(s \otimes t) = \bar{s} \otimes t$, for all $s,t \in \mathbb{Z}[\pi]$. Then T is an additive bijection and $T(gs \otimes gt) = \bar{s}\bar{g} \otimes gt$, for all $g \in \pi$. Hence T induces an additive isomorphism from the quotient of $\mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi]$ by the diagonal action of π to $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi] \cong \mathbb{Z}[\pi]$, which maps $s \otimes t$ to $\bar{s}t$. The images of $\mathbb{Z}[\pi] \otimes \overline{\Delta}$ and $\overline{\Delta} \otimes \mathbb{Z}[\pi]$ under T are $\overline{\Delta}$ and Δ, respectively. We obtain the symmetric product $\mathbb{Z}[\pi] \odot \mathbb{Z}[\pi]$ by factoring out the tensor square $\mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi]$ by all sums of terms of the form $s \otimes t - t \otimes s$. The image of all such sums in $\mathbb{Z}[\pi]$ is the subgroup U. (Note that U is not usually an ideal!) Since $\mathbb{Z}[\pi] \odot_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi] \cong \mathbb{Z}[\pi]/U$ and $U + \overline{\Delta} = U + \Delta$, we see that $\Pi \odot_\pi \Pi \cong \mathbb{Z}[\pi]/(U + \overline{\Delta})$.

This may be extended to other 2-dimensional duality groups as follows. Suppose that P is an $a \times b$ presentation matrix for Π. View $\mathbb{Z}[\pi]^b$ as a module of row vectors, with standard basis $\{e_1, \dots, e_b\}$. Define a function $T : \mathbb{Z}[\pi]^b \otimes \mathbb{Z}[\pi]^b \to M_b(\mathbb{Z}[\pi])$ by $T(se_i \otimes te_j) = \bar{s}te_{ij}$, the matrix with (i,j) entry $\bar{s}t$ and all other entries 0. Then $T(\mathbb{Z}[\pi]^b \otimes \mathrm{Im}(P))$ is $Row(P)$, the left ideal in $M_b(\mathbb{Z}[\pi])$ consisting of matrices with all rows in $\mathrm{Im}(P)$, while $T(\mathrm{Im}(P) \otimes \mathbb{Z}[\pi]^b)$ is the right ideal $Row(P)^\dagger$, the conjugate transpose of $Row(P)$. Let V be the subgroup generated by $M - M^\dagger$, for all M in $M_b(\mathbb{Z}[\pi])$. Then $\Pi \otimes_\pi \Pi \cong M_b(\mathbb{Z}[\pi])/(V + Row(P) + Row(P)^\dagger)$.

Suppose now that π is solvable. Then it is a Baumslag-Solitar group $\mathbb{Z}*_m$, with a one-relator presentation $\langle a,t \mid tat^{-1}a^{-m}\rangle$, for some $m \neq 0$ [26]. In this case we have a more explicit model for $\Pi \odot_\pi \Pi$.

Theorem 30. *Let $\pi = \mathbb{Z}*_m$ and let $w : \pi \to \mathbb{Z}^\times$ be a homomorphism. Let $\Pi = E^2\mathbb{Z}$. If $|m| > 1$ then $\Pi \odot_\pi \Pi$ is torsion free.*

Proof. We may assume that π has the presentation $\langle a,t \mid tat^{-1}a^{-m}\rangle$. Let $A = \langle\langle a\rangle\rangle$. Then $\pi \cong A \rtimes \mathbb{Z}$. Let $a_n = t^n a t^{-n}$ in A, for all $n \in \mathbb{Z}$, and let $a^x = a^k_{-n}$, for all $x = \frac{k}{m^n} \in \mathbb{Z}[\frac{1}{m}]$. Then $a^0 = 1$, $a^1 = a$ and $a^x a^y = a^{x+y}$ for all $x,y \in \mathbb{Z}[\frac{1}{m}]$, and $x \mapsto a^x$ determines an isomorphism from $\mathbb{Z}[\frac{1}{m}]$ to A. Every element of π is uniquely of the form $t^p a^x$, for some $p \in \mathbb{Z}$ and $x \in \mathbb{Z}[\frac{1}{m}]$, and $(t^p a^x)^{-1} = t^{-p} a^{-m^p x}$. If m is even then $w(a^x) = 1$ for all x; if m is odd then $w(a^x) = w(a^{mx})$ for all x.

