Some remarks on projective Anosov flows in hyperbolic 3-manifolds

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Abstract We explore some constructions of projectively Anosov flows on hyperbolic 3-manifolds that may lead to new ways to construct pairs of transverse taut contact forms and foliations.

1 Introduction

This (informal) note is to report some discussions that took place at the Matrix conference 'Dynamics, foliations and the geometry of 3-manifolds' during September 2018. We thought that some outcomes of the discussion could be relevant and seem to open several questions that we intend to pursue in the future. The discussions[1] revealed strong connections between the work of participants coming from very different fields so we thought it could be a good idea to advertise them. We warn the reader that the results and examples exposed here need to be expanded and revised carefully, we hope to do so in the near future.

There is a strong link between pairs of negative and positive contact structures and projectively Anosov flows. This was first noticed by Mitsumatsu [Mit, Mit_1] (see also [Asa] and references therein) who in particular used this as well as Eliashberg-Thurston [ET] approximating theorem to show that any 3-manifold ad-

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[1] Other people also participated in some of the discussions, we thank in particular Sergio Fenley for his comments.

mits a projectively Anosov flows given by the vector field of intersection of the two contact planes. The contact structures are open, and it follows from a result of Arroyo-Rodriguez Hertz ([ARH]) that generic projectively Anosov flows are hyperbolic (Axiom A with strong transversality). Recently, it has been shown that even C^0-foliations can be approached by contact structures (Bowden, Kazez-Roberts [Bow, KR]).

It is natural to ask when do these flows can be modified by isotopy to become Anosov flows. See e.g. the work of Gourmelon-Potrie ([GP]) where the same problem is considered in the case of projectively Anosov diffeomorphisms of surfaces. Of course, not every projectively Anosov flow can be deformed into an Anosov flow (e.g. the construction above gives projectively Anosov flows in the sphere S^3 which does not admit Anosov flows) but for example, if the 3-manifold is hyperbolic, this could in principle be the case. In fact, Thurston asked [Thu] if a hyperbolic 3-manifold admitting 3 transverse taut foliations should admit an Anosov flow.

Taut foliations give rise (by approximation) to tight contact structures. When one has a pair of transverse positive and negative contact forms, there is a notion of *tautness* (due to Colin-Firmo [CF]) which is adapted to the setting we are in. Tautness should be an essential hypothesis of the positive and negative contact structures giving rise to projectively Anosov flows that can be deformed into Anosov flows. In this note we provide some examples in the negative direction showing that certain pairs of transverse taut foliations cannot be approached by contact structures giving rise to projectively Anosov flows that can be deformed into Anosov (or pseudo-Anosov) flows within projectively Anosov flows.

The key point is the use of an adaptation of the *hyperbolic plugs* introduced in the paper of Beguin-Bonatti-Yu [BBY] to construct such examples. We end the note by rising several questions and future directions.

2 Projectively Anosov plugs

In this section we introduce projectively hyperbolic plugs à là Beguin-Bonatti-Yu [BBY] and explain some of their basic properties in certain situations.

We will then construct some solid torus plugs. These will be attracting plugs (i.e. the vector field points inward) and will have an associated foliation (it could have branching) in the boundary, associated to the weak stable direction. In the following sections we will glue them with some repelling plugs to obtain the examples we want to present.

As explained in the introduction, we want to have plugs with Morse-Smale dynamics and for which the (branching) foliations have no Reeb-components. The specific ones we construct are adapted to the examples we will present here, but clearly, it makes sense both to construct more as to make a theoretical study of which such plugs are possible and understand obstructions[2].

2.1 Definition of projectively hyperbolic plug

Important concept is that of *projectively hyperbolic Reebless plug*: I.e. the pA dynamics has no compact leafs in the branching foliation.

We will work under the simplifying assumption that our plugs are either attracting or repelling. This simplifies immensely many technical work of [BBY] but one can of course expect to extend some of these definitions to more general settings. Similarly, we will make some assumptions on the dynamics in the maximal invariant subset of the plug which are convenient for our purposes, but of course one can imagine extending these definitions to other settings.

