New preasymptotic estimates for approximation of periodic Sobolev functions

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Dedicated to Ian Sloan on the occasion of his 80th birthday

Abstract Approximation of Sobolev embeddings is a well-studied subject in high-dimensional approximation, with many application to different branches of mathematics. E.g., for isotropic Sobolev spaces $H^s(\mathbb{T}^d)$ of fractional smoothness $s > 0$ on the d-dimensional torus it is known that the approximation numbers a_n of the embedding $H^s(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$ behave like $a_n \sim n^{-s/d}$ as $n \to \infty$, where the (weak) equivalence $\sim$ holds only up to multiplicative constants which are not known explicitly. However, for practical purposes it is more relevant to know the preasymptotic behaviour of the a_n, i.e. for small n, say $n \le 2^d$. In this range the dependence on n is only logarithmic.

The main results in this note are sharp two-sided preasymptotic estimates for approximation of isotropic Sobolev functions on $\mathbb{T}^d$. In particular we give explicit constants, which show the exact dependence on the dimension d, the smoothness s, and further parameters of the norm. This improves the known results in the literature. Moreover, we prove a new preasymptotic estimate for approximation of Sobolev functions of dominating mixed smoothness.

1 Introduction

Approximation of Sobolev functions is a classical topic in functional analysis, with numerous applications to other branches of mathematics like approximation theory or numerical analysis. The quality of such approximations can be expressed in terms of approximation numbers a_n of embeddings of the corresponding Sobolev spaces. The error is measured in the norm of the target space, mostly in the L_2- or L_∞-norm. The approximation numbers a_n coincide with the worst case error that can be achieved by linear algorithms that are allowed to use n pieces of (general linear) information on the functions that we wish to approximate.

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The asymptotic rate of the approximation numbers a_n as $n \to \infty$ is known in many cases, but often only up to non-specified multiplicative constants, i.e. the dependence of these constants on the smoothness parameter of the spaces and the dimension of the underlying domain has not yet been investigated very carefully. For practical purposes, however, this is a crucial point. Without further information on the constants, the asymptotic rate alone is fairly useless in numerical computations.

Another effect which is important for numerical approximation of functions on high-dimensional domains, e.g. on the d-dimensional unit cube or the d-dimensional torus (periodic case), is that one usually has to 'wait exponentially long' until the asymptotic rate 'becomes visible'. More precisely, for small n, say $n \leq 2^d$, the behaviour of the approximation numbers is often quite different from the asymptotic behaviour. In many practical problems the dimension d is very large, possibly up to hundreds or thousands. But for large or even moderate dimensions, 2^d pieces of information might already be well beyond the capacity of a computer. Therefore it is necessary to have good estimates in this so-called preasymptotic range. Only quite recently this topic has attracted more attention, but still there are only few papers devoted to a systematic study of the preasymptotic behaviour, among them [5], [4] and [6] for periodic functions, and [1] for functions on spheres. Prior to these papers, preasymptotic estimates appeared only occasionally in the literature. For detailed comments on this issue we refer to Section 4.5 of [6].

The aim of this note is to provide some new preasymptotic estimates for approximation of periodic functions from two classical types of Sobolev spaces, namely isotropic spaces and spaces of dominating mixed smoothness. In both cases we consider arbitrary fractional smoothness $s > 0$, and in the isotropic case we consider in addition a family of equivalent norms. Our results exhibit an interesting dependence on the parameters of these norms, showing again how sensitive appproximation problems are with respect to change of norms. Our main results improve the known results in the literature: In the isotropic case we obtain explicit constants, which were not available so far (c.f. [5] and [4]), and in the mixed case we improve the exponent that was given in [6].

The paper is organized as follows: In Section 2 we collect some known facts on approximation numbers, and give the definitions of the Sobolev spaces that we consider in this paper. Sections 3 and 4 contain our main results on preasymptotic estimates for approximation in isotropic Sobolev spaces resp. Sobolev spaces of dominating mixed smoothness.

Notation. Throughout the paper we use standard notation. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural, integer, real and complex numbers, respectively. For sequences (a_n) and (b_n) of positive real numbers the weak equivalence $a_n \sim b_n$ means that there are constants $C, c > 0$ such that $c\,b_n \leq a_n \leq C\,b_n$ for all $n \in \mathbb{N}$.

2 Approximation in periodic spaces of Sobolev type

The approximation numbers of a (bounded linear) operator $T : X \to Y$ between two Banach spaces are defined by

$$a_n(T) := \inf\{\|T - A\| : \mathrm{rank}(A) < n\} \quad , \quad n \in \mathbb{N}\,.$$

Since T is compact if $\lim_{n\to\infty} a_n(T) = 0$, the rate of decay of $a_n(T)$ as $n \to \infty$ describes the 'degree' of compactness of T. If T is a *compact* operator between *Hilbert* spaces, then the approximation numbers coincide with the well-known singular numbers,

$$a_n(T) = s_n(T) := \lambda_n((T^*T)^{1/2})\,,$$

where the eigenvalues $\lambda_n((T^*T)^{1/2})$ are arranged in non-increasing order and counted according to their multiplicities. We will work only in this Hilbert space situation. For further properties of approximation numbers we refer to the monographs [7] or [3].

