

Notes on Global Product Structure

Andy Hammerlindl

Abstract These notes present a careful explanation of parts of a proof showing that if an Anosov diffeomorphism has global product structure and the universal cover has polynomial growth of volume, then the Anosov diffeomorphism is topologically conjugate to an infranilmanifold automorphism. The original proof published by Brin and Manning relied on an incorrectly stated result of Auslander about the properties of maps on infranilmanifolds. The current notes show how the proof can be adapted to avoid relying on this incorrect result.

1 Introduction

These notes give a careful explanation of parts of the proof of the following result.

Theorem 1.1 (Brin-Manning) *If an Anosov diffeomorphism f has Global Product Structure, and the universal cover has polynomial growth of volume, then f is topologically conjugate to an infranilmanifold automorphism.*

The original proof of Brin and Manning [3] relies on an incorrect statement of Auslander regarding infranilmanifold automorphisms whereas these current notes avoid using this incorrect statement.

These notes explain in detail only the first part of the proof of the theorem, which is the construction of a semi-conjugacy. They do not include the proof that the semi-conjugacy is injective and therefore a true conjugacy since the original arguments for this step hold without modification. I have ordered the steps of the proof so that as much is proved as possible before introducing infranilmanifolds.

Andy Hammerlindl
School of Mathematical Sciences, Monash University, Victoria 3800 Australia e-mail:
andy.hammerlindl@monash.edu

My original motivation in preparing these notes was to convince myself of the correctness of the proof. Almost all of the following repeats arguments already given by Brin, Manning, Franks, Dekimpe, and others [3, 4, 5, 6].

2 The proof

Lemma 2.1 *If $f : M \rightarrow M$ has Global Product Structure, then for every $\varepsilon > 0$, there is $k \geq 1$ such that*

$$f^{-k}(B_\varepsilon(x)) \cap B_\varepsilon(y) \neq \emptyset$$

for all $x, y \in M$. Here, $B_\varepsilon(x) := \{y \in M : d(x, y) < \varepsilon\}$.

Proof. For the lifted foliations W^u and W^s on the universal cover \tilde{M} , the intersection

$$[\tilde{x}, \tilde{y}] = W^s(\tilde{x}) \cap W^u(\tilde{y})$$

depends continuously on $\tilde{x}, \tilde{y} \in \tilde{M}$. As W^u and W^s are tangent to continuous subbundles E^u and E^s , $d_s(\tilde{x}, [\tilde{x}, \tilde{y}])$ and $d_u(\tilde{y}, [\tilde{x}, \tilde{y}])$ are continuous as well and are bounded for $(\tilde{x}, \tilde{y}) \in K \times K$ where K is a compact fundamental domain of the covering $\tilde{M} \rightarrow M$. Projecting down, there is $R > 0$ such that for all $x, y \in M$, there is

$$z \in W^s(x) \cap W^u(y) \subset M$$

with $d_s(x, z) < R$ and $d_u(y, z) < R$. One can then find $n \geq 1$, independent of x and y , such that

$$d_s(f^n(x), f^n(z)) < \varepsilon \quad \text{and} \quad d_u(f^{-n}(y), f^{-n}(z)) < \varepsilon.$$

This is enough to prove the lemma with $k = 2n$. \square

Corollary 2.2 *With f, ε, k as in the last lemma, for any periodic point $f^m(x) = x$, there is a closed ε -pseudo orbit*

$$x, fx, \dots, f^{m-1}x, y, fy, \dots, f^{k-1}y, x.$$

Proof. Take $y \in B_\varepsilon(x) \cap f^{-k}(B_\varepsilon(x)) \neq \emptyset$. \square

Lemma 2.3 *If f has Global Product Structure, there is k such that $\#\text{Fix}(f^m) \leq \#\text{Fix}(f^{m+k})$ for all $m \geq 1$.*

Proof. f is expansive; there is $\delta > 0$ such that if $d(f^n(x), f^n(x')) < \delta$ for all $n \in \mathbb{Z}$, then $x = x'$. It also has a periodic shadowing property; there is $\varepsilon > 0$ such that every closed ε -pseudo orbit $x_0, x_1, \dots, x_i = x_0$ is $\frac{1}{3}\delta$ -shadowed by a true orbit $x = f^i(x)$.