The function which sends a_n to a_{n+1} determines an automorphism α of the commutative domain $D = \mathbb{Z}[A] \cong \mathbb{Z}[a_n | n \in \mathbb{Z}]/(a_{n+1} - a_n^m)$, and $\mathbb{Z}[\pi]$ is isomorphic to the twisted Laurent extension $D_\alpha[t,t^{-1}]$. (An explicit isomorphism is given by the function which sends $t^p a_n \in \oplus_{p\in\mathbb{Z}} t^p D$ to $t^{n+p} a t^{-n} \in \mathbb{Z}[\pi]$ for all $n,p \in \mathbb{Z}$.)

We shall assume henceforth that m is positive, for simplicity of notation. Let $J_0 = \{1,\dots,m-1\}$, let $J_s = \{\frac{d}{m^s} \mid 0 < d < m^{s+1},\ (d,m) = 1\}$, for all $s \geq 1$, and let $J = \cup_{s\geq 0} J_s$. Then $E = D/D(a^m - w(a)^m)$ is freely generated as an abelian group by the image of $\{a^x \mid x \in J\}$.

The images of the free derivatives of the relator $r = tat^{-1}a^{-m}$ in $\mathbb{Z}[\pi]$ are $\frac{\partial r}{\partial a} = t - \mu_m$, where $\mu_m = \Sigma_{i=0}^{i=m-1} a^i$, and $\frac{\partial r}{\partial t} = 1 - a^m$. Hence

$$\Pi \cong \mathbb{Z}[\pi]/\mathbb{Z}[\pi](a^m - w(a)^m, t\overline{\mu_m} - w(t)) \cong (\oplus_{k\in\mathbb{Z}} t^k E)/\sim,$$

where

$$t^k a^x \sim w(t) t^k a^x t \overline{\mu_m} = w(t) t^{k+1} a^{\frac{x}{m}} \overline{\mu_m}, \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in J.$$

As an abelian group, $\Pi \cong \varinjlim t^p E$, the direct limit as $p \to +\infty$ of the family of D-linear monomorphisms $\sigma : t^p E \to t^{p+1} E$ given by $\sigma(t^p a^x) = w(t) t^{p+1} a^{\frac{x}{m}} \overline{\mu_m}$, for all $p \in \mathbb{Z}$ and $x \in J$. It follows easily that

$$\Pi \odot \Pi \cong \varinjlim (t^k E \odot t^k E) = (\oplus_{p\in\mathbb{Z}} t^p E \odot t^p E)/\sim,$$

where $t^k a^x \odot t^k a^y \sim t^{k+1} a^{\frac{x}{m}} \overline{\mu_m} \odot t^{k+1} a^{\frac{y}{m}} \overline{\mu_m}$, for all $k \in \mathbb{Z}$ and $x,y \in J$.

Setting $z = y - x$ gives

$$t^k a^x (1 \odot a^z) \sim t^{k+1} a^{\frac{x}{m}} (\overline{\mu_m} \odot \overline{\mu_m} a^{\frac{z}{m}}).$$

(Here π acts diagonally on $\Pi \odot \Pi$.) We may expand the term in parentheses as

$$\overline{\mu_m} \odot \overline{\mu_m} a^{\frac{z}{m}} = \Sigma_{i,j=0}^{i,j=m-1} w(a)^i a^{-i} (1 \odot w(a)^{i-j} a^{i-j} a^{\frac{z}{m}}).$$

Define a function $f : E \to \Pi \odot \Pi$ by $f(e) = 1 \odot e = e \odot 1$ for $e \in E$. Then f is additive and $f(a^x) = w(a)^m a^x f(a^{m-x})$ for all x, since $a^x \odot 1 = a^x(1 \odot w(a)^m a^{m-x})$. The induced map from E to $\Pi \odot_\pi \Pi$ is onto, and

$$\Pi \odot_\pi \Pi \cong E/N,$$

where N is the subgroup generated by

$$\{a^z - w(a^{m-z})a^{m-z},\ a^z - w(t)m\Sigma_{k=0}^{m-1} w(a)^k a^{k+\frac{z}{m}},\ \forall z \in J\}.$$