An *attracting projectively hyperbolic Reebless plug* (V,X) will be a compact (not necessarily connected) 3-manifold V with boundary ∂V and a vector field X pointing inwards on ∂V so that:

- The maximal invariant set $\Lambda = \bigcap_{t>0} X_t(V)$ is projectively hyperbolic, i.e. the differential DX_t preserves two continuous invariant two dimensional bundles E and F defined on Λ so that for every vector v not in E it follows that the angle between $DX_t v$ and F decreases exponentially (see [Asa, ARH]).
- There are no compact boundaryless invariant surfaces in Λ.
- The dynamics in Λ is *Axiom A*.

Remark 1. In the case that the plug contains no attracting periodic orbits, the assumptions imply that the maximal invariant set is a lamination by C^1-surfaces.

If (V,X) is an attracting projectively hyperbolic Reebless plug, it provides extra information on how the E direction intersects the boundary ∂V of V. Notice that the bundles E and F are only defined on the maximal invariant set. However, using cone-fields, one sees that E extends in a unique way on the whole plug as an invariant bundle which contains the direction of the vector field X. In particular E is transverse to the boundary of the plug, and induces by intersection a (non-sigular) 1-dimensional bundle on ∂V. As a consequence (if the manifold is oriented) each boundary component is a torus $\mathbb{T}^2$. What we want the plug to verify:

- The maximal invariant set is projectively hyperbolic and Axiom A. In particular, a projectively hyperbolic tight plug on the solid torus has Morse-Smale dynamics.
- Other than that, definitions are the same as in [BBY].

[2] An easy obstruction is that an attracting plug cannot have the foliation defined in the boundary having no Reeb annuli, as this would allow a contractible loop transverse to the weak stable foliation, giving rise to a Reeb component in the solid torus.

One should state a result analogous to [BBY] which in the attractor/repelling setting goes back to Franks and Williams:

Theorem 1. *Let (V,X) an attracting projectively hyperbolic plug and with boundary components $\partial_{in}V = S_1 \cup \ldots \cup S_k$ and (W,Y) a repelling projectively hyperbolic plug with boundary components $\partial_{out}W = \hat{S}_1 \cup \ldots \cup \hat{S}_k$. Assume there are diffeomorphisms $\varphi_i : S_i \to \hat{S}_i$ so that the image of the induced foliation of S_i by X is mapped transverse to the induced foliation on $\hat{S}_i$ by φ. Then, one can put a differentiable structure in $M = V \sqcup W/_{\{\varphi_i\}}$ so that the flow X and Y glue well and induce a projectively Anosov flow on M.*

Proof (Idea of the proof). The fact that the gluing can be made a smooth manifold and induce a flow is standard (see [BBY] for details). Also, in this setting, showing that the resulting flow is projectively Anosov is rather easy as a cone-field criteria suffice (when the pieces are not attracting and repelling this becomes more subtle, see [BBY]).

2.2 A plug with six Reeb components

Consider a vector field X in the disk $\mathbb{D}$ as in figure 1.

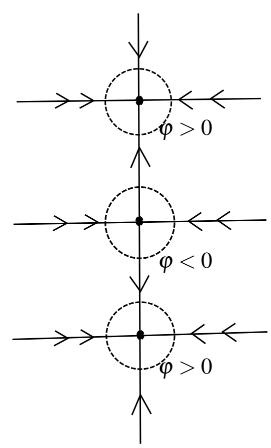

Fig. 1 A vector field in the disk

To get a vector field in the solid torus, multiply $\mathbb{D} \times S^1$ and consider the vector field

$$Y(x,y,\theta) = X(x,y) + \varphi \frac{\partial}{\partial \theta},$$

where φ is a smooth bump function also as in the figure 1. Transversally, (in coordinates (y,θ) of the plane $x=0$) the flow looks exactly like in figure 2. It is clear that since the contraction in the direction $\frac{\partial}{\partial x}$ is stronger than in the $\frac{\partial}{\partial y}$ direction, this flow is projectively hyperbolic.