Suppose $x = f^m(x)$. By the previous corollary, there is an ε -pseudo orbit

$$x, fx, \dots, f^{m-1}x, y, fy, \dots, f^{k-1}y, x.$$

and this is $\frac{1}{3}\delta$ -shadowed by some $z = f^{m+k}(z)$. In particular, $d(f^i(x), f^i(z)) < \frac{1}{3}\delta$ for $0 \leq i < m$. Say by the same process that $x' = f^m(x')$ leads to a point $z' = f^{m+k}(z')$.

If $z = z'$, then $d(f^i(x), f^i(x')) < \frac{2}{3}\delta$ for $0 \leq i < m$. Hence, for all $i \in \mathbb{Z}$, and therefore $x = x'$. \square

Remark 1. Brin and Manning use $\frac{1}{3}\delta$, but it seems that $\frac{1}{2}\delta$ would suffice.

Lemma 2.3 will later be used to establish hyperbolicity of a Lie group automorphism. For this, we will also need an elementary result about complex numbers.

Lemma 2.4 *Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are such that no λ_i is a root of unity. Define*

$$a_m := \prod_{i=1}^n |1 - \lambda_i^m|.$$

If there is $k \geq 1$ such that $a_m \leq a_{m+k}$ for all $m \geq 1$, then $|\lambda_i| \neq 1$ for all i .

For completeness, we include a proof at the end of the paper.

Lemma 2.5 *If A is a hyperbolic automorphism of a nilpotent Lie group L , then $\alpha_b : L \rightarrow L, x \mapsto A(x) \cdot b$ has a fixed point for any $b \in L$.*

Proof. By the Anosov closing lemma, if $g : M \rightarrow M$ is Anosov, there are constants $\delta, \varepsilon > 0$ such that if $d(g(x), x) < \varepsilon$, then there is $y = f(y)$ with $d(x, y) < \delta$. (This is just shadowing of a constant pseudo orbit.) The values of δ and ε depend on bounds on the angle between E^u and E^s and bounds related to expansion and contraction given in the definition of an Anosov diffeomorphism. The closing lemma is proven locally, so it holds for M non-compact so long as E^u and E^s are uniformly continuous and these bounds hold uniformly on M .

To prove the lemma, equip L with a metric such that $d(x \cdot y, x \cdot z) = d(y, z)$ for all $x, y, z \in L$. The closing lemma holds for all α_b and with δ and ε independent of b . If α_b has a fixed point x , then $d(\alpha_{b'}(x), x) < \varepsilon$ for all $b' \in B_\varepsilon(b)$, so every such b' has a fixed point. Since $\alpha_1 = A$ has a fixed point, and $B_\varepsilon(B_{n\varepsilon}(1)) = B_{(n+1)\varepsilon}(1)$ we can show that any α_b has a fixed point.

Alternatively, one can prove this result using induction on the nilpotency class of the Lie group and the fact that the result is true for the abelian Lie group \mathbb{R}^n . \square

Assumption 2.6 *For the rest of this note, assume that $f : M \rightarrow M$ is an Anosov diffeomorphism with Global Product Structure and the universal cover \tilde{M} has polynomial growth of volume.*

Lemma 2.7 *\tilde{M} is homeomorphic to \mathbb{R}^n for some n .*

Proof. By Global Product Structure, \tilde{M} is homeomorphic to an unstable leaf direct product with a stable leaf. As each of these is homeomorphic to \mathbb{R}^n for some n , so is the direct product. \square

Lemma 2.8 *$\pi_1(M)$ is torsion free.*

Proof. Suppose not. Then there would be a deck transformation $\gamma: \tilde{M} \rightarrow \tilde{M}$ of finite period. As \tilde{M} is homeomorphic to \mathbb{R}^n , this gives a fixed-point free homeomorphism of \mathbb{R}^n which is periodic. This is ruled out by a classic result of P. A. Smith. (This is repeating an argument given by Franks [6]. Franks cites [2] as a reference.) \square