Let $\mathbb{T}$ be the torus, i.e. the interval $[0,2\pi]$ where the endpoints are identified. In this note we consider Sobolev spaces on the d-dimensional torus $\mathbb{T}^d$, equipped with the *normalized* Lebesgue measure. Hence the Fourier coefficients of $f \in L_2(\mathbb{T}^d)$ are

$$\widehat{f}(k) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx \quad , \quad k \in \mathbb{Z}^d\,.$$

For a given sequence $\mathbf{w} = (w(k))_{k\in\mathbb{Z}^d}$ of positive weights, bounded away from zero, let $H^{\mathbf{w}}(\mathbb{T}^d)$ be the space of all $f \in L_2(\mathbb{T}^d)$ such that the norm

$$\|f|H^{\mathbf{w}}(\mathbb{T}^d)\| := \left(\sum_{k\in\mathbb{Z}^d} w(k)^2 |\widehat{f}(k)|^2\right)^{1/2} \tag{1}$$

is finite. Clearly, $H^{\mathbf{w}}(\mathbb{T}^d)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathbf{w}} := \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)\, \overline{\widehat{g}(k)}\, w(k)^2\,,$$

and $\{f_k\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis (ONB) in $H^{\mathbf{w}}(\mathbb{T}^d)$, where $f_k(x) := \frac{e^{ikx}}{w(k)}$ for $x \in \mathbb{T}^d$ and $k \in \mathbb{Z}^d$, and $kx := \sum_{j=1}^{d} k_j x_j$. For the concrete spaces that we will consider in this paper, the weights satisfy the additional conditions

$$1 = w(0) \le w(k) \quad \text{for all } k \in \mathbb{Z}^d \qquad \text{and} \qquad \lim_{|k|\to\infty} w(k) = \infty\,. \tag{2}$$

This ensures that there is a compact embedding $H^{\mathbf{w}}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$ of norm one. Setting $e_k(x) := e^{ikx}$, we have for every $f \in H^{\mathbf{w}}(\mathbb{T}^d)$

$$f=\sum_{k\in\mathbb{Z}^d}\langle f,f_k\rangle_{\mathbf{w}}f_k=\sum_{k\in\mathbb{Z}^d}\frac{1}{w(k)}\langle f,f_k\rangle_{\mathbf{w}}e_k \tag{3}$$

with convergence in $L_2(\mathbb{T}^d)$. The series in (3) is a Schmidt representation of the embedding $I_d : H^{\mathbf{w}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, since $\{e_k\}_{k\in\mathbb{Z}^d}$ is an ONB in $L_2(\mathbb{T}^d)$. Let $(\sigma_n)_{n\in\mathbb{N}}$ denote the non-increasing rearrangement of $\big(1/w(k)\big)_{k\in\mathbb{Z}^d}$. Then

$$a_n(I_d : H^{\mathbf{w}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = s_n(I_d : H^{\mathbf{w}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \sigma_n\,. \tag{4}$$

From a theoretical point of view the story would be finished here, but in concrete cases it is a highly non-trivial task to *find* this rearrangement. In particular, this requires subtle combinatorial estimates.

Now we give the definitions of the Sobolev spaces that we are going to investigate. As in the general case (1) above, their norms are again weighted ℓ_2-sums of Fourier coefficients.

Definition 2.1 *(Isotropic spaces)*
Let $d \in \mathbb{N}$, $s > 0$ and $0 < p < \infty$. Then the Sobolev space $H^{s,p}(\mathbb{T}^d)$ is defined as the collection of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|H^{s,p}(\mathbb{T}^d)\| := \left(\sum_{k\in\mathbb{Z}^d}\Big(1+\sum_{j=1}^d |k_j|^p\Big)^{2s/p}|\widehat{f}(k)|^2\right)^{1/2} < \infty\,. \tag{5}$$

Remark 2.2 For fixed $s > 0$, all these norms are equivalent. That means, in fact all spaces $H^{s,p}(\mathbb{T}^d)$, $0 < p < \infty$, coincide, i.e. the superscript p only indicates which norm we are using.

Moreover let us point out that for integer smoothness $s = m \in \mathbb{N}$ this family of norms is closely related to the most common classical norms on the isotropic Sobolev space $H^m(\mathbb{T}^d)$. For $p = 2$ we have the inequalities

$$\frac{1}{\sqrt{m!}}\|f|H^{m,2}(\mathbb{T}^d)\| \le \left(\sum_{|\alpha|\le m}\|D^\alpha f|L_2(\mathbb{T}^d)\|^2\right)^{1/2} \le \|f|H^{m,2}(\mathbb{T}^d)\|\,.$$

The term in the middle is the original norm on $H^m(\mathbb{T}^d)$. Note that the equivalence constants depend only on the smoothness parameter m but not on the dimension d.

For $p = 2m$ we have even an equality with another classical equivalent norm on $H^m(\mathbb{T}^d)$ that is often used and works only with the highest derivatives in each coordinate direction,

$$\|f|H^{m,2m}(\mathbb{T}^d)\| = \left(\|f|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^d \left\|\frac{\partial^m f}{\partial x_j^m}\Big|L_2(\mathbb{T}^d)\right\|^2\right)^{1/2}.$$

For the proof of these facts see section 2 of [5].

Definition 2.3 *(Spaces of dominating mixed smoothness)*
Let $d \in \mathbb{N}$ and $s > 0$. Then the Sobolev space $H^s_{\mathrm{mix}}(\mathbb{T}^d)$ is defined as the collection of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|H^s_{\mathrm{mix}}(\mathbb{T}^d)\| := \left(\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}\big(1+|k_j|\big)^{2s}|\widehat{f}(k)|^2\right)^{1/2} < \infty. \tag{6}$$

In [6] we considered some other (quite natural) equivalent norms on $H^s_{\mathrm{mix}}(\mathbb{T}^d)$, but for simplicity we restrict ourselves in this note just to the norm given in Definition 2.3.

3 Preasymptotics for approximation in isotropic Sobolev spaces

Approximation of isotropic Sobolev embeddings is a topic with a long history, and many authors have contributed to this subject, see the monographs by Temlyakov [9] and Tikhomirov [10], and the references therein. By Theorems 4.1 and 4.2 in Chapter 2 of [9] we have for the Sobolev spaces $H^{s,p}(\mathbb{T}^d)$ the two-sided estimates

$$c_{s,p}(d)\,n^{-s/d} \le a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_{s,p}(d)\,n^{-s/d} \quad , \quad n \in \mathbb{N}$$

with certain unspecified positive constants $c_{s,p}(d)$ and $C_{s,p}(d)$. Only recently the dependence of these constants on the parameters s,p and, more importantly, on the dimension d was investigated. For fixed $s > 0$ and some values of the parameter p this was done in [5], further results can be found in [8, 1, 4, 12]. Quite surprisingly, it turned out that for fixed s and p the constants decay polynomially in d.

From a computational point of view, it is even more relevant to have estimates in the preasymptotic range, i.e. for small n, say $n \le 2^d$. In Theorem 4.6 of [5] , for $s > 0$, $d \in \mathbb{N}$ and $2d+2 \le n \le 2^d$, an almost matching two-sided estimate was shown,

$$\left(\frac{\ln 2}{\ln(4n)}\right)^s \le a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{\ln(2d+1)}{\ln n}\right)^s. \tag{7}$$

But the proof worked only for the parameter value $p = 1$, the only case where an exact formula for the cardinalities

$$C_p(r,d) := \mathrm{card}\{k \in \mathbb{Z}^d : |k_1|^p + \ldots + |k_d|^p \le r\} \quad , \quad r,d \in \mathbb{N} \tag{8}$$

was available. Somewhat later a connection to entropy numbers was found which allowed to deal with the full parameter range $0 < p < \infty$, see Theorem 1 of [4]. For all $s,p > 0$ and $d \le n \le 2^d$ the equivalence

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim_{s,p} \left(\frac{\ln(1+d/\ln n)}{\ln n}\right)^{s/p} \tag{9}$$

was shown, where the symbol $\sim_{s,p}$ means that the equivalence constants depend only on s and p, but not on d and n. However, in [4] these constants were *not explicitly given*.

The aim of this section is to determine *explicit* constants in (9). Concerning the proof technique, we return to combinatorial estimates, the crucial quantities will be the cardinalities $C_p(r,d)$ defined in (8). Their role is explained by the following fact: If $r \le d$, then

$$(r+1)^{-s/p} \in \{(1+|k_1|^p + \dots |k_d|^p)^{-s} : k \in \mathbb{Z}^d\},$$

and hence, applying (4) to the weight that determines $H^{s,p}(\mathbb{T}^d)$, we have

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (r+1)^{-s/p} \quad , \quad \text{if } n = C_p(r,d)\,. \tag{10}$$

Before we can prove the main results in this section we need some preliminary estimates for the cardinalities $C_p(r,d)$. Our first result implies that, for $d \ge 4$ and all p, the range $2 \le n \le C_p(\lfloor d/2 \rfloor, d)$ is sufficiently large, i.e. contains the preasymptotic range $2 \le n \le 2^d$.

Lemma 3.1 *Let $0 < p < \infty$ and $d, r \in \mathbb{N}$. Then we have*

$$C_p(r,d) \ge \left(1 + \frac{2d}{r}\right)^r \qquad \textit{for all} \quad r \le d \tag{11}$$

and

$$C_p(\lfloor d/2 \rfloor, d) \ge 2^d \qquad \textit{for all} \quad d \ge 4\,. \tag{12}$$

Proof. As in formula (3.3) in [5] we have for $r \le d$

$$C_p(r,d) = 1 + \sum_{\ell=1}^{r} 2^\ell \binom{d}{\ell} \operatorname{card}\left\{(m_i) \in \mathbb{N}^\ell : \sum_{i=1}^{\ell} m_i^p \le r\right\}. \tag{13}$$

For the convenience of the reader we sketch how to prove that. To find all vectors $(k_j) \in \mathbb{Z}^d$ with $\sum_{j=1}^d |k_j|^p \le r$ we note that the number ℓ of non-zero coordinates $k_j \ne 0$ is at most r. If $\ell = 0$, we only have the null vector. For $\ell = 1, \dots, r$ we consider all vectors $(m_i) \in \mathbb{N}^\ell$ with $\sum_{i=1}^\ell m_i^p \le r$. Now select ℓ coordinates $j \in \{1, \dots, d\}$ and place the m_i's there, this gives a vector (k_j). There are $\binom{d}{\ell}$ possibilities for selecting these positions. Moreover we can flip the sign in all non-zero coordinates, altogether there are 2^ℓ choices of signs $\pm$. In this way we obtain all vectors $(k_j) \in \mathbb{Z}^d$ with $\sum_{j=1}^d |k_j|^p \le r$, which proves formula (13). Clearly all cardinalities in (13) are at least one, just take all $m_i = 1$. This implies

$$C_p(r,d) \ge \sum_{\ell=0}^{r} 2^\ell \binom{d}{\ell} \ge \sum_{\ell=0}^{r} 2^\ell \binom{r}{\ell} \left(\frac{d}{r}\right)^\ell = \left(1 + \frac{2d}{r}\right)^r.$$

Here we used the inequality $\binom{d}{\ell} \ge (\frac{d}{r})^\ell \binom{r}{\ell}$ for $\ell = 0, 1, \dots, r$. For $\ell = 0$ this is trivial, and for $1 \le \ell \le r$ we have

$$\frac{\binom{d}{\ell}}{\binom{r}{\ell}\left(\frac{d}{r}\right)^{\ell}} = \prod_{j=1}^{\ell} \frac{(d-\ell+j)r}{(r-\ell+j)d} \geq 1\,,$$

since all factors are ≥ 1. This proves (11).

To prove (12) we distinguish two cases.

Case 1. Let d be an even integer, i.e. $d = 2m$ for some $m \in \mathbb{N}$. Then, taking only the last summand in (13) and using $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$, we get

$$C_p(\lfloor d/2 \rfloor, d) = C_p(m, 2m) \geq 2^m \binom{2m}{m} \geq 2^m \cdot \left(\frac{2m}{m}\right)^m = 2^{2m} = 2^d\,.$$

Case 2. Let now $d \geq 5$ be an odd integer, i.e. $r = 2m+1$ for some $m \geq 2$. Taking now the last two summands in (13) we obtain

$$\begin{aligned} C_p(\lfloor d/2 \rfloor, d) &= C_p(m, 2m+1) \\ &\geq 2^{m-1}\binom{2m+1}{m-1} + 2^m \binom{2m+1}{m} \\ &\geq 2^{m-1}\left(\frac{2m+1}{m-1}\right)^{m-1} + 2^m \left(\frac{2m+1}{m}\right)^m \\ &= 2^{m-1} \cdot 2^{m-1}\left(1 + \tfrac{3}{2(m-1)}\right)^{m-1} + 2^m \cdot 2^m \left(1 + \tfrac{1}{2m}\right)^m \\ &\geq 2^{2m-2}\left(1 + \tfrac{3}{2}\right) + 2^{2m}\left(1 + \tfrac{1}{2}\right) = 2^{2m}\left(\tfrac{5}{8} + \tfrac{3}{2}\right) \geq 2^d\,, \end{aligned}$$

where we used in the last line the estimate $(1+x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x > 0$, which follows from the binomial formula. This finishes the proof. $\square$

Our first main result in this section is a lower estimate for the approximation numbers a_n of the embedding $H^{s,p}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$ in the preasymptotic range $n \leq 2^d$.

Theorem 3.2 *Let $s > 0$, $0 < p < \infty$, $d \geq 4$ and $2d+1 \leq n \leq 2^d$. Then*

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \left(\frac{\ln\left(1 + \frac{3d}{\ln n}\right)}{3 \ln n}\right)^{s/p}. \qquad (14)$$

Proof. Denote $a_n := a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$. Fix $n \in \mathbb{N}$ with $2d+1 \leq n \leq 2^d$. From the definition (8) of $C_p(r,d)$ we see that $C_p(1,d) = 2d+1$. By Lemma 3.1 there is an $r \in \mathbb{N}$ with $2 \leq r \leq d/2$ such that $C_p(r-1,d) \leq n \leq C_p(r,d)$. Moreover

$$\ln n \geq \ln C_p(r-1,d) \geq (r-1)\ln\left(1 + \frac{2d}{r-1}\right) \geq (r-1)\ln 5\,,$$

since $d \geq 2r$. Consequently

$$\frac{1}{r-1} \geq \frac{\ln 5}{\ln n}\,.$$

Inserting this in the previous inequality we get

$$\ln n \geq (r-1)\ln\Big(1+\frac{2\ln 5\cdot d}{\ln n}\Big) \geq (r-1)\ln\Big(1+\frac{3d}{\ln n}\Big)$$

and

$$\frac{1}{r-1} \geq \frac{\ln\big(1+\frac{3d}{\ln n}\big)}{\ln n}\,.$$

Since $r \geq 2$, we have $r+1 \leq 3(r-1)$, and altogether this yields

$$a_n \geq a_{C_p(r,d)} = \frac{1}{(r+1)^{s/p}} \geq \frac{1}{(3(r-1))^{s/p}} \geq \left(\frac{\ln\big(1+\frac{3d}{\ln n}\big)}{3\ln n}\right)^{s/p}.$$

The proof is finished. $\square$

Now we pass to upper estimates. Here we distinguish between two cases for the parameter p, we begin with $1 \leq p < \infty$. The first estimate (15) in the theorem below is a slight improvement of (7), it holds for all parameter values $p \geq 1$ and not only for $p = 1$, while the second estimate (16) provides explicit constants for the correct order in n and d, see (9).

Theorem 3.3 *Let $d \in \mathbb{N}$, $d \geq 4$, $s > 0$, $1 \leq p < \infty$ and $2d+1 \leq n \leq 2^d$. Then*

$$a_n\big(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \leq \left(\frac{\frac{1}{2}+\ln d}{\ln n}\right)^{s/p} \qquad \textit{and} \tag{15}$$

$$a_n\big(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \leq \left(\frac{3+2\ln\big(\frac{d}{\ln n}\big)}{\ln(2\sqrt{\pi}n)}\right)^{s/p}. \tag{16}$$

Proof. Since $p \geq 1$, we have $|m|^p \geq |m|$ for all $m \in \mathbb{Z}$. So, for any $r \in \mathbb{N}$,

$$C_p(r,d) = \operatorname{card}\Big\{k \in \mathbb{Z}^d : \sum_{j=1}^{d} |k_j|^p \leq r\Big\} \leq C_1(r,d) = \sum_{\ell=0}^{\min\{r,d\}} 2^\ell \binom{r}{\ell}\binom{d}{\ell}.$$

The formula for $C_1(r,d)$ was shown in Lemma 3.2 in [5].
Given any $n \in \mathbb{N}$ with $2d+1 \leq n \leq 2^d$, we have

$$C_p(r-1,d) \leq n \leq C_p(r,d)$$

for some $r \in \mathbb{N}$ with $2 \leq r \leq \frac{d}{2}$, and hence

$$a_n := a_n\big(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \leq a_{C_p(r-1,d)} = r^{-s/p}\,. \tag{17}$$

Now let us prove (15). If $r = 2$, then $d \geq 4$, and hence

$$C_1(2,d) = 1+2d+2d^2 = d^2\Big(2+\frac{2}{d}+\frac{1}{d^2}\Big) \leq \frac{41}{16}d^2 \leq ed^2\,.$$

If $r\ge 3$, then $d\ge 6$. Using $\binom{d}{\ell}\le\frac{d^\ell}{\ell!}$ we get

$$\begin{aligned}C_1(r,d)&=\sum_{\ell=0}^{r}2^\ell\binom{d}{\ell}\binom{r}{\ell}\le\max_{\ell\ge 0}\frac{2^\ell}{\ell!}\cdot\sum_{\ell=0}^{r}d^\ell\binom{r}{\ell}=2(d+1)^r\\&\le 2^{r/3}\left(1+\tfrac{1}{d}\right)^r d^r\le\left(\tfrac{7}{6}\sqrt[3]{2}\right)^r d^r\le(\sqrt{e}d)^r,\end{aligned}$$

which gives

$$\ln n\le\ln C_p(r,d)\le\ln C_1(r,d)\le r(\tfrac{1}{2}+\ln d).$$

Together with (17) this implies (15).
To prove inequality (16) we first observe that

$$\binom{d}{\ell}\le\frac{d^\ell}{\ell!}=\frac{r^\ell}{\ell!}\left(\frac{d}{r}\right)^\ell\le\max_{0\le\ell\le r}\frac{r^\ell}{\ell!}\cdot\left(\frac{d}{r}\right)^\ell=\frac{r^r}{r!}\left(\frac{d}{r}\right)^\ell\le\frac{e^r}{\sqrt{2\pi r}}\left(\frac{d}{r}\right)^\ell$$

by Sterling's formula. This implies

$$n\le C_p(r,d)\le C_1(r,d\le\frac{e^r}{2\sqrt{\pi}}\sum_{\ell=0}^{r}\left(\frac{2d}{r}\right)^\ell\binom{r}{\ell}=\frac{e^r}{2\sqrt{\pi}}\left(1+\frac{2d}{r}\right)^r$$

and

$$\ln(2\sqrt{\pi}n)\le r+r\ln\left(1+\frac{2d}{r}\right)\le r\left(1+\ln\left(\frac{d}{2r}+\frac{2d}{r}\right)\right)=r\ln\left(\frac{5ed}{2r}\right).\qquad(18)$$

As one can easily verify, the function $f(x)=\frac{\ln x}{\sqrt{x}}$ is decreasing for $x\ge e^2$, whence

$$\ln x\le\sqrt{\frac{x}{x_0}}\ln x_0\qquad\text{whenever } x\ge x_0\ge e^2.$$

Applying this with $x:=\frac{5ed}{2r}\ge x_0:=5e$ we obtain

$$\ln n\le\ln(2\sqrt{\pi}n)\le r\sqrt{\frac{d}{2r}}\ln(5e)=\sqrt{\frac{dr}{2}}\ln(5e),$$

which implies

$$\frac{2d}{r}\le 2d\cdot\frac{d}{2}\left(\frac{\ln(5e)}{\ln n}\right)^2=\left(\frac{d\ln(5e)}{\ln n}\right)^2.$$

Since $n\le 2^d$, we have $1\le\left(\frac{d\ln 2}{\ln n}\right)^2$, and therefore

$$1+\frac{2d}{r}\le\underbrace{\left((\ln 2)^2+(1+\ln 5)^2\right)}_{\le 7.29\le 7.38\le e^2}\left(\frac{d}{\ln n}\right)^2\le e^2\left(\frac{d}{\ln n}\right)^2.$$

Inserting this in (18) we arrive at

$$\frac{1}{r} \leq \frac{1+\ln\left(1+\frac{2d}{r}\right)}{\ln(2\sqrt{\pi}n)} \leq \frac{3+2\ln\left(\frac{d}{\ln n}\right)}{\ln(2\sqrt{\pi}n)},$$

and in view of (17) the proof is finished. □

Remark 3.4 Clearly the approximation numbers strongly depend on the chosen norm in the Sobolev space. As shown by (14), (15) and (16), in the preasymptotic range the norm parameter p appears *even in the exponent* s/p. This is an interesting effect, which is in contrast to the asymptotic behaviour, where the rate $a_n \sim n^{-s/d}$ is the same for all norms (considered in this paper), while the norm parameter p influences only the hidden equivalence constants, see e.g. Section 4 of [5].

Remark 3.5 Comparing the two estimates in the theorem, it is easy to see that for small n and d the first inequality (15) gives a better bound than (16). However, for large d and moderate n the bound in (16) is better. For example, if $d \geq 26$ and $2^{\sqrt{26d}} \leq n \leq 2^d$, then

$$\frac{e^3 d^2}{(\ln n)^2} \leq \frac{e^3 d^2}{26d \cdot (\ln 2)^2} < 1.61d < \sqrt{e}d\,,$$

and therefore

$$\frac{3+2\ln\left(\frac{d}{\ln n}\right)}{\ln(2\sqrt{\pi}n)} < \frac{\ln\left(\frac{e^3 d^2}{(\ln n)^2}\right)}{\ln n} < \frac{\frac{1}{2}+\ln d}{\ln n}.$$

Now let $0 < p < 1$. We will prove similar results as in the case $p \geq 1$, and also the proof strategy is the same. The heart of the proof is to find good upper bounds for the cardinalities $C_p(r,d)$. But the difficulty is that for $p < 1$ they are larger than $C_1(r,d)$, and no exact formula is available. Therefore some modifications are required, which will involve volume arguments. As a preparation we state in the following lemma some auxiliary results that we will need later.

Lemma 3.6 (i) *Let* $0 < p < \infty$ *and* $d \in \mathbb{N}$. *Then the volume (d-dimensional Lebesgue measure) of* $B_p^d := \left\{x \in \mathbb{R}^d : \sum_{j=1}^d |x_j|^p \leq 1\right\}$ *is*

$$\mathrm{vol}_d\left(B_p^d\right) = \frac{2^d \Gamma(1+1/p)^d}{\Gamma(1+d/p)}.$$

(ii) *For all real numbers* $x > 1$ *and* $a > 0$ *the Gamma function satisfies*

$$\Gamma(1+x) \leq x^x \qquad \text{and} \qquad \Gamma(1+x) \geq a^x e^{-a}.$$

Proof. Formula (i) is well known, it can be found e.g. in [11].
For a proof of the first inequality in (ii) see e.g. [5].
The second inequality in (ii) is easy to prove. We have

$$\frac{\Gamma(1+x)}{a^x} = \int_0^\infty \left(\frac{t}{a}\right)^x e^{-t}\,dt \geq \int_a^\infty e^{-t}\,dt = e^{-a}. \quad \square$$

Now we give upper preasymptotic estimates for the case $0 < p < 1$.

Theorem 3.7 *Let $d \in \mathbb{N}, d \geq 4$, $s > 0$, $0 < p < 1$ and $2d+1 \leq n \leq 2^d$. Moreover let $c := \frac{3\sqrt{\pi}}{2} = 2.6586\ldots$. Then we have*

$$a_n\big(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \leq e^s \left[\frac{\frac{2}{p} + \ln\left(\frac{d}{\ln cn}\right)}{\ln cn}\right]^{s/p} .$$

Proof. Let $n \in \mathbb{N}$, $2d+1 \leq n \leq 2^d$. By Lemma 3.6 there is an integer r with $2 \leq r \leq d/2$ and $C_p(r-1,d) \leq n \leq C_p(r,d)$. This implies

$$a_n := a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq a_{C_p(r-1,d)} = r^{-s/p} .$$

Now we must estimate r in terms of n. According to (13) we have

$$C_p(r,d) = 1 + \sum_{\ell=1}^{r} 2^\ell \binom{d}{\ell} A_p(r,\ell) \tag{19}$$

where $A_p(r,\ell) = \operatorname{card} \mathscr{A}_p(r,\ell)$ and

$$\mathscr{A}_p(r,\ell) = \Big\{(m_i) \in \mathbb{N}^\ell : \sum_{i=1}^{r} m_i^p \leq r\Big\} .$$

Our first aim is to estimate the cardinalities $A_p(r,\ell)$. To this end we consider the pairwise disjoint cubes $Q_k := k + (-1,0]^\ell$, $k \in \mathbb{N}^\ell$. We have the inclusion

$$\bigcup_{k \in \mathscr{A}_p(r,\ell)} Q_k \subseteq r^{1/p} B_p^\ell \cap \mathbb{R}_+^\ell ,$$

where $B_p^\ell = \{x \in \mathbb{R} : \sum_{j=1}^\ell |x_j|^p \leq 1\}$ is the unit ball in $\mathbb{R}^\ell$ equipped with the p-norm. Taking volume (Lebesgue measure in $\mathbb{R}^\ell$) we get

$$A_p(r,\ell) \leq r^{\ell/p} \frac{\operatorname{vol}\big(B_p^\ell\big)}{2^\ell} = r^{\ell/p} \frac{\Gamma(1+1/p)^\ell}{\Gamma(1+\ell/p)} .$$

Here we used the volume formula from part (i) of Lemma 3.6. Moreover, by part (ii) of the same Lemma we have

$$\Gamma(1+1/p) \leq (1/p)^{1/p} \qquad \text{and} \qquad \Gamma(1+\ell/p) \geq (r/p)^{\ell/p} e^{-r/p} ,$$

which implies $A_p(r,\ell) \leq e^{r/p}$. Inserting this in (19) we obtain

$$C_p(r,d) \leq e^{r/p} \sum_{\ell=0}^{r} 2^\ell \binom{d}{\ell} \leq e^{r/p} \sum_{\ell=0}^{r} \underbrace{\frac{(2d)^\ell}{\ell!}}_{:=b_\ell} .$$

The sum can easily be bounded by comparing with a geometric series. Since $2 \leq r \leq d/2$, the summands satisfy $\frac{b_\ell}{b_{\ell-1}} = \frac{2d}{\ell} \geq 4$. By induction we get $b_\ell \leq 4^{\ell-r} b_r$, and using Sterling's formula we obtain

$$\sum_{\ell=0}^{r} b_\ell \leq b_r \sum_{\ell=0}^{r} 4^{\ell-r} \leq \frac{4b_r}{3} \leq \frac{4}{3} \cdot \frac{(2d)^r}{r!}$$
$$\leq \frac{4}{3\sqrt{2\pi r}} \left(\frac{2ed}{r}\right)^r \leq \frac{2}{3\sqrt{\pi}} \left(\frac{2ed}{r}\right)^r .$$

This implies, with constant $c := \frac{3\sqrt{\pi}}{2}$,

$$\ln cn \leq \ln C_p(r,d) \leq r \ln\left(\frac{2e^{1+1/p} d}{r}\right) .$$

Now we set

$$\alpha := \frac{p}{p+1} \quad , \quad \text{which gives} \quad \frac{1}{\alpha} = 1 + 1/p \quad \text{and} \quad \frac{1}{1-\alpha} = p+1 .$$

It is easy to show that the function $f(x) := x^{-\alpha} \ln x$ attains its maximum over $[1,\infty)$ at $e^{1/\alpha}$, whence

$$\ln x \leq x^\alpha f(e^{1/\alpha}) = \frac{x^\alpha}{\alpha e} \qquad \text{for all } x \geq 1 .$$

Applying this with $x := e^{1+1/p} \frac{2d}{r} = e^{1/\alpha} \frac{2d}{r}$ we get

$$\ln cn \leq r \ln x \leq r \frac{x^\alpha}{\alpha e} = \frac{r}{\alpha} \left(\frac{2d}{r}\right)^\alpha .$$

Multiplying this inequality with $\left(\frac{2d}{r}\right)^{1-\alpha}$ and dividing by $\ln cn$ we obtain

$$\left(\frac{2d}{r}\right)^{1-\alpha} \leq \frac{2d}{\alpha \ln cn} .$$

Recall that $\frac{1}{\alpha} = 1 + 1/p$ and $\frac{1}{1-\alpha} = p+1$, and therefore

$$\frac{2d}{r} \leq (1+1/p)^{p+1} \left(\frac{2d}{\ln cn}\right)^{p+1} \leq e^{\frac{p+1}{p}} \left(\frac{2d}{\ln cn}\right)^{p+1} \leq \left(e^{1/p} \frac{2d}{\ln cn}\right)^{p+1} .$$

We conclude that

$$\ln cn \leq r \ln\left(e^{1/\alpha} \frac{2d}{r}\right) \leq r \ln\left(\left(e^{2/p} \frac{2d}{\ln cn}\right)^{p+1}\right) = r(p+1) \left[\frac{2}{p} + \ln\left(\frac{2d}{\ln cn}\right)\right] .$$

Observing that $(p+1)^{1/p} \leq e$, we finally arrive at the desired estimate

$$a_n \leq r^{-s/p} \leq e^s \left[\frac{\frac{2}{p} + \ln\left(\frac{2d}{\ln cn}\right)}{\ln cn} \right]^{s/p},$$

and the proof is finished. □

4 Preasymptotics for approximation in mixed Sobolev spaces

In this final section we give some new preasymptotic estimates for the approximation numbers of L_2-embeddings of Sobolev spaces $H^s_{\mathrm{mix}}(\mathbb{T}^d)$ of dominating mixed smoothness as defined in Definition 2.3. In particular this definition implies

$$\|I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\| = 1 \qquad \text{for all } d \in \mathbb{N} \text{ and } s > 0\,.$$

For the long history of mixed Sobolev spaces we refer to the comments in [6]. It is well known that

$$a_n(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim n^{-s}(\ln n)^{s(d-1)}\,.$$

Note that the function $f(x) := x^{-1}(\ln x)^{(d-1)}$ is increasing for $1 \leq x \leq e^{d-1}$, and $f(e^{d-1}) = \big((d-1)/e\big)^{d-1} \gg 1$. Hence this equivalence relation is useless in the preasymptotic range $n \leq 2^d$, since $a_n(I_d) \leq \|I_d\| = 1$ for all $n \in \mathbb{N}$.

In Theorem 4.9 of [6] it was shown that for all $s > 0$ and $d \in \mathbb{N}$ with $d \geq 2$ and $8 \leq n \leq \frac{d}{2}4^d$

$$a_n(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{e^2}{n}\right)^{s/(2+\log_2 d)}. \tag{20}$$

We shall improve this estimate as follows.

Theorem 4.1 *Let $s > 0$ and $d \in \mathbb{N}$. Then we have for all $n \geq 6$*

$$a_n(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{16}{3n}\right)^{s/(1+\log_2 d)}. \tag{21}$$

Proof. Our strategy is similar to the proof of Theorem 4.9 in [6]. For $r, d \in \mathbb{N}$ we consider the cardinalities

$$C(r,d) := \operatorname{card}\left\{k \in \mathbb{Z}^d : \prod_{j=1}^{d}(1+|k_j|) \leq r\right\}. \tag{22}$$

Let $m := \min\{d, \lfloor \log_2 r \rfloor\}$. By formula (3.1) in [6] we have

$$C(r,d) = 1 + \sum_{\ell=1}^{m} 2^\ell \binom{d}{\ell} A(r,\ell)\,,$$

where

$$A(r,\ell) = \operatorname{card} \mathscr{A}(r,\ell) \quad \text{with} \quad \mathscr{A}(r,\ell) = \Big\{ k \in \mathbb{N}^\ell : \prod_{j=1}^{\ell} (1+k_j) \le r \Big\} .$$

For the disjoint cubes $Q_k := k + [0,1)^\ell$ with $k \in \mathbb{N}^\ell$ we get the inclusion

$$\bigcup_{k \in \mathscr{A}(r,\ell)} Q_k \subseteq \mathscr{H}(r,\ell) := \Big\{ x \in \mathbb{R}^\ell : x_j \ge 1, \prod_{j=1}^{\ell} x_j \le r \Big\} .$$

Taking volume in $\mathbb{R}^\ell$ this implies

$$A(r,\ell) \le \operatorname{vol}_\ell(\mathscr{H}(r,\ell)) .$$

The set $\mathscr{H}(r,\ell)$ is of hyperbolic cross type, and for its volume we use the estimate, for all integers $\ell \in \mathbb{N}$ and all real numbers $r \ge 1$,

$$v_\ell(r) := \operatorname{vol}_\ell(\mathscr{H}(r,\ell)) \le \frac{r^2}{4} . \tag{23}$$

Inequality (23) can easily be verified by induction on ℓ. Indeed, for $\ell = 1$ we have

$$v_1(r) = r - 1 \le \frac{r^2}{4} ,$$

and if (23) holds for some $\ell \in \mathbb{N}$ and all $r \ge 1$, then the recursion formula

$$v_{\ell+1}(r) = \int_1^r v_\ell(r/t)\, dt$$

implies

$$v_{\ell+1}(r) \le \frac{r^2}{4} \cdot \int_1^r \frac{dt}{t^2} \le \frac{r^2}{4} \cdot \int_1^\infty \frac{dt}{t^2} = \frac{r^2}{4} .$$

If $r \ge 2$, then $1 \le \frac{r^2}{4}$, and combining (22) and (23) we obtain

$$C(r,d) \le \frac{r^2}{4} \sum_{\ell=0}^{m} 2^\ell \binom{d}{\ell} \le \frac{r^2}{4} \sum_{\ell=0}^{m} \frac{(2d)^\ell}{\ell!} = \frac{r^2}{4} \sum_{\ell=0}^{m} b_\ell , \tag{24}$$

where we used $\binom{d}{\ell} \le \frac{d^\ell}{\ell!}$ and have set $b_\ell := \frac{(2d)^\ell}{\ell!}$. Since $m \le d$, we have

$$\frac{b_\ell}{b_{\ell-1}} = \frac{2d}{\ell} \ge 2 \qquad \text{for all } \ell = 1, \dots, m .$$

Similarly as in the proof of Theorem 3.7, by comparing with a geometric series, this implies

$$\sum_{\ell=0}^{m} b_\ell \le 2b_m = 2\cdot \frac{4^m}{m!}\cdot\left(\frac{d}{2}\right)^m \tag{25}$$

Observing that $\max_{m\in\mathbb{N}} \frac{4^m}{m!} = \frac{4^3}{3!} = \frac{32}{3}$, formulae (24) and (25) yield for $d \ge 2$

$$C(r,d) \le \frac{r^2}{4}\cdot\frac{64}{3}\cdot\left(\frac{d}{2}\right)^m \le \frac{16r^2}{3}\left(\frac{d}{2}\right)^{\log_2 r} = \frac{16}{3}\, r^{1+\log_2 d}\,. \tag{26}$$

This is also true for $d = 1$, since $C(r,1) = \operatorname{card}\{k \in \mathbb{Z} : 1+|k| \le r\} = 2r+1 \le 3r$. For every $n \in \mathbb{N}$ with $n \ge 2$ there is an integer $r \ge 2$ such that

$$C(r-1,d) < n \le C(r,d)\,.$$

Then

$$a_n(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = a_{C(r,d)}(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = r^{-s}\,.$$

The first equality follows from the fact that $k \mapsto w(k) := \prod_{j=1}^{d}(1+|k_j|)$ is a map from $\mathbb{Z}^d$ *onto* $\mathbb{N}$, hence all elements of the non-increasing rearrangement $(\sigma_n)_{n\in\mathbb{N}}$ of $(1/w(k)^s)_{k\in\mathbb{Z}^d}$ are of the form r^{-s} for some $r \in \mathbb{N}$. But the σ_n are exactly the approximation numbers, see (4). Now $n \le C(r,d)$ and (26) finally imply the desired estimate

$$a_n(I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = r^{-s} \le \left(\frac{16}{3n}\right)^{s/(1+\log_2 d)}\,,$$

and the proof is finished. $\square$

We conclude the paper by some comments.

Remark 4.2 Our new inequality (21) improves (20) in several respects:

- The range of n is larger.
- The factor $\frac{16}{3}$ is smaller than e^2.
- Most importantly, the exponent $s/(1+\log_2 d)$ is better than $s/(2+\log_2 d)$.

Remark 4.3 In this note we only dealt with L_2-approximation, but it would be desirable to have corresponding results for L_∞-approximation, too. This problem was addressed in [2], where several sharp *asymptotic* estimates were established, including exact constants and their dependence on the dimension d and the smoothness parameter s. However, it seems much harder to prove *preasymptotic* estimates, this remains an open problem.

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