Since $a^z - w(a^{m-z})a^{m-z} \in N$, the images $[a^z]$ of the elements a^z with $0 \leq z \leq \frac{m}{2}$ generate the quotient E/N. Given that $[a^z] = w(a)^{m-z}[a^{m-z}]$, the conditions

$$[a^z] = w(t)m\Sigma_{k=0}^{m-1} w(a)^k [a^{k+\frac{z}{m}}]$$

and

$$[a^{m-z}] = w(t)m\Sigma_{k=0}^{m-1} w(a)^k [a^{k+\frac{m-z}{m}}]$$

are equivalent.

Let F_s be the subgroup of $\Pi \odot_\pi \Pi$ generated by $\{[a^z] \mid m^{s-1}z \in \mathbb{Z}\}$, for $s \geq 1$. If $|m| > 1$ then the conditions $[a^z] = [w(t)m\Sigma_{k=0}^{m-1} w(a)^k [a^{k+\frac{z}{m}}]$ in E/N, for $z \in J$, imply that F_s is generated by $\{[a^0]\} \cup \{[a^z] \mid 0 < 2z \leq m,\ m^{s-1}z \in \mathbb{Z},\ m^{s-2}z \notin \mathbb{Z}\}$, for all $s \geq 1$, with a single relation of the form $(1 - w(t)m)[a^0] = m^s\sigma$, where σ is a sum of the generators $[a^z]$ with $z \in J_s$ such that $0 < 2z < m$, and coefficients not divisible by $(1-m)$. Hence F_s is torsion free, for all $s \geq 1$. Since $\Pi \odot_\pi \Pi$ is the increasing union $\cup_{s\geq 0} F_s$, it is also torsion free.

If $m = \pm 1$ and $w = 1$ then $\Pi \odot_\pi \Pi \cong \mathbb{Z}$. However, if $m = \pm 1$ and $w \neq 1$ then $\Pi \odot_\pi \Pi = \mathbb{Z}/2\mathbb{Z}$, and so the theorem does not extend to this case.

Note that the argument of the final paragraph implies that every generator of $\Pi \odot_\pi \Pi$ is m-divisible, and that $\Pi \odot_\pi \Pi$ is a free $\mathbb{Z}[\frac{1}{m}]$-module of infinite rank.

Corollary 31 *If $\pi = \mathbb{Z}*_m$ with $|m| > 1$ then $\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\Pi)$ is torsion free.*

Proof. If m is even this follows immediately from the theorem and the short exact sequence of Lemma 18, since $H^2(\pi;\mathbb{F}_2) = 0$ then. If m is odd we may apply the final part of Lemma 18. Letting x be the image of $1 \in \mathbb{Z}[\pi]$, we see that $\gamma_\Pi(x)$ generates $\Pi/(2, I_w)\Pi = H^2(\pi;\mathbb{F}_2)$, while the image of $f(1) = x \odot x$ in $\Pi \otimes_\pi \Pi$ is not 2-divisible.

It is not immediately obvious that the models for $\Pi \odot_\pi \Pi$ in Lemma 19 and Theorem 30 agree when $\pi \cong \mathbb{Z}*_m$. However (assuming for simplicity that $m \geq 1$ and $w = 1$), the relations

$$t^k a^x \sim_1 t^k a^x t \mu_m = t^{k+1} a^{\frac{x}{m}} \mu_m \quad \text{and} \quad t^k a^x \sim_2 (t^k a^x)^{-1} = t^{-k} a^{-m^k x}$$

together imply that $\Pi \odot_\pi \Pi$ is generated by the image of E and that

$$a^z \ \sim_1\ t a^{\frac{z}{m}} \mu_m = \Sigma_{i=0}^{i=m-1} t a^i a^{\frac{z}{m}} \ \sim_2\ \Sigma_{i=0}^{i=m-1} t^{-1} a^{-mi-z} = mt^{-1}a^{-z}$$

$$\sim_1\ ma^{-\frac{z}{m}} \mu_m \ \sim_2\ m\Sigma_{i=0}^{i=m-1} a^{-i} a^{\frac{z}{m}} = ma^{\frac{z}{m}} \mu_m,$$

for all $z \in J$. This is enough to see that $\mathbb{Z}[\pi]/(U + \overline{\Delta})$ is a quotient of E/N, as an abelian group, when $\Delta = (a^m - 1, t - \mu_m)\mathbb{Z}[\pi]$.