Lemma 2.9 *The maximal normal nilpotent subgroup N of $\pi_1(M)$ has finite index.*

Proof. By Gromov, $\pi_1(M)$ has a nilpotent group H of finite index [7]. Then, there is a subgroup $K \leq H$ of finite index and normal (in $\pi_1(M)$). As it is a subgroup of H , it is nilpotent. The Hirsch-Plotkin radical N of $\pi_1(M)$ contains K and is therefore of finite index. Hence, it the desired maximal subgroup. \square

Corollary 2.10 *N is characteristic; it is preserved by every automorphism.*

Lift $f: M \rightarrow M$ to a map $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$. Viewing $\pi_1(M)$ as the group of all deck transformations, the choice of lift defines a group automorphism f_* of $\pi_1(M)$ by

$$\tilde{f}(\gamma(x)) = f_*(\gamma)(\tilde{f}(x))$$

for all $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$. By the above corollary, $f_*(N) = N$.

We use the following results on nilmanifolds as given by Malcev [9].

Theorem 2.11 (Malcev) *If N is a torsion-free nilpotent finitely-generated group, there is*

- *a connected, simply connected, nilpotent Lie group L ,*
- *a subgroup $\Lambda < L$, and*
- *a group isomorphism $T: \Lambda \rightarrow N$,*

such that L/Λ is a compact manifold (a nilmanifold). Every group automorphism of Λ extends to a unique Lie group automorphism of L .

Any such L is unique up to Lie group isomorphism. In general, Λ and T are not unique.

Lemma 2.12 *There is a nilpotent Lie group L , a subgroup Λ , an isomorphism $T: \Lambda \rightarrow N$, and a Lie group automorphism $A: L \rightarrow L$ such that*

- *$A(\Lambda) = \Lambda$, and*
- *$A(x) = T^{-1}f_*T(x)$ for $x \in \Lambda$.*

Proof. This is the theorem of Malcev applied to our specific situation. \square

Lemma 2.13 *A is hyperbolic.*

Remark 2. As the choice of T in lemma 2.12 was arbitrary, this will show that A is hyperbolic for any choice of T .

Note that we do not assume E^u is orientable on M or that f preserves an orientation of E^u if it exists. However, in order to prove lemma 2.13, we will temporarily make an additional assumption on orientation. Recall that any bundle over a simply connected manifold is orientable, as there is no double cover to lift to.

Assumption 2.14 For now, assume that on the universal cover, \tilde{f} preserves the orientation of E^u . If need be, replace f and A with f^2 and A^2 .

As A is hyperbolic if and only if A^2 is hyperbolic, this assumption can be made freely for the purposes of proving lemma 2.13.

Viewing $\gamma \in \pi_1(M)$ as a diffeomorphism $\tilde{M} \rightarrow \tilde{M}$ whose derivative maps the lifted unstable bundle E^u to itself, define a group homomorphism $\theta : \pi_1(M) \rightarrow \{+1, -1\}$ by whether γ preserves or reverses the orientation of E^u . Then $N^+ := \ker(\theta) \cap N$ defines a finite-index normal nilpotent subgroup of $\pi_1(M)$.

By assumption 2.14, f_* restricts to an automorphism of N^+ .

Define a manifold \hat{M} as a quotient \tilde{M}/N^+ . That is, $x, y \in \tilde{M}$ are identified if there is $\gamma \in N^+ \subset \pi_1(M)$ such that $\gamma(x) = y$. Since $\gamma(x) = y$ implies $f_*(\gamma)(\tilde{f}(x)) = \tilde{f}(y)$, \tilde{f} quotients to a map $\hat{f} : \hat{M} \rightarrow \hat{M}$. Further, $\pi_1(\hat{M})$ is isomorphic to N^+ and \tilde{f} can be viewed as a lift of \hat{f} to \tilde{M} . As such, the induced group automorphism \hat{f}_* of $\pi_1(\hat{M})$ can be identified with the restriction of f_* to N^+ .

Lemma 2.15 *There is a subgroup $\Lambda^+ \subset \Lambda$ such that*

- $T(\Lambda^+) = N^+$,
- $A(\Lambda^+) = \Lambda^+$,
- $A(x) = T^{-1}\hat{f}_*T(x)$ for $x \in \Lambda^+$.

Proof. This follows from lemma 2.12 and the fact that $f_*(N^+) = N^+$ (under assumption 2.14). \square

Let $\hat{g} : L/\Lambda^+ \rightarrow L/\Lambda^+$ be the quotient of A to the nilmanifold L/Λ^+ .

Lemma 2.16 *There is a homotopy equivalence $\hat{h} : \hat{M} \rightarrow L/\Lambda^+$ with lift $\tilde{h} : \tilde{M} \rightarrow L$ such that the maps induced by \hat{f} , \hat{g} , and \hat{h} on the fundamental groups satisfy $\hat{h}_*\hat{f}_* = \hat{g}_*\hat{h}_*$.*

Proof. By the construction of \hat{g} , there is an isomorphism between $\pi_1(\hat{M})$ and $\pi_1(L/\Lambda^+)$ that conjugates \hat{f}_* and \hat{g}_* . Since \tilde{M} is homeomorphic to \mathbb{R}^n and L/Λ^+ is a nilmanifold, both spaces are of type $K(\pi, 1)$ the lemma follows from standard results in algebraic topology. \square

Remark 3. Many treatments of Eilenberg-MacLane spaces assume that the fundamental group is defined with regard to a pointed space (X, x_0) . This complicates matters, as we have not yet proved that \hat{f} has a fixed point, and so may not be a based map. However, one can isotope \hat{f} to a map \hat{f}_1 which does have a fixed point, prove the lemma for \hat{f}_1 , and then use the isotopy to prove it for \hat{f} .

Lemma 2.17 *Let $L(\cdot)$ denote the Lefschetz number of a diffeomorphism of a manifold. Then,*

$$\#\text{Fix}(\hat{f}^m) = |L(\hat{f}^m)| = |L(\hat{g}^m)| = \prod_{i=1}^n |1 - \lambda_i^m|$$

where λ_i are the eigenvalues of A .

Proof. The left-most equality is a standard result for Anosov diffeomorphisms which preserve the orientation of E^u . (See [11].) By the previous lemma, the maps induced by \hat{f} and \hat{g} on the homology groups are conjugate. Therefore, their traces are the same, and the resulting Lefschetz numbers are the same. The last equality is proved in [10]. \square

Proof (of lemma 2.13). None of the λ_i in lemma 2.17 can be a root of unity, for then some iterate of \hat{f} would be an Anosov map without periodic points. The result then follows as a combination of lemmas 2.3, 2.4, and 2.17. \square

End of assumption 2.14. As we have proved lemma 2.13, we no longer need the assumption.

Lemma 2.18 *f has a fixed point.*

Proof. Suppose that instead of using N^+ and Λ^+ , we had used N and Λ to define maps $\check{f} : \check{M}/N \rightarrow \check{M}/N$ and $\check{g} : L/\Lambda \rightarrow L/\Lambda$. As \check{f} may not preserve an orientation of E^u , we do not compare $\#\text{Fix}(\check{f}^m)$ to $|L(\check{f}^m)|$. However, the other two equalities given by lemma 2.17 hold in this case, and using $m = 1$ and that A is hyperbolic, we get that $L(\check{f})$ is non-zero. Then \check{f} has a fixed point and this projects to a fixed point for f . \square

Let $\text{Aff}(L)$ denote the affine transformations of L , those functions of the form $x \mapsto b \cdot \beta(x)$, where β is an automorphism of the Lie group and $b \in L$.

Theorem 2.19 (Auslander-Schenkman) *If Γ is a torsion-free finitely-generated group with nilpotent Hirsch-Plotkin radical N of finite index, then Γ can be viewed as a subgroup of $\text{Aff}(L)$, where L is the Lie group given by theorem 2.11 corresponding to N . Moreover, L/Γ is a compact manifold.*

This follows from section 2 of [1].

Remark 4. There is a subtle issue here. When regarded as a subgroup of Γ , N consists of affine maps. These maps are of the form $L \rightarrow L$, $x \mapsto a \cdot x$ for some $a \in L$. If N is regarded as a subgroup of L , its elements are no longer maps, but simply elements $a \in L$. These two distinct interpretations may have contributed to Auslander providing an incorrect proof for the extension of an automorphism of Γ . To try to avoid confusion, we will not regard N as a subgroup of L , and instead use the symbol $\Lambda \subset L$ and say that they are identified by an isomorphism $T : \Lambda \rightarrow N$.

We now follow the proof given in the paper of Lee and Raymond [8], but in regards to our specific situation. By the previous theorem, $\pi_1(M)$ can be identified with a subgroup Γ of $\text{Aff}(L)$ and f_* then defines an automorphism $\psi : \Gamma \rightarrow \Gamma$ for which $\psi(N) = N$. This restriction to N defines an automorphism $T^{-1}\psi T$ on Λ which extends to $A : L \rightarrow L$ by the result of Malcev. Continuing on, the proof of Lee and Raymond shows that ψ is conjugation by an element of $\text{Aff}(L)$. They give a formula in the proof: using their notation, ψ is conjugation by $(b, \mu(b^{-1})A)$ [8, page 75]. This can be re-written as $\psi(\gamma) = \alpha\gamma\alpha^{-1}$ where $\alpha(x) = A(x) \cdot b$.

Lemma 2.20 α is hyperbolic.

Proof. The above construction of A using the results of Malcev is exactly the same as in lemma 2.12, and therefore A is hyperbolic by lemma 2.13. \square

Corollary 2.21 α has a fixed point.

Proof. This is a specific case of lemma 2.5. \square

If $x_0 \in L$ is a fixed point of α , then $\alpha(x) = A(x \cdot x_0^{-1}) \cdot x_0$, so $\alpha = \beta A \beta^{-1}$, where $\beta(x) = x \cdot x_0$. Note that $\beta \in \text{Aff}(L)$. For $\gamma \in \Gamma$, the formula $\psi(\gamma) = \alpha \gamma \alpha^{-1}$ expands to

$$\psi(\gamma) = \beta A \beta^{-1} \gamma \beta A^{-1} \beta^{-1} \Rightarrow \beta^{-1} \psi(\gamma) \beta A = A \beta^{-1} \gamma \beta.$$

Define $\bar{\Gamma} = \{\beta^{-1} \gamma \beta : \gamma \in \Gamma\}$ and $\bar{\psi} : \bar{\Gamma} \rightarrow \bar{\Gamma}$ by $\bar{\psi}(\beta^{-1} \gamma \beta) = \beta^{-1} \psi(\gamma) \beta$, so that the above formula can be rewritten simply as

$$\bar{\psi}(\bar{\gamma}) A = A \bar{\gamma}.$$

The manifold $P = L/\bar{\Gamma}$ is a compact manifold, and the Lie group automorphism A quotients down to an Anosov diffeomorphism $g : P \rightarrow P$. Identifying $\pi_1(P)$ with $\bar{\Gamma}$, there is a commutative diagram

$$\begin{array}{ccccccc} \pi_1(P) & \longrightarrow & \bar{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(M) \\ \downarrow g_* & & \downarrow \bar{\psi} & & \downarrow \psi & & \downarrow f_* \\ \pi_1(P) & \longrightarrow & \bar{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(M) \end{array}$$

where all arrows are isomorphisms and the top and bottom rows are the same. Thus, f_* and g_* are conjugate.

As M and P are $K(\pi, 1)$ and f and g have fixed points, we can apply the results of Franks to find a semi-conjugacy $h : M \rightarrow P$ such that $hf = gh$ [6]. Beyond this point, the proof given by Brin and Manning is fairly easy to follow (see also [5]), and there are no subtleties with respect to infranilness.

3 Proof of lemma 2.4

We now proceed to prove lemma 2.4. First, consider only the eigenvalues which lie on the unit circle.

Assumption 3.1 Assume for the next two lemmas that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are not roots of unity, and $|\lambda_i| = 1$ for all i . The λ_i need not be distinct.

Lemma 3.2 If $\{n_j\}$ is a subsequence of \mathbb{N} such that $\lambda_1^{n_j} \rightarrow 1$, then there is $p \in \mathbb{N}$ such that $\lambda_1^{pn_j} \rightarrow 1$ and $\lambda_i^{pn_j} \not\rightarrow \lambda_i$ for all i .

Proof. Clearly, $(\lambda_1^{n_j})^p \rightarrow 1$ for any p . Suppose for some i and distinct p, q , that $\lambda_i^{pn_j} \rightarrow \lambda_i$ and $\lambda_i^{qn_j} \rightarrow \lambda_i$. Then $(\lambda_i^{pn_j})^q \rightarrow \lambda_i^q$ and $(\lambda_i^{qn_j})^p \rightarrow \lambda_i^p$, and so $\lambda_i^p = \lambda_i^q$. This contradicts the fact that λ_i is not a root of unity. Therefore, for all but finitely many p , the lemma is satisfied. \square

Lemma 3.3 *There is a subsequence $\{m_j\}$ of \mathbb{N} such that for all i , $L_i = \lim_{j \rightarrow \infty} \lambda_i^{m_j}$ exists and $L_i \neq 1$. Further, $\lim_{j \rightarrow \infty} \lambda_1^{m_j+1} = 1$.*

Proof. As λ_1 is not a root of unity, there is a subsequence n_j such that $\lambda_1^{n_j} \rightarrow 1$. Since each sequence $\lambda_i^{n_j}$ lies in a compact subset of \mathbb{C} , by replacing n_j with a further subsequence, we may assume $\lambda_i^{n_j}$ converges for all i . Choosing p as in the previous lemma, $m_j = pn_j - 1$ is the desired subsequence in the statement of this lemma. \square

We now consider eigenvalues of any modulus.

Lemma 3.4 *Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are such that no λ_i is a root of unity. Further suppose $|\lambda_1| = 1$. Define*

$$a_m := \prod_{i=1}^n |1 - \lambda_i^m|.$$

Then there is a subsequence $\{a_{m_j}\}$ such that $\frac{a_{m_j+1}}{a_{m_j}} \rightarrow 0$.

Proof. If $|\lambda_i| < 1$, then

$$\frac{|1 - \lambda_i^{m+1}|}{|1 - \lambda_i^m|} \rightarrow 1.$$

If $|\lambda_i| > 1$, then

$$\frac{|1 - \lambda_i^{m+1}|}{|1 - \lambda_i^m|} \rightarrow |\lambda_i|.$$

By the previous lemma, there is $\{m_j\}$ such that

$$\lim_{j \rightarrow \infty} \frac{|1 - \lambda_i^{m_j+1}|}{|1 - \lambda_i^{m_j}|}$$

exists for those i with $|\lambda_i| = 1$, and for $i = 1$ in particular, the limit is zero. Then,

$$\lim_{j \rightarrow \infty} \frac{a_{m_j+1}}{a_{m_j}} = \prod_i \lim_{j \rightarrow \infty} \frac{|1 - \lambda_i^{m_j+1}|}{|1 - \lambda_i^{m_j}|} = 0.$$

\square

Proof (of lemma 2.4). If we define \tilde{a}_m similarly to a_m , but using $\tilde{\lambda}_i = \lambda_i^k$ in place of λ_i , then $\tilde{a}_m = a_{km}$ and the hypothesis $a_m \leq a_{m+k}$ for all m implies $\tilde{a}_m \leq \tilde{a}_{m+1}$ for all m . Therefore, it is enough to prove the lemma in the case $k = 1$. Suppose that $|\lambda_i| = 1$ for some i . Without loss of generality, $i = 1$. By the previous lemma, there is a sequence of terms $a_{m_j+1}/a_{m_j} \geq 1$ which tends to zero, a contradiction. \square

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