A slicing obstruction from the $10/8+4$ theorem

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Abstract Using the $10/8+4$ theorem of Hopkins, Lin, Shi, and Xu, we derive a smooth slicing obstruction for knots in the three-sphere using a spin 4-manifold whose boundary is 0–surgery on a knot. This improves upon the slicing obstruction bound by Vafaee and Donald that relies on Furuta's 10/8 theorem. We give an example where our obstruction is able to detect the smooth non-sliceness of a knot by using a spin 4-manifold for which the Donald-Vafaee slice obstruction fails.

1 Introduction

A knot in the three–sphere is smoothly slice if it bounds a disk that is smoothly embedded in the four–ball. Classical obstructions to sliceness include the Fox–Milnor condition [3] on the Alexander polynomial, the $\mathbb{Z}_2$–valued Arf invariant [14], and the Levine–Tristram signature [8, 15]. Furthermore, modern Floer homologies and Khovanov homology produce powerful sliceness obstructions. Heegaard Floer concordance invariants include τ of Ozsváth–Szabó [10], the $\{-1,0,+1\}$–valued invariant ε of Hom [6], the piecewise–linear function $\Upsilon(t)$ [11], the involutive Heegaard Floer homology concordance invariants $\overline{V_0}$ and $\underline{V_0}$ [5], as well as ϕ_i homomorphisms of [1]. Rasmussen [13] defined the s–invariant using Khovanov–Lee homology, and Piccirillo recently used the s–invariant to show that the Conway knot is not slice [12].

We study an obstruction to sliceness derived from handlebody theory. We call a four-manifold a two-handlebody if it can be obtained by attaching two-handles to a four-ball. In [2] Donald and Vafaee used Furuta's $10/8$ theorem [4] to obtain a slicing obstruction. This obstruction is able to detect nontrivial torsion elements

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in the concordance group as well as find topologically slice knots which are not smoothly slice.

We apply the recent $10/8+4$ theorem of Hopkins, Lin, Shi, and Xu [7], which improves on Furuta's inequality, to improve the Donald–Vafaee slicing obstruction.

Theorem 1. *Let $K \subset S^3$ be a smoothly slice knot and X be a spin two-handlebody with $\partial X \cong S^3_0(X)$. If $b_2(X) \neq 1$, 3, or 23, then*

$$b_2(X) \geq \frac{10}{8}|\sigma(X)|+5.$$

We will give an example in Proposition 1 of a knot K and a spin two-handlebody with boundary $S^3_0(K)$ where our obstruction detects the non-sliceness of K and the Donald-Vafaee slice obstruction fails using this spin 2-handlebody.

2 The slicing obstruction

In [2] Donald and Vafaee used Furuta's 10/8 theorem to obtain a slicing obstruction.

Theorem 2 ([2]). *Let $K \subset S^3$ be a smoothly slice knot and X be a spin 2-handlebody with $\partial X \cong S^3_0(X)$. Then either $b_2(X) = 1$ or*

$$4b_2(X) \geq 5|\sigma(X)|+12$$

Recently, Hopkins, Lin, Shi, and Xu have improved Furuta's theorem with the following 10/8+4 theorem.

Theorem 3 ([7]). *Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality*

$$b_2(M) \geq \frac{10}{8}|\sigma(M)|+4.$$

Using the above theorem, we prove Theorem 1.

Proof (Proof of Theorem 1). The proof is identical to the proof of [2, Theorem 1.1], except one applies the 10/8+4 theorem of [7] instead of Furuta's 10/8 theorem.

If K is smoothly slice, then $S^3_0(K)$ embeds smoothly in S^4 (see for example [9, Theorem 1.8]). The embedding splits S^4 into two spin 4-manifolds U and V with a common boundary $S^3_0(K)$. Since $S^3_0(K)$ has the same integral homology as $S^1 \times S^2$, the Mayer-Vietoris sequence shows that U and V have the same integral homology as $S^2 \times D^2$ and $S^1 \times D^3$, respectively.

Let X be a spin 2-handlebody with $\partial X \cong \partial V \cong S^3_0(K)$ (where $\cong$ denotes orientation-preserving diffeomorphism), and let $W = X \cup_{S^3_0(K)} -V^4$. We restrict the spin structure on X to the boundary $S^3_0(K)$ and extend this spin structure on $S^3_0(K)$ over the manifold V. Then W is spin since the spin structures of X and

V agree on the boundary and spin structures behave well with respect to gluing. By Novikov additivity, W has signature $\sigma(W)=\sigma(X)+\sigma(V)$. Since $\sigma(V)=0$, we have $\sigma(W)=\sigma(X)$. As in [2] we will show that $b_2(W)=b_2(X)-1$. The Euler characteristic satisfies $\chi(W)=\chi(X)=1+b_2(X)$, where the first equality uses $\chi(V)=\chi(S^3_0(K))=0$ and the second equality holds since X is a 2-handlebody. Since $H_1(W,X;\mathbb{Q})\cong H_1(V,Y;\mathbb{Q})=0$, it follows from the exact sequence for the pair (W,X) that $b_1(W)=b_3(W)=0$. Therefore, $b_2(W)=b_2(X)-1$.

If $b_2(X)\neq 1$, 3, or 23, then W cannot be homeomorphic to S^4, $S^2\times S^2$, or $K3$. The result follows by applying the Hopkins, Lin, Shi, and Xu theorem [7]. □

Remark 1. This improves upon the slicing obstruction by Donald and Vafaee (under some restrictions on the second Betti number of the spin 2-handlebody).