Can we extend the argument of Theorem 30 in any way? In particular, does the hypothesis of Theorem 27 hold for ascending HNN extensions $F*_\varphi$ with base F

a finitely generated free group and φ an endomorphism such that $p \prec \varphi(p)$ for all $1 \prec p$ with respect to some left ordering $\prec$ on F? When φ is an automorphism π is a semidirect product $F(r) \rtimes_\varphi \mathbb{Z}$, and the result of Theorem 27 holds by Theorem 22. If φ has odd order and $w = 1$ then it can be shown that $\Pi \odot_\pi \Pi$ is 2-torsion free. However, as we have seen, the argument of Theorem 27 itself must be changed in order to accommodate other semidirect products $F(r) \rtimes_\varphi \mathbb{Z}$ and orientation characters w.

20 4-manifolds and 2-knots

In this section we shall invoke surgery arguments, and so "4-manifold" and "s-cobordism" shall mean TOP 4-manifold and (5-dimensional) TOP s-cobordism, respectively. We continue to assume that π is a 2-dimensional duality group.

Suppose that π is either the fundamental group of a finite graph of groups, with all vertex groups $\mathbb{Z}$, or is square root closed accessible, or is a classical knot group. (This includes all PD_2-groups, semidirect products $F(n) \rtimes \mathbb{Z}$ and the solvable groups $\mathbb{Z}*_m$.) Then $Wh(\pi) = 0$, $L_5(\pi, w)$ acts trivially on the s-cobordism structure set $S^s_{TOP}(M)$ and the surgery obstruction map $\sigma_4(M) : [M, G/TOP] \to L_4(\pi, w)$ is onto, for any closed 4-manifold M realizing (π, w). (See Lemma 6.9 and Theorem 17.8 of [34].)

If, moreover, $w_2(\widetilde{M}) = 0$ then every 4-manifold homotopy equivalent to M is s-cobordant to M, by Theorem 6.7, Lemma 6.5 and Lemma 6.9 of [Hi]. If $w_2(\widetilde{M}) \neq 0$ there are at most two s-cobordism classes of homotopy equivalences. After stabilization by connected sum with copies of $S^2 \times S^2$ there are two s-cobordism classes, distinguished by their KS smoothing invariants (see [43]).

If π is solvable then 5-dimensional s-cobordisms are products and stabilization is unnecessary, so homotopy equivalent 4-manifolds with fundamental group π are homeomorphic if the universal cover is Spin, and there are two homeomorphism types otherwise, distinguished by their KS invariants.

The Baumslag-Solitar group $\mathbb{Z}*_m$ has such a graph-of-groups structure and is solvable, so the 5-dimensional TOP s-cobordism theorem holds. Thus if m is even the closed orientable 4-manifold M with $\pi_1(M) \cong \mathbb{Z}*_m$ and $\chi(M) = 0$ is unique up to homeomorphism. If m is odd there are two such homeomorphism types, distinguished by whether $v_2(M) = 0$ or $v_2(M) \neq 0$.

Let π be a finitely presentable group with $c.d.\pi = 2$. If $H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong \mathbb{Z}$ and $H_2(\pi;\mathbb{Z}) = 0$ then $\mathrm{def}(\pi) = 1$ [34, Theorem 2.8]. If moreover π is the normal closure of a single element then $\pi \cong \pi K = \pi_1(S^4 \setminus K)$, for some 2-knot $K : S^2 \to S^4$. (If the Whitehead Conjecture is true every knot group of deficiency 1 has cohomological dimension at most 2.) Since π is torsion free it is indecomposable, by a theorem of Klyachko [44]. Hence π has one end.