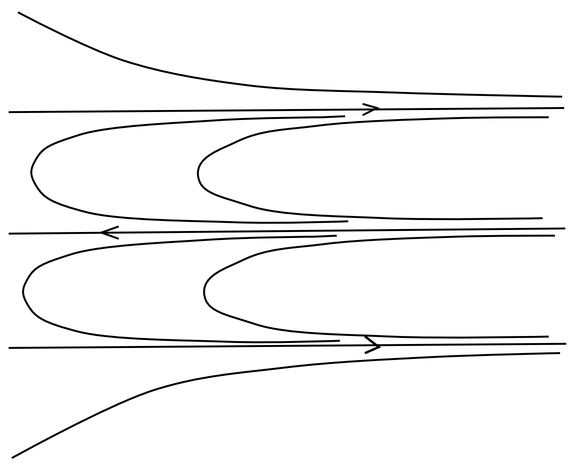

Fig. 2 The flow in the $x = 0$ plane

The maximal invariant set in the solid torus is a band with three periodic orbits, two attracting ones flowing in one direction and one saddle in the middle (flowing in the oposite direction). This produces two Reeb annuli of orbits that, when pushed along the weak stable direction produce four Reeb-annuli in the entering torus. The upper and lower attracting parts of the attracting sinks produce other two Reeb annuli for the weak stable foliation intersected with the entering torus. In total, the weak stable foliation intersected with the entering tours has six Reeb annuli, all oriented in the same direction.

Remark 2. This can be easily extended to create projectively hyperbolic plugs with $6+4n$ (with $n \geq 0$) Reeb annuli in the entering tours oriented in the same direction. Other variations are also possible (see section 6 for further discussions).

2.3 A plug compatible with an incoherent repeller

Here we'll present a projectively hyperbolic Reebless plug on the solid torus that will be possible to glue to the examples in Section 9 of [BBY] (see in particular Theorem 1.10 in [BBY] and the proof of Lemma 9.8 with its Figure 17).

The notion of *incoherent repeller* (or attractor) was introduced by Christy [Ch] (see [BBY, Subsection 1.4.2]) and corresponds to a certain configuration induced in the boundary of a neighborhood of a hyperbolic repeller that forbids the existence of Birkhoff sections. The configuration is depicted in figure 3.

For simplicity we just construct one specific example whose entry and exit torus look like in figure 3. Notice that the attracting solid torus with that entrance foliation can be made similarly to the zipped tori constructed in [BBY, Example 7.13].

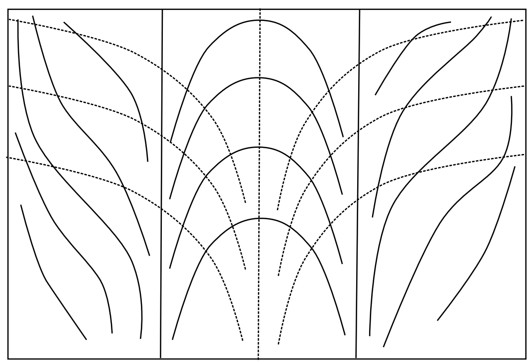

Fig. 3 Solid lines correspond to the incoherent repeller and in dashed ones represent the foliation in the attracting solid torus.

3 Filling a pseudo-Anosov flow

Here we show how to construct a projectively Anosov flow in a hyperbolic 3-manifold by making a DA-construction in a suspension pseudo-Anosov flow and including the plug constructed in subsection 2.2.

First, notice that there exists a pseudo Anosov homeomorphism of a genus two surface admitting a unique singular point, which by necessity needs to be a 6-prong saddle (see for example [FM], in particular figure 11.6 and the criteria in Theorem 14.4).

Gluing such plug with the one constructed in subsection 2.2 according to the conditions of Theorem 1 gives the desired projectively Anosov flow.

Now we provide some arguments for the tautness of the foliations induced by the bundles of the projectively Anosov splitting (a different approach is presented in section 5). The arguments are not symmetric for the E and F direction, particularly because by construction, the bundle E contains a uniformly contracting subbundle, and therefore it is uniquely integrable and its integral surfaces are planes or cylinders (depending on whether it contains a closed orbit or not). This implies that the foliation tangent to E is taut. For F one needs to be a bit more careful. We will only show a weaker form of tautness, through every surface tangent to F there is a closed transversal intersecting the surface, but certainly one should be able to push the arguments to get a stronger version (for the subtleties with the definitions, we refer the reader to [KR_2]).

First notice that F contains an expanding direction, and is therefore uniquely integrable everywhere except at the attracting points (c.f. Figure 4) in the maximal invariant set of the solid torus attracting plug (which is an invariant annulus). This implies that if there is a surface tangent to F which is closed, it has to be a torus and intersect at least one of the two attracting periodic orbits[3]. Notice that in any case, every surface tangent to F cannot be completely contained in the solid torus, so it accumulates somewhere in the repeller. This implies that the surface has some recurrence and this allows to construct a closed transversal through it. This completes the sketch of the proof.

4 Filling incoherent repellers

This follows by gluing the plug in subsection 2.3 with an example given by Theorem 1.10 (see in particular Figure 17 in the proof of Lemma 9.8 of [BBY]). Similarly to the previous section, applying Theorems 1, one sees that one obtains a projectively Anosov flow.

We now argue for the tautness of the bundles. Tautness of E is simpler as again it contains a uniformly contracting subbundle. The F direction can also be handled similarly to the case of the previous example using the fact that every surface tangent to F will accumulate in the repeller which is an essential lamination.

It remains to show that this can be done in a hyperbolic 3-manifold. We sketch an argument showing that this should be possible (but details should be checked more carefully).

The example is made with some atoroidal pieces and some seifert pieces of the form pair of pants times S^1 so that the flow is 'horizontal'. As the attractor is transitive, there is a periodic orbit which 'fills' every seifert piece. Now, doing Dehn-Goodman surgery along the periodic orbit carefully one should be able to obtain a

[3] In this case, as the manifold is hyperbolic this implies that it must bound a solid torus, therefore, there should be a Reeb component in the attracting plug which does not have one by assumption.

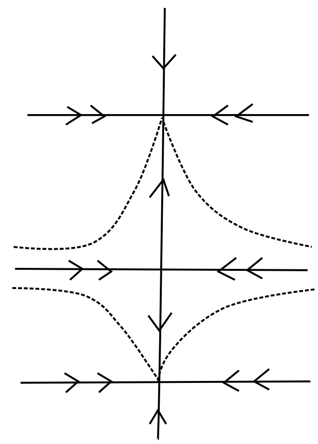

Fig. 4 The F direction is only defined in the maximal invariant set before gluing. After gluing it is shown in dotted lines how it make look in a transversal, in particular, it may have (and indeed has) merging points in the attracting periodic orbits.

hyperbolic 3-manifold (in particular, most surgeries in the atoroidal parts will remain atoroidal and any dehn-surgery in a filling curve in the seifert part will make the resulting manifold atoroidal, notice also that the periodic orbit should traverse all the torus of the JSJ decomposition thus kill them after Dehn surgery, see Foulon-Hasseblatt [FH, Appendix] for similar arguments).

Notice that Dehn-Goodman surgery does not affect the bundles much outside the closed orbit where it is performed and therefore tautness remains unchanged by this procedure.

The bundles need not be orientable for this construction, but this can be achieved by taking a finite cover (obviously, being a hyperbolic manifold is stable under taking finite lifts).

5 A foliation/contact interpretation of the examples

The example given by filling a DA blow up of a suspension pseudo-Anosov flow given in section 3, has the interesting property that it gives a pair of transverse foliations that are without compact leaves, and are hence taut. Moreover, it gives a template for building these purely in terms of filling the blown-up stable and unstable laminations of the suspension flow by configurations of monkey saddles.

Blow-up of laminations

Blowing up the singular leaves of the stable/unstable laminations gives a pair or transverse laminations $\widehat{\Lambda}_s, \widehat{\Lambda}_u$. These correspond to the stable respectively unstable laminations obtained after doing attracting resp. repelling DA blow-ups at all singular orbits.

Filling by monkey saddles

The complements of the laminations $\widehat{\Lambda}_s, \widehat{\Lambda}_u$ consist of ideal polygon bundles, where the number of sides of the polygon corresponds to the number of prongs of the corresponding singular orbit. These can be filled by monkey saddle foliations consisting of simply connected leaves assuming that the number of sides is even. However, if this is done in the standard way the resulting foliation will not even have trivial Euler class and cannot be associated to any flow with dominated splitting.

We assume that the number of sides is 6 as in the examples above. Then we add 3 cylindrical leaves whose ends connect different ends of a complementary bundle region. Taking the complement of these leaves, we obtain 4 complementary ideal polygon regions, two of which have two sides and two of which have 4 sides. Filling each of these with monkey saddles in such a way that each of the leaves are stable in that they have attracting resp. holonomy, we obtain the foliation from the example in section 3.

Pairs of complementary foliations

As the flow the flow given in Section 3 has no repelling closed orbits and the dynamics of are Axiom A the stable invariant plane field E is actually integrable. The corresponding foliation $\widehat{\mathscr{F}_u}$ is obtained by the splitting and filling operation above applied to the lamination $\widehat{\Lambda}_u$ without closed leaves.

It is not hard to see that there is a branching foliation (c.f. [BI]) tangent to the unstable invariant plane field F that branches precisely at the two attracting orbits (recall figure 4). Pulling these leaves apart we thus obtain a complementary foliation $\widehat{\mathscr{F}'_u}$, one of whose leaves corresponds to the invariant annulus described in Figure 2. One has an analogous construction of $\widehat{\mathscr{F}_s}$ and complementary foliation $\widehat{\mathscr{F}'_s}$. Both of these foliations are without compact leaves due to the dynamics of the flow: Any compact leaf would have to be a torus and there are no invariant tori.

These foliations can then be approximated by contact structures which are then transverse and universally tight cf. [Bow], [KR], giving a tight dominated splitting. It is not clear what this new flow has to do with the one given in Section 3, however under some genericity they ought to be semi-conjugate.

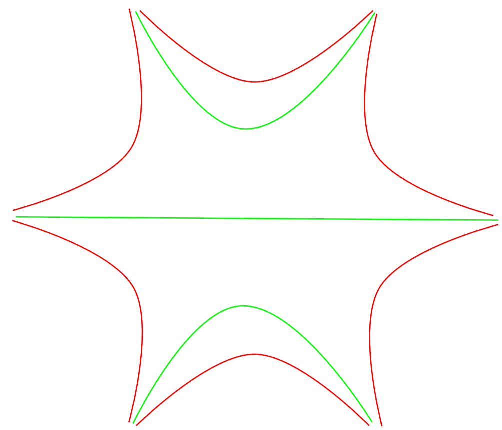

Fig. 5 Splitting up a complementary region. The splitting leaves are shown in green.

Remark 3. By stacking these regions on top of each other it is easy to do this when all prongs are of order $6+4n$ and all foliations are orientable. In this way we obtain orientable foliations without compact leaves that have trivial Euler class. This construction will work for any pseudo-Anosov flow satisfying the assumption on prongs.

6 Questions and future directions

Clearly, the examples presented here are just a sample of what can be done using these techniques and are far from exhaustive. In particular, some questions that should be addressed in addition to completing several of the arguments sketched above are the following:

- Is it possible to construct attracting plugs with Morse-Smale dynamics and without closed invariant surfaces so that the boundary behaviour is arbitrary[4]? What are the possible obstructions? Can these be characterised?
- Can one obtain general criteria for gluing projectively hyperbolic Reebless plugs so that they remain Reebless? taut?
- Is it possible to characterise those pairs of transverse contact structures that can be deformed into Anosov flows?
- In the partially hyperbolic setting there is some additional information on the contact structures. Is it true that if a hyperbolic 3-manifold admits a partially hyperbolic diffeomorphism, then it admits an Anosov flow? See [BFFP] for some progress in this direction.

[4] As explained above a simple obstruction is that the foliation in the boundary has to have at least one Reeb annuli, but in principle we do not know of other obstruction.

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