We give an example of a knot and a spin 4-manifold where one can apply our obstruction. Let K' be a knot that is the closure of the braid word

$$K'=(\sigma_{12}\sigma_{11}\cdots\sigma_1)^{12}(\sigma_7\sigma_8\cdots\sigma_{12})^{-7}(\sigma_1\sigma_2\cdots\sigma_{10})^{-11}b,$$

where

$$b=(\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_3\sigma_2)(\sigma_1)^{-2}(\sigma_3\sigma_4)^{-1}\sigma_5^{-2}.$$

See Figure 1. The knot K' is presented as a generalized twisted torus knot. It is the closure of a braid formed by taking a $(13,12)$ torus knot and then adding one negative full twist on seven strands, one negative full twist on eleven strands, and the braid b. As noted in [2] the obstruction from Theorem 1 is generally easier to apply to knots like this because they can be unknotted efficiently by blowing up to remove full twists.

Proposition 1. *The knot K' is not smoothly slice.*

Proof. Add a 0-framed 2-handle to ∂D^4 along K' and blow up three times as follows. Blow up once negatively across thirteen strands on the top, then blow up positively across seven and eleven strands indicated by the two boxes labeled with -1 in Figure 1. This gives a manifold with second Betti number 4 and signature 1. The characteristic link has one component, with framing $-13^2+7^2+11^2=1$. Four Reidemeister I moves immediately show that this knot is isotopic to the knot in Figure 5 of [2], which in turn is isotopic to the figure eight knot.

At this point, we follow the procedure that Donald and Vafaee use to show that the figure eight knot is not slice. They apply a sequence of blow-ups, blow-downs, and handleslides until the characteristic link is empty and then apply their slice obstruction. Starting with the 1-framed figure eight knot, the same sequence of blow-ups, blow-downs, and handleslides can be applied until the characteristic link is empty.

This procedure is shown in Figure 2 and detailed below.

1. First blow up negatively twice as indicated in Figure 2(B). This gives $b_2=6$ and $\sigma=-1$.
2. Slide one of the two blow up curves over the other, resulting in Figure 2(C).

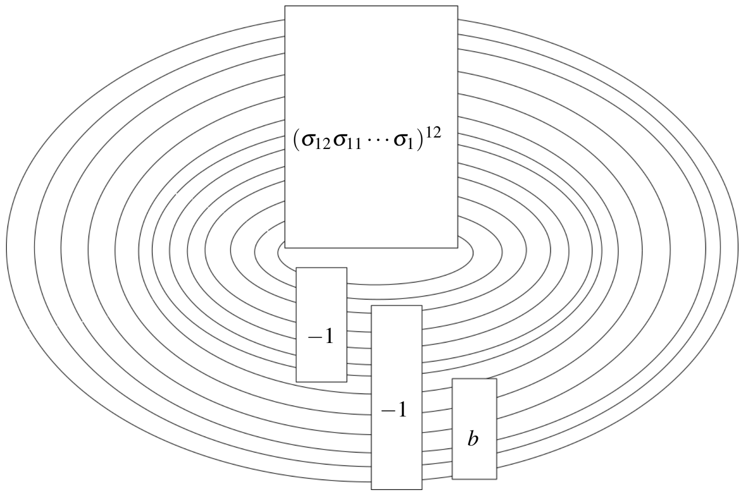

Fig. 1 The knot K' is a generalized twisted torus knot obtained from the torus knot $T_{13,12}$ by adding one negative full twist around seven adjacent strands, one negative full twist around eleven adjacent strands, and the braid b.

3. Figure 2(D) shows just the characteristic link, a split link whose components are a 1-framed trefoil and a -2-framed unknot. Blowing up negatively once around the three strands of the trefoil changes the characteristic link to a two-component unlink with framings -8 and -2 as in Figure 2(E). This is inside a four-manifold with $\sigma = -2$ and second Betti number $b_2 = 7$.
4. Positively blowing up the meridians eight times changes both framings in the characteristic link to -1 and gives $b_2 = 15$ and $\sigma = 6$.
5. Blow down negatively twice, resulting in an empty characteristic link.

The result is a spin 4-manifold X with boundary $S^3_0(K')$ with $b_2(X) = 13$ and $\sigma(X) = 8$. Thus, Theorem 1 concludes that K' is not smoothly slice. □

We observe that with the spin two-handlebody in the above proof, the Donald-Vafaee slice obstruction fails to detect the smooth non-sliceness of K'.

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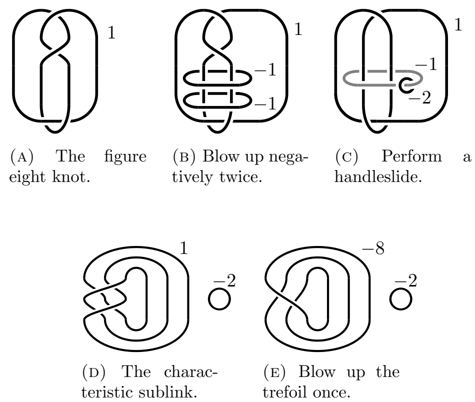

(A) The figure eight knot. (B) Blow up negatively twice. (C) Perform a handleslide.

(D) The characteristic sublink. (E) Blow up the trefoil once.

Fig. 2 A sequence of blow-ups and handleslides that shows $S^3_0(K')$ bounds a spin manifold with $b_2 = 13$ and $\sigma = 8$. These diagrams come from the figure eight example in [2] with different framing coefficients.

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