Let $M = M(K)$ be the closed 4-manifold obtained by surgery on the 2-knot K. Then $\pi_1(M) \cong \pi = \pi K$ and $\chi(M) = \chi(\pi) = 0$, and so M is a minimal model for π. If K is reflexive it is determined by M and the orbit of its meridian under the automorphisms of π induced by self-homeomorphisms of M. If $\pi = F(n) \rtimes \mathbb{Z}$ the homotopy type of M is determined by π, as explained in §4 above. Since $H^2(M;\mathbb{F}_2) = 0$ it follows that M is s-cobordant to the fibred 4-manifold with $\#^n(S^2 \times S^1)$ and fundamental group π. Knots with Seifert surface a punctured sum $\#^n(S^2 \times S^1)_o$ are reflexive. Thus if K is fibred (and $c.d.\pi = 2$) it is determined (among all 2-knots) up to s-concordance and change of orientations by π together with the orbit of its meridian under the automorphisms of π induced by self-homeomorphisms of the corresponding fibred 4-manifold. (This class of 2-knots includes all Artin spins of fibred 1-knots. See §6 of [34, Chapter 17] for more on 2-knots with $c.d.\pi = 2$.)

A stronger result holds for the group $\pi = \mathbb{Z}*_2$. This is the group of Fox's Example 10, which is a ribbon 2-knot [23]. In this case π determines the homotopy type of $M(K)$, by Theorems 30 and 27. Since metabelian knot groups have an unique conjugacy class of normal generators (up to inversion) Fox's Example 10 is the unique 2-knot (up to TOP isotopy and reflection) with this group. (If K is any other nontrivial 2-knot such that πK is torsion free and elementary amenable then $M(K)$ is homeomorphic to an infrasolvmanifold. See [34, Chapters 16-18].)

Let $\Lambda = \mathbb{Z}[\mathbb{Z}]$. There is a hermitian pairing B on a finitely generated free Λ-module which is not extended from the integers, and a closed orientable 4-manifold M_B with $\pi_1(M) \cong \mathbb{Z}$ and such that the intersection pairing on $\pi_2(M_B)$ is equivalent to B. In particular, M_B is not the connected sum of $S^1 \times S^3$ with a 1-connected 4-manifold [30]. Let $N_B \subset M_B$ be an open regular neighbourhood of a loop representing a generator of $\pi_1(M_B)$. Suppose that X is a closed 4-manifold with fundamental group π and that there is an orientation preserving loop $\gamma \subset X$ whose image in π/π' generates a free direct summand. (For instance, there is such a loop if X is the total space of an S^2-bundle over an aspherical closed surface F with $\beta_1(F) > 1$). Then γ has a regular neighbourhood homeomorphic to N_B, and we may identify these regular neighbourhoods to obtain $N = M_B \cup_{S^1 \times D^3} X$. The inclusion of $\langle g \rangle$ into π and the projection of π onto Z mapping g to 1 determines a monomorphism $\gamma : \Lambda \to \mathbb{Z}[\pi]$ and a retraction $\rho : \mathbb{Z}[\pi] \to \Lambda$. In particular, $\Lambda \otimes_{\mathbb{Z}[\rho]} (\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\gamma]} B) \cong B$. It follows that as B is not extended from $\mathbb{Z}$ neither is $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\gamma]} B$. Therefore N is not the connected sum of E with a 1-connected 4-manifold.

21 Some questions

We shall collect here some of the questions that have arisen en route.

1. Are strongly minimal PD_4-complexes always of v_2-type II or III?
2. If X has v_2-type I and $c.d.\pi = 2$ is there a minimal model $f : X \to Z$ with $v_2(Z) = 0$?
3. Must a strongly minimal PD_4-complex with π a nontrivial free product be a connected sum?
4. Can we say more about PD_4-complexes with π infinitely ended and $\Pi = 0$?
5. Are there strongly minimal PD_4-complexes with $E^3\mathbb{Z} \neq 0$?
6. Do strongly minimal PD_4-complexes always have $k_1 = 0$?

7. If X is a PD_4-complex such that $\pi = \pi_1(X)$ has one end and $\Pi = \pi_2(X)$ is projective, must π be a PD_4-group?
8. To what extent do k_2 and v_2 determine each other?
9. In Theorem 23 must Y be a PD_4-complex?
10. Can we extend Theorems 27 and 30 to encompass the known results for π a semidirect product $F(r) \rtimes \mathbb{Z}$ (at least if $w = 1$)?
11. Can we relax the running hypothesis that π should have one end?

The final four questions are of most interest for the present work